The book leaves much to be done but this fact only enhances its interest. It should be productive of many extensions along the lines of economic interpretation as well as of mathematical research. In fact the authors suggest a number of directions in which research might profitably be pursued.

ARTHUR H. COPELAND

Principles of stellar dynamics. By S. Chandrasekhar. The University of Chicago Press, 1942. 10+251 pp.

The primary field of this book is astronomy and not mathematics, although the latter is used as an essential tool. The readers of this review, professional mathematicians almost exclusively, will have a normal human interest in the major astronomical aspects of the book, but their critical scrutiny is bound to be concentrated on how the astronomical problems are formulated mathematically and what sort of mathematics has been proposed for their solution. For this reason, and partly also in the interest of brevity, this review treats only of the mathematical aspects of the book.

In the first chapter is given a detailed discussion of the kinematical concepts appropriate to the study of stellar systems. Since these systems contain a large number of stars, it becomes necessary to introduce a method similar to that employed in hydrodynamics, where the motion of a fluid is described by a vector field, representing at each point and for each instant of time the velocity of the fluid. In hydrodynamics the velocity of the fluid at a point is conceived as the velocity of the "fluid particle" at the point in question. But this notion of a "particle" at the point in question is difficult to make precise, especially if one assumes the fluid to consist of a large number of small atoms with relatively large empty spaces between them. Nevertheless such a concept (in which the stars play the role of the atoms) is characteristic of stellar dynamics as distinguished from celestial (particle) mechanics, which considers systems containing but a relatively small number of bodies.

The components $U_0(x, y, z, t)$, $V_0(x, y, z, t)$, $W_0(x, y, z, t)$ of the vector field thus introduced do not, of course, necessarily represent the components of velocity of a star which might happen to be at the point (x, y, z), but rather the velocity of the centroid of stars in a "small volume" about the point (x, y, z). The components of velocity of an individual star are written in the form $U=U_0+u$, $V=V_0+v$, $W=W_0+w$, where the vector (u, v, w) is called the *residual* velocity. The statistical consideration of these residual velocities is a characteristic of stellar dynamics and gas theory as distinguished from classical hydrodynamics. Associated with each point of space is a distribution density function $F(u, v, w)$, such that $Fdudvdw$ is regarded as approximately measuring the number of stars per unit volume, the components of whose residual velocities lie in the intervals $(u, u+du)$, $(v, v+dv)$, and $(w, w+dw)$ respectively. Considerable space is devoted to a review of observed astronomical phenomena to justify the assumption that such functions U_0, V_0, W_0, F can be introduced in the manner indicated. Still more space is devoted to the study of the exact form of the distribution density F. The fact that it can often be assumed to be Gaussian (but not necessarily spherically symmetric as in the Maxwellian special case) is known as Schwarzschild's law. In cases when the distribution is not Gaussian, there is a hint that the given stellar system can be considered as the superposition of two or more systems each one of which obeys Schwarzschild's law. An example of such a situation is that afforded by the phenomenon of the so-called high velocity stars in our galaxy.

The most general form for the distribution density function, $F(u, v, w) = F[x, y, z, t; u, v, w] = F[x, y, z, t; U-U_0, V-V_0, W-W_0]$, considered in this book is a generalization of the Schwarzschild form, namely

$$F(u, v, w) = \Psi(Q + \sigma),$$

where Q is a positive definite quadratic form in u, v, w, whose coefficients together with σ are functions of x, y, z, and t.

The motion of a single star is assumed to be governed to a high degree of approximation by equations of the familiar type,

$$\ddot{r} = \text{grad } \Phi \qquad (r = ix + jy + kz). \tag{1}$$

Here Φ, a function of x, y, z, and t, is a so-called "smoothed-out" potential function, depending upon the general distribution of the other stars of the system. This assumption is evidently justified as long as the star in question is relatively far from the other stars, but it loses its significance if the star has a near encounter with another star. To indicate that the approximation (1) is indeed likely to be valid over extraordinarily long periods of time is the fundamental object of the work on the "relaxation time" of Chapter II, a subject to which we shall presently return.

Since $\Psi(Q+\sigma)$, considered as a function of x, y, z, U, V, W, may be interpreted as a distribution density function of the star in six-dimensional phase space, it is at once obvious that

$$\int\int\int\int\int\int \Psi(Q + \sigma)dxdydzdUdVdW$$

is an integral invariant of (1). Hence, by Liouville's theorem, $\Psi(Q+\sigma)$, and, in fact, $Q+\sigma$ itself, is a first integral of (1), the condition for which is the familiar partial differential equation,

$$(2)\quad \frac{\partial\Psi}{\partial x}U+\frac{\partial\Psi}{\partial y}V+\frac{\partial\Psi}{\partial z}W+\frac{\partial\Psi}{\partial U}\frac{\partial\Phi}{\partial x}+\frac{\partial\Psi}{\partial V}\frac{\partial\Phi}{\partial y}+\frac{\partial\Psi}{\partial W}\frac{\partial\Phi}{\partial z}=0.$$

From this equation there are two alternative ways of proceeding. One method is to derive (with J. H. Jeans) the equations of hydrodynamics. But these equations can not carry the full implications of the kinematical hypotheses, especially the one regarding the existence of a distribution function for the residual velocities. Indeed, the four hydrodynamical equations (that is, the equation of continuity plus the three equations expressing Newton's second law of motion for a fluid) are only the first four of a series of equations obtained by multiplying equation (2) by $U^pV^qW^r$, integrating with respect to U, V, and W over all values, and reducing by appropriate partial integrations. If the distribution function has the proper behaviour at infinity, one thus obtains an infinite number of equations by letting p, q, and r assume independently of each other the values $0, 1, 2, 3, \cdots$, whereas the four hydrodynamical equations are only those corresponding to $(p=q=r=0)$, $(p=1,\ q=r=0)$, $(q=1,\ p=r=0)$, $(r=1,\ p=q=0)$. Accordingly the author, although the hydrodynamical equations are mentioned and even briefly applied in Chapter IV, uses for the most part a more powerful method which will now be described.

