INEQUALITIES CONNECTING SOLUTIONS OF CREMONA'S EQUATIONS

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1. **Introduction.** Let a complete and regular linear system $\Sigma_{p,d}$ of plane curves of *dimension* d, the *genus* of the general curve being p, be determined by its *order* x_0, and its multiplicities $x_1, \cdots, x_\rho$ at a set of ρ *general* base points. $x=(x_0; x_1, \cdots, x_\rho)$ is called the *characteristic* of $\Sigma_{p,d}$ and satisfies Cremona's equations:

$$(1)\qquad \begin{aligned} x_0^2 - x_1^2 - x_2^2 - \cdots - x_\rho^2 &\equiv (xx) = d + p - 1, \\ 3x_0 - x_1 - x_2 - \cdots - x_\rho &\equiv (lx) = d - p + 1. \end{aligned}$$

On the other hand, an integer solution x of (1) may or may not determine a linear system. If an x does determine a $\Sigma_{p,d}$, it is said to be *proper*. In this definition is included the usual convention that $(0; -1, 0, \cdots, 0)$ is a proper characteristic of the set of directions at a base point [1].[1]

If a system $\Sigma_{p,d}$ of characteristic x is subjected to a Cremona transformation C with F-points at the base points of Σ, $\Sigma \rightarrow \Sigma'_{p,d}$ whose characteristic x' at the F-points of C^{-1} is given by:

$$(2)\qquad L: \begin{aligned} x_0' &= (cx) \equiv c_0x_0 - c_1x_1 - c_2x_2 - \cdots - c_\rho x_\rho, \\ x_i' &= (\overset{i}{f}x) \equiv \overset{i}{f_0}x_0 - \overset{i}{f_1}x_1 - \overset{i}{f_2}x_2 - \cdots - \overset{i}{f_\rho}x_\rho, \end{aligned}$$
$$i = 1, 2, \cdots, \rho.$$

Here c is the characteristic of the homaloidal net of C^{-1} and the f^i are the characteristics of the P-curves of this net. Thus proper characteristics c of $p=0$, $d=2$ and proper characteristics f of $p=d=0$ play a central role in the theory and will be prominent in this article. The collection of all transformations L for a given ρ forms a group, G_ρ. G_ρ is generated by transformations L for which c is of type $(2; 1110 \cdots 0)$, and for any $L \in G_\rho$ the forms (xx), (lx) and (xy) are invariant.

In this paper attention is restricted to characteristics of $x_0>0$, and $p \geqq 0$ and $d \geqq 0$. We shall designate this as *property* A and obtain inequalities implied by (1) and property A. The inequalities are interesting in themselves and lead to a criterion for distinguishing proper characteristics.

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[1] Numbers in brackets refer to the references cited at the end of the paper.

2. Inequalities involving the characteristics of homaloidal nets.

THEOREM 1. *If x has property* A, *then* $2x_0-x_1-x_2-x_3\geqq 0$. *Moreover, the equals signs hold only for* $p=d=0$; $x=(1;110)$, $(1;101)$ *or* $(1;011)$.

Since $(xx)=d+p-1\geqq -1$, it may be shown that $x_0\geqq x_i$, $i=1,\cdots,\rho$. Indeed, set $x_i=x_0-a$ in $(xx)\geqq -1$:

$$2ax_0 - a^2 - x_1^2 - \cdots - x_{i-1}^2 - x_{i+1}^2 - \cdots - x_\rho^2 \geqq -1,$$

or

$$a(2x_0 - a) \geqq -1.$$

Since x_0 is a positive integer, a may not be negative. Thus the integers a_1, a_2, a_3 in $x_1=x_0-a_1$, $x_2=x_0-a_2$, $x_3=x_0-a_3$ are non-negative. Substituting these in the quadratic relation yields:

$$-2x_0^2 + 2x_0(a_1 + a_2 + a_3) - a_1^2 - a_2^2 - a_3^2 - x_4^2 - \cdots - x_\rho^2 \geqq -1.$$

