Uncleft rings are extensions of cleft rings, and the study of extensions naturally starts with inseparable fields. (Received April 1, 1944.)

119. John Williamson: *Hadamard's determinant theorem and the sum of four squares.*

A square matrix H of order n is called an Hadamard or an H-matrix (Jacques Hadamard, *Résolution d'une question relative aux déterminants*, Bull. Sci. Math. (2) vol. 17 (1893) Part I, pp. 240–246) if each element of H has the value ± 1 and if $|H|$ has the maximum possible value $n^{n/2}$. In the first part by an adaptation of methods used by R. E. A. C. Paley, *On orthogonal matrices*, Journal of Mathematics and Physics, Massachusetts Institute of Technology, vol. 12 (1933) pp. 311–320, it is shown that: (1) if there exists an H-matrix of order $m>1$, there exists an H-matrix of order $m(p^h+1)$ where p is an odd prime, and (2) there exists an H-matrix of order $N(N-1)$ where $N=2^t k_1 k_2, \cdots, k_r$ and $k_i = p_i^{h_i}+1 \equiv 0 \mod 4$, p_i an odd prime. In the second part it is shown that an H-matrix of order $4n$ exists if there exist four polynomials $A_i(x) = \sum_{j=0}^{n-1} a_{ij} x^j$, $i=1$, 2, 3, 4, satisfying the following conditions: $a_{ij} = a_{i,n-j} = \pm 1$, $\sum_{i=1}^{4}(A_i(\omega))^2 = 4n$ for every nth root ω of unity. Such polynomials $A_i(x)$ are determined for specific small values of n and in particular for $n=43$ thus showing the existence of an H-matrix of order 172, a result not previously known. (Received March 22, 1944.)

ANALYSIS

120. R. P. Agnew: *Summability of subsequences.*

If A is a regular (real or complex) matrix method of summability and x_n is a bounded complex sequence, then there exists a subsequence y_n of x_n such that the set L_Y of limit points of the transform Y_n of y_n includes the set L_x of limit points of the sequence x_n. (Received February 2, 1944.)

121. E. F. Beckenbach and Maxwell Reade: *Further results on mean-values and harmonic polynomials.*

In this paper the authors study the relation between the "vertex averages" used by Walsh (J. L. Walsh, Bull. Amer. Math. Soc. vol. 42 (1936) pp. 923–930) and the "peripheral" and "areal averages" used by Beckenbach and Reade (E. F. Beckenbach and Maxwell Reade, Trans. Amer. Math. Soc. vol. 53 (1943) pp. 230–238). From the relation noted it follows that some of the results of Walsh are equivalent to those obtained by Beckenbach and Reade, and moreover, by following the methods outlined by the latter authors, it is possible to extend Walsh's results to more general "vertex averages." (Received March 27, 1944.)

122. E. F. Beckenbach and Maxwell Reade: *Regular solids and harmonic polynomials.*

Suppose D is a domain containing a regular solid V_0 and ϕ is a class of functions $f(x, y, z)$ defined and continuous on D. It is assumed that if V is similar and parallel to V_0 then the value of $F(x, y, z)$ at the center of V is the mean of values of $f(x, y, z)$ at the vertices. The class ϕ is shown to consist of certain harmonic polynomials. For the five regular solids these classes are given in terms of three spherical harmonics and their partial derivatives. The solution of the problem, suggested by J. L. Walsh (Bull. Amer. Math. Soc. vol. 42 (1936) pp. 923–930), of determining the class of

functions satisfying the above mean-value property for all regular solids in D similar but not necessarily parallel to a given one, follows from the preceding results. (Received March 20, 1944.)