Remembering that $\Psi=\Psi(Q+\sigma)$, where Q is a quadratic form in $U-U_0$, $V-V_0$, $W-W_0$, one sees that equation (2) after division by Ψ' has the form of a cubic polynomial in U, V, W, equated to zero. Since U, V, W are independent variables, we can equate to zero each one of the coefficients of this polynomial, thus obtaining twenty partial differential equations in the eleven unknown functions U_0, V_0, W_0, Φ, σ, and the six coefficients of Q. The study of these twenty equations in Chapter III appears to the reviewer to be the central theme of the entire work. The problem is evidently that of determining the most general conditions under which equations (1) admit an integral quadratic in the velocities. Although various special cases of this problem are well known in classical dynamics or have been considered in an astronomical context by various authors including Eddington and Jeans, the completely general problem seems to have been first formulated and studied by Chandrasekhar.

A very interesting and general result of the investigation of the twenty partial differential equations is to the effect that, for stellar systems in steady states and possessing some distribution function of the form $\psi(Q+\sigma)$, the potential Φ is necessarily characterized by helical symmetry. If, in addition, the stellar system is of finite extent, this helical symmetry reduces to axial symmetry. The case of non-steady states has not been completely treated except for certain special cases.

In Chapter IV the discussion centers on the case when the residual velocities have a spherical distribution. In this way the author is led to a theory of the spiral structure of extragalactic nebulae. He is careful to emphasize, however, that the class of spiral orbits predicted is so wide that the theory is really too general to give any indication as to why certain forms of spiral orbits are preferred to the exclusion of others.

In the fifth and last chapter there seems to be a departure from the central theme of the work as described above. The subject here is the classical n body problem. After a standard discussion of the elementary first integrals and of Lagrange's identity, statistical considerations are introduced for the purpose of obtaining information on the dispersion of velocities in clusters, the rate of disintegration of clusters by the escape of stars, and the time of relaxation of a cluster.

Although a cluster is usually regarded as a much smaller and more compact group of stars than a stellar system in the sense of the preceding chapters, a formula from Chapter II for the relaxation time of a stellar system is applied in Chapter V in the determination of the relaxation time of a cluster. The method used in Chapter II is to estimate the effect of two body encounters in the long history of a representative star.

Since a cluster may be a small part of a much larger stellar system, a generalization of the n body problem is also considered, in which superimposed on the potential of the cluster itself is the general "tidal" potential of the rest of the stellar system; but the only instance of such a situation considered in any detail is a case in which the tidal potential has both a plane and axis of symmetry and in which the center of gravity of the cluster describes a circular orbit.

It is statistically impossible in a work of such scope and originality to avoid a certain number of blemishes. Some which have attracted the attention of the reviewer are the following:

In concluding Chapter I, the author states, as if it were an important principle, that due to the considerations of this chapter "we are able to express the distribution function $\Psi(x, y, z; U, V, W; t)$ in the form $\Psi(x, y, z; U-U_0, V-V_0, W-W_0; t)$, where U_0, V_0, W_0 are functions of position and time only." Clearly this is not what the author really means, since the principle stated is trivially true with $U_0 \equiv V_0 \equiv W_0 \equiv 0$.

In Chapter II it would be well to insert an explanation of the term "impact parameter," as this term is not used in any of a half dozen standard treatises on dynamics which a reader might consult. Although what may be taken as a strictly mathematical definition is given in Appendix I, equation 26, there is no explanation of the word "impact" nor is it clear without further discussion why the integrated average over the impact parameter must be weighted proportionally to the parameter (cf. the factor DdD in the formula (2.313) instead of simply dD).

The remarks inserted in the bibliography at the end of Chapter III (pp. 133, 134) involving the expansion of the potential function in a Taylor's series give a deceiving sense of generality. Evidently the results to be obtained by choosing the origin so that the linear terms disappear would be valid only in the neighborhoods of the presumably rather rare critical points, unless it were possible to prove that the Taylor's series had a large domain of convergence.

In Chapter IV (pp. 185, 186) it would be desirable to note that $\overline{U}$, $\overline{V}$, and $\overline{W}$ are identical respectively with U_0, V_0, and W_0; it would be equally desirable to indicate the well known relation existing between $\overline{U^2}$, $\overline{UV}$, and so on, U_0, $U_0 V_0$, and so on, and the strain tensor.

In spite of these criticisms, the reviewer found the book to be extremely interesting, and he feels that it has reached the highest level of scientific merit.

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The mathematics of physics and chemistry. By H. Margenau and G. M. Murphy. New York, Van Nostrand, 1943. 12+581 pp. $6.50.

In this textbook Professor Margenau and Murphy have assembled a very useful collection of mathematical principles as applied in prewar fundamental research in physics and chemistry. Mathematicians may not, in general, be in sympathy with the authors' deliberate compromising of rigor of derivation to maintain an emphasis on applications. It is doubtful that the book will prove successful as a textbook without prerequisites including the conventional course in advanced calculus. On the other hand, it does fulfill a long standing need, particularly evident in smaller universities, for a textbook suitable as the basis for a mathematics course at this level for graduate students of physics and chemistry.