Now $a_1, a_2, a_3, x_4, \cdots, x_\rho$ cannot all vanish, for this would imply that $-2x_0^2\geqq -1$. Thus:

$$2x_0(a_1 + a_2 + a_3) - 2x_0^2 > -1$$

or

$$a_1 + a_2 + a_3 - x_0 > -1/2x_0.$$

It follows that $a_1+a_2+a_3-x_0\geqq 0$ and thus that

$$2x_0 - (x_0 - a_1) - (x_0 - a_2) - (x_0 - a_3) \geqq 0.$$

If x is a characteristic with property A and $2x_0-x_1-x_2-x_3=0$, then the image of x under

$$\begin{aligned} & x_0' = 2x_0 - x_1 - x_2 - x_3, & \\ \mathrm{A}_{123}\colon\ & x_i' = x_i + (x_0 - x_1 - x_2 - x_3), & i = 1, 2, 3, \\ & x_j' = x_j, & j = 4, \cdots, \rho, \end{aligned}$$

has $x_0'=0$ and satisfies the same Cremona equations. Thus

$$\begin{aligned} -x_1'^2 - x_2'^2 - \cdots - x_\rho'^2 &= d + p - 1, \\ -x_1' - x_2' - \cdots - x_\rho' &= d - p + 1. \end{aligned}$$

This is possible only for d, $p=0$, 0; 1, 0 and 0, 1. A canvass of the cases reveals that d, $p=0$, 0 and $x'=(0;\ -1\ 0\cdots 0)$ comprise all possibilities. Thus $x=(1;110)$, $(1;101)$, $(1;011)$ are the only values of x for which the equals sign holds.

Since $(2; 1110 \cdots 0)$ is the characteristic of a homaloidal net of conics, the form of the inequality clearly suggests the following generalization:

THEOREM 2. *If x has property* A *and c is the characteristic of a homaloidal net, then $(cx) \geqq 0$. Moreover, the equals sign holds only for the characteristics of the principal curves of the homaloidal net.*

Consider first characteristics x of $p+d>0$. In this case, Theorem 1 asserts that any x' obtained from x under A_{ijk} has $x_0' > 0$. Since c is the characteristic of a homaloidal net, c is the image of $(1; 0, 0, \cdots, 0)$ under a sequence of transformations of the form A_{ijk}. Let $x \to x'$ under the sequence that sends $c \to c' = (1; 0, \cdots, 0)$. Since $x_0' > 0$, it follows that $(c'x') > 0$. Thus $(cx) > 0$, for this bilinear relation is invariant under G_ρ.

If $p=0$, $d=0$, a modification of the argument is required since in this case x_0' might vanish under some A_{ijk}. But in this case x is by Theorem 1 a proper characteristic. Thus an improper characteristic x of $p=d=0$ always goes into a characteristic of $x_0' > 0$ under A_{ijk} and the argument above applies. For proper characteristics x of $p=d=0$, it is clear that $(cx) \geqq 0$, else the rational curve would have too many intersections with the homaloidal net. If $(cx)=0$, x is the characteristic of a rational curve meeting the curves of the net only at the base points, and hence is the characteristic of a principal curve of the net.

3. Inequalities involving characteristics of rational curves.

LEMMA. *If x has property* A *and x^* denotes the same characteristic with x_ρ deleted, then x^* has property* A.

A simple computation yields for p', d' of x^*:

$$d' = d + x_\rho(x_\rho + 1)/2, \qquad p' - 1 = p - 1 + x_\rho(x_\rho - 1)/2.$$

Since $x_\rho(x_\rho+1)/2$ and $x_\rho(x_\rho-1)/2$ are non-negative functions of the integer x_ρ, the conclusion follows.

THEOREM 3. *If x has property* A *and $p+d>0$, and f is a proper characteristic of $p=d=0$ and $(fx)<0$, then $x_0>f_0$.*

Since f is proper, there is [2] an $L \in G_\rho$ such that $\bar{f}=L(f)=(0; 0, \cdots, 0, -1)$. $\bar{x}=L(x)$ has $\bar{x}_0>0$ by Theorem 2 and $(fx)=(\bar{f}\bar{x})<0$. But $(\bar{f}\bar{x})=\bar{x}_\rho<0$. Thus $\bar{x}$ may be written in the form

$$\bar{x} = \bar{x}^* + k\bar{f},$$

where k is a positive integer, and $\bar{x}^*$ is $\bar{x}$ with $\bar{x}_\rho$ deleted. Now consider the image of $\bar{x}$ under L^{-1}.

$$L^{-1}(\bar{x}) = L^{-1}(\bar{x}^* + k\bar{f}) = L^{-1}(\bar{x}^*) + kL^{-1}(\bar{f}),$$

or

$$x = L^{-1}(\bar{x}^*) + kf.$$

Now $\bar{x}^*$ has $\bar{x}_0^* > 0$, and $p' + d' > 0$ by the lemma. Hence by Theorem 2 its image $(\bar{x}^*)'$ has $(\bar{x}_0^*)' > 0$. Since

$$x_0 = (\bar{x}_0^*)' + kf_0,$$

it follows that $x_0 > f_0$.