123. Stefan Bergman: *Certain classes of analytic functions of two real variables and their properties.*

The author derives conditions in order that the operator $T(f) = K(z, \bar{z})f(z) + \int_{c^1} E(z, \bar{z}, t)f[n(z, t)]dt$ transforms analytic functions into complex solutions, u, of $L(u) = u_{z\bar{z}} + au_z + bu_{\bar{z}} + cu = 0$, $z = x+iy$, $\bar{z} = x-iy$, x, y real. Here K, E, n are some fixed functions, c^1 is a fixed curve in the complex t plane, and f ranges over the totality of analytic functions of a complex variable which are regular at the origin. Choosing $K = 0$, $n = z(1-t^2)/2$, $c^1 = [-1 \leq t \leq 1]$ he shows that to every L the function $E(z, \bar{z}, t)$ can be determined in such a way that $E(z, 0, t) = (1-t^2)^{-1/2}$ and therefore $u(z, 0) = \int_{-1}^{1} f(z(1-t^2)/2)dt/(1-t^2)^{1/2}$. The above choice of K, E, n, c^1 defines a particular operator, P. The author shows that to every real solution U, of $L(U) = 0$, there exists a function f such that $U = \mathrm{Re}\ [P(f)]$. The singularities and the value distribution of the complex solutions, $u = P(f)$, of $L(u) = 0$ are studied. The sum of the indices of points where $u(z, \bar{z}) = b$ ($\bar{z}$ conjugate to z) and which are situated in the circle $x^2 + y^2 \leq r^2$ is denoted by $n[r, (u(z, \bar{z}) - b)^{-1}]$. Let $u(z, \bar{z}) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} A_{mn}z^m\bar{z}^n$ be an entire function. An upper bound for the growth of $n[r, (u(z, \bar{z}) - b)^{-1}]$ considered as function of r, which is independent of b, is given in terms of the A_{n0}, $n = 0, 1, 2, \cdots$, for a large class of differential equations, L. (Received March 27, 1944.)

124. Stefan Bergman: *The determination of some properties of a function satisfying a partial differential equation from its series development.*

Let $u = \sum D_{mn}z^m\bar{z}^n$, $D_{nm} = \overline{D}_{nm}$, $z = x+iy$, $\bar{z} = x-iy$, be a solution of the differential equation $L(U) = U_{z\bar{z}} + aU_z + \bar{a}U_{\bar{z}} + cU = 0$, c real, where a and c are entire functions of x and y. For certain types of equations $L(U) = 0$ the author determines the upper bounds for the growth of $|U|$ in terms of the subsequence $\{D_{m0}\}$, $m = 0, 1, 2, \cdots$. Let $Q^{(n)}(z)$, $n = 1, 2, \cdots$, be a set of analytic functions of a complex variable z. The point $2a$ is said to be a branch point of the type $\{Q^{(n)}\}$ of an analytic function $f(z)$ if, in the neighborhood of the point $2a$, $f(z) = \sum_{n=0}^{\infty} a_n z^n$ can be represented in the form $f(z) = \pi^{1/2}\sum_{n=0}^{\infty} 2^{-2n}\Gamma(2n+1)[\Gamma(n+1)]^{-1}Q^{(n)}(z)I^{1/2+n}[z^{-1/2}(a - z/2)]^{-1} + g(z)$ where $g(z)$ is regular at $z = 2a$ and I^k denotes the integral of the kth order. Necessary and sufficient conditions in terms of the a_n are given in order that f has one and only one singularity on the circle of convergence, which singularity is a branch point of the type $\{Q^{(n)}\}$. Generalization of this result to the case where $u(z, \bar{z})$ are certain complex solutions of $L(u) = 0$ are given. (Received March 9, 1944.)

