A GENERALIZATION OF STEINER'S FORMULAE

E. VIDAL ABASCAL

Let C be an arbitrary convex curve in the plane of length L and area F; and let C_ρ be a curve parallel to C at a distance ρ from it, of length L_ρ and area F_ρ. Then according to Steiner's classical result:

$$L_\rho = L + 2\pi\rho, \qquad F_\rho = F + \rho L + \pi\rho^2.$$

In this paper we develop a generalization of these formulae for curves lying on a curved surface whose curvature $K(v^1, v^2)$ (referred to geodesic parallel coordinates) is a function of v^2 alone. Explicit formulae are derived in the case of surfaces of constant curvature. In this treatment it is necessary to put certain restrictions on the curve C and the distance ρ to replace Steiner's assumption of convexity. These restrictions (which are discussed below) are stated in their most obvious form, and a discussion of methods of relaxing them is deferred to a later paper. Our chief results are contained in the formulae (12) and (15) below.

Let the curve C be a simple, closed, bounding, and differentiable curve on the surface S. Choose a coordinate system in which $v^1=0$ is the curve C, and in which $v^2=$constant are the geodesics orthogonal to C. Further let v^2 be the arc length of C measured positively for motion on the curve which keeps the bounded area to the left, and let v^1 be the arc length of geodesics normal to C measured positively outward from C. Choose the unit normals to C so that they point toward the interior of C. Then we have:

$$ds^2 = (dv^1)^2 + g_{22}(v^1, v^2)(dv^2)^2; \qquad g_{22}(0, v^2) = 1. \tag{1}$$

For the moment we ignore the question of determining the region of S within which such a coordinate system is valid, and proceed to compute $(g_{22})^{1/2}$. In this coordinate system we have the following relations (see L. P. Eisenhart *An introduction to differential geometry*, pp. 181 and 188)

$$\frac{\partial^2 (g_{22})^{1/2}}{\partial v^1 \partial v^1} + K(g_{22})^{1/2} = 0, \tag{2}$$

$$\kappa_g(v^2) = \left[\frac{\partial (g_{22})^{1/2}}{\partial v'}\right]_{v^1=0}, \tag{3}$$

Received by the editors December 26, 1946.

where κ_g is the geodesic curvature of C.

We assume that K is a differentiable function of v^2, that it is independent of v^1, and that it is never zero. Then the integration of (2) gives:

$$(g_{22})^{1/2} = f(v^2) \sin \left[v^1(K(v^2))^{1/2}\right] + h(v^2) \cos \left[v^1(K(v^2))^{1/2}\right]. \tag{4}$$

If $K(v^2)$ is negative, complex numbers are introduced, and f and h must be so chosen that the resulting value of $(g_{22})^{1/2}$ is real. From (3) we find that

$$\begin{aligned} \kappa_g(v^2) = \Big\{ & f(v^2)(K(v^2))^{1/2} \cos \left[v^1(K(v^2))^{1/2}\right] \\ & - h(v^2)(K(v^2))^{1/2} \sin \left[v^1(K(v^2))^{1/2}\right]\Big\}_{v^1=0} \end{aligned} \tag{5}$$

or $\kappa_g(v^2) = f(v^2)(K(v^2))^{1/2}$. Hence

$$f(v^2) = \frac{\kappa_g(v^2)}{(K(v^2))^{1/2}}. \tag{6}$$

Furthermore equation (4) must be valid along C, on which $v^1 = 0$ and $(g_{22})^{1/2} = 1$. Therefore from (4), $h(v^2) = 1$. Hence:

$$(g_{22})^{1/2} = \frac{\kappa_g(v^2)}{(K(v^2))^{1/2}} \sin \left[v^1(K(v^2))^{1/2}\right] + \cos \left[v^1(K(v^2))^{1/2}\right]. \tag{7}$$

The chosen coordinate system will fail to be valid whenever (1) $(g_{22})^{1/2} \leqq 0$; or (2) when v^1 is so large that the region described overlaps itself. The second difficulty may be overcome by considering overlapping portions to be on separate covering sheets of S (as in a Riemann surface), but we must assume that $(g_{22})^{1/2} > 0$. We let C_ρ be the closed curve $v^1 = \rho$ (const.) and restrict ourselves to the interior of C_ρ. Hence we require that:

$$\frac{\kappa_g(v^2)}{(K(v^2))^{1/2}} \sin \left[v^1(K(v^2))^{1/2}\right] + \cos \left[v^1(K(v^2))^{1/2}\right] > 0 \tag{8}$$

for all v^2 and for $0 \leqq v^1 \leqq \rho$. Without further assumptions on K, κ_g, and ρ no simplification of (8) is possible. However, for constant K the validity of (8) may be inferred from other simple assumptions as follows:

Case 1. $K = \text{constant} > 0$. Then if $\kappa_g(v^2) > 0$ and $0 \leqq \rho \leqq \pi 2K^{1/2}$ each term of (8) is positive, so (8) holds.