THEOREM 4. *If x has property* A *and $p+d>0$, and f is a proper characteristic such that $p=d=0$ and $x_0 \leqq f_0$, then $(fx) \geqq 0$.*

Theorem 4 follows from Theorem 3 by formal reasoning and offers a generalization of a property of proper characteristics. For if x is a proper characteristic, $(fx) \geqq 0$ follows from the fact that the curves of the system may not have more than $f_0 x_0$ intersections with the irreducible rational curve associated with f. The significance of the theorem is that all characteristics $f_0 \geqq x_0 > 0$, $p \geqq 0$, $d \geqq 0$, $p+d>0$ must enjoy this same property.

There are examples of characteristics with property A and $p+d>0$ which even have $x_i > 0$, $i = 1, \cdots, \rho$, for which there is an f of $f_0 < x_0$ such that $(fx) < 0$. An early example is $(5; 3^2 1^6)$ and $(1; 1^2 0^6)$.

4. Applications.

THEOREM 5. *Let x be a characteristic of property* A *and $p+d>0$, such that $(fx) \geqq 0$ for all proper f of $p=d=0$ and $f_0 < x_0$; then $x_i \geqq 0$ and, moreover, if x' is the image of x under any $L \in G_\rho$, then $x_0' > 0$ and $x_i' \geqq 0$ for $i = 1, \cdots, \rho$.*

Since $\bar{f} = (0; 0^{\rho-1} - 1)$ is a proper f of $f_0 = 0 < x_0$ and $(\bar{f}x) \geqq 0$, it follows that $x_i \geqq 0$. By Theorem 4, $(fx) \geqq 0$ for all proper f, $p=d=0$ of $f_0 \geqq x_0$. Then $(fx) \geqq 0$ for all proper f. These characteristics f are simply permuted by any $L \in G_\rho$. Thus if $x' = L(x)$, it follows that $(fx') \geqq 0$ for all proper f. Since these include $(0; 0^{\rho-1} - 1)$, it follows as before that $x_i' \geqq 0$. Theorem 2 asserts that $x_0' > 0$.

The following important result is now easily established:

THEOREM 6. *Let c be a solution of* (1) *for $p=0$, $d=2$, $c_0>0$ such that $(fc) \geqq 0$ for all proper f of $p=d=0$ and $f_0 < c_0$, then c is the characteristic of a homaloidal net.*

As before, $c_i \geqq 0$ and it is known [3] that in such a case $c_0 - c_1 - c_2 - c_3 < 0$ if c_1, c_2, c_3 are the greatest of the numbers c_i and $c_0 > 1$. Thus under A_{123}, $c \rightarrow c'$ of $c_0' < c_0$ and by Theorem 5, $c_i' \geqq 0$, $i = 1, 2, \cdots, \rho$. Thus this reduction may be continued until $c_0'' = 1$, in which case $c'' = (1; 0, \cdots, 0)$. Under the given hypotheses, c is then the image of $(1; 0, \cdots, 0)$ under some $L \in G_\rho$ and must be proper.

This result has been conjectured much earlier and indeed was proved [4] by the writer, but the proof given on that occasion was quite elusive and unsatisfactory. Fragmentary results indicate that Theorem 5 has other important applications to cases where a generalization of Noether's inequality is possible. It would be desirable to avoid the restriction $p + d > 0$. It is possible that Theorem 5 might still be true if one removed this restriction and added at the end of the theorem "or else x' is of the type $(0; 0, \cdots, 0, -1)$."

References

1. A. B. Coble, *Cremona's diophantine equations*, Amer. J. Math. vol. 56 (1934) pp. 459–461. See especially p. 461.

2. J. L. Coolidge, *Algebraic plane curves*, 1931, p. 399.

3. Max Noether, *Ueber Flachen welche Schaaren rationaler Curven besitzen*, Math. Ann. vol. 3 (1871) p. 167.

4. G. B. Huff, *A sufficient condition that a C-characteristic be geometric*, Proc. Nat. Acad. Sci. U.S.A. vol. 29 (1943) p. 198.

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