Case 2. $K = \text{constant} < 0$. Then (8) can more properly be written:

$$\frac{\kappa_g(v^2)}{(-K)^{1/2}} \sinh \left[v^1(-K)^{1/2}\right] + \cosh \left[v^1(-K)^{1/2}\right] > 0. \tag{8'}$$

And here if $\kappa_g(v^2) > 0$ and $\rho \geqq 0$, the inequality is valid.

These are the assumptions which correspond to Steiner's requirement that C be convex, and henceforth we consider only curves C and values of ρ for which they are verified.

Then from (7) the length of C_ρ is given by:

$$L_\rho = \int_C (g_{22}(\rho, v^2))^{1/2} dv^2 \tag{9}$$

or

$$L_\rho = \int_C \frac{\kappa_g(v^2)}{(K(v^2))^{1/2}} \sin\left[\rho(K(v^2))^{1/2}\right] dv^2 + \int_C \cos\left[\rho(K(v^2))^{1/2}\right] dv^2. \tag{10}$$

(We note that (10) holds even if C does not bound. However, the assumption that C bounds is essential for further developments.)

When K is constant, (10) may be simplified by the use of the Gauss-Bonnet formula:

$$\int_C \kappa_g(v^2) dv^2 = 2\pi - K \iint_{\text{Interior of } C} (g_{22})^{1/2} dv^1 dv^2 = 2\pi - KF. \tag{11}$$

Hence

$$L_\rho = 2\pi \frac{\sin\left[\rho K^{1/2}\right]}{K^{1/2}} - FK^{1/2} \sin\left[\rho K^{1/2}\right] + L \cos\left[\rho K^{1/2}\right]. \tag{12}$$

When K is negative (12) may more appropriately be written:

$$\begin{aligned} L_\rho = 2\pi \frac{\sinh\left[\rho(-K)^{1/2}\right]}{(-K)^{1/2}} &+ F(-K)^{1/2} \sinh\left[\rho(-K)^{1/2}\right] \\ &+ L \cosh\left[\rho(-K)^{1/2}\right]. \end{aligned} \tag{12'}$$

We note that as $K \to 0$, (12) and (12') approach Steiner's formula.

Finally to find F_ρ, the area of C_ρ, we consider

$$F_\rho = F + \int_C \left\{ \int_0^\rho (g_{22}(v^1, v^2))^{1/2} dv^1 \right\} dv^2 \tag{13}$$

or

$$\begin{aligned} F_\rho = F &+ \int_C \left\{ \int_0^\rho \kappa_g(v^2) \frac{\sin\left[v^1(K(v^2))^{1/2}\right]}{(K(v^2))^{1/2}} dv^1 \right\} dv^2 \\ &+ \int_C \left\{ \int_0^\rho \cos\left[v^1(K(v^2))^{1/2}\right] dv^1 \right\} dv^2. \end{aligned} \tag{14}$$

When K is constant, (14) simplifies as follows owing to (11):

$$F_\rho = L\frac{\sin[\rho K^{1/2}]}{K^{1/2}} - 2\pi\left(\frac{\cos[\rho K^{1/2}] - 1}{K}\right) + F\cos[\rho K^{1/2}]. \tag{15}$$

When K is negative (15) may more appropriately be written:

$$\begin{aligned} F_\rho = L\frac{\sinh[\rho(-K)^{1/2}]}{(-K)^{1/2}} &- 2\pi\left(\frac{\cosh[\rho(-K)^{1/2}] - 1}{K}\right) \\ &+ F\cosh[\rho(-K)^{1/2}]. \end{aligned} \tag{15'}$$

We again note that if $K \to 0$, formulae (15) and (15') approach Steiner's result.

Observatorie de Santiago de Compostela