125. B. H. Bissinger and Fritz Herzog: *An extension of some previous results on generalized continued fractions.*

In a previous paper (abstract 49-11-253) the authors have extended certain measure theoretical results concerning simple continued fractions to the f-expansion $x = f(a_1 + f(a_2 + \cdots)$, where f belongs to a certain class F_1 of functions. In this paper the following results are obtained: when $f \in F_1$ the set of all x, $0 < x < 1$, for which $a_n \neq k_n$ for all n (the k_n being a given sequence of positive integers) is of measure zero if and only if $\sum f'(k_n)$ diverges; for almost all x, $0 < x < 1$, the a_n assume every positive integer infinitely often; moreover, every finite sequence of positive integers

occurs in successive places infinitely often in the f-expansion of almost all x, $0 < x < 1$. In the case $f(t) = 1/t$ the above results represent corresponding statements about simple continued fractions. (Received April 1, 1944.)

126. Augusto Bobonis: *A sufficiency theorem for differential systems.*

The purpose of the present paper is to establish necessary and sufficient conditions for linear differential systems and boundary conditions of the form $y' = (A + \lambda B)y$, $M(\lambda)y(a) + N(\lambda)y(b) = 0$ to have an infinite number of characteristic values. These are obtained on the assumptions that the matrix $\|B\|$ has constant rank and the parameter λ enters linearly in both differential equations and boundary conditions. By defining a problem of Lagrange, and the use of the extremizing properties of the characteristic values, Reid (Trans. Amer. Math. Soc. vol. 44 (1938) pp. 508–521) established necessary and sufficient conditions for a system, not involving the linear parameter in the boundary conditions, to have an infinity of characteristic values. These conditions are obtained here by defining a Bolza problem and use of the extremizing properties of the characteristic values of the accessory boundary value problem arising from this problem of Bolza. (Received March 7, 1944.)

127. H. D. Brunk: *Dirichlet series meromorphic in a half-plane.* Preliminary report.

Applications are made of theorems given in another paper in obtaining information concerning the distribution of singular points (in particular, poles) of functions, given by Dirichlet series, which satisfy certain conditions of boundedness type in certain half-planes. For example, one theorem states that if the function $f(s)$ whose Dirichlet series expansion is $\sum a_n e^{-\lambda_n s}$ satisfies a certain condition of a boundedness type in an open half-plane containing its half-plane of absolute convergence, if the sequence $\{\lambda_n\}$ satisfies certain "density" conditions, and if the coefficients $\{a_n\}$ satisfy an essential condition, and if further $f(s)$ has as its only singularities in an open half-plane containing its half-plane of absolute convergence poles of finite maximum order at certain points $s_0 + 2\pi k i$ (k taking on only integral values) on its axis of absolute convergence, $f(s)$ having at least one pole on this axis, then $f(s)$ is essentially a linear combination of Taylor-Dirichlet series (series of the form $\sum b_n e^{-ns}$). (Received March 1, 1944.)

128. H. D. Brunk: *Theorems of composition for Dirichlet series.* Preliminary report.

Theorems generalizing for Dirichlet series Hadamard's familiar multiplication theorem for Taylor series have been given by Mandelbrojt (Acta Math. vol. 55 (1930) pp. 1–32). Let $f(s)$, $\phi(s)$, and $H(s) = H(f, \phi)$ represent the (uniform) functions whose Dirichlet series expansions are, respectively, $\sum a_n e^{-\lambda_n s}$, $\sum b_n e^{-\lambda_n s}$, and $\sum a_n b_n e^{-\lambda_n s}$. One theorem of Mandelbrojt imposes certain types of boundedness conditions on $f(s)$ and $\phi(s)$, satisfied in certain half-planes, with neighborhoods of their singular points extracted. The result gives a region in which $H(s)$ is holomorphic. The author of the present paper uses Mandelbrojt's method in generalizing this theorem, to give a theorem in which no such conditions are imposed on the functions $f(s)$ and $\phi(s)$. A region in which $f(s)$ is holomorphic is obtained, defined in terms of regions in which $f(s)$ and $\phi(s)$ are bounded. This theorem is then used to give a theorem closely resembling that of Mandelbrojt, and is also used with a theorem of Schottky to define regions in which $H(s)$ is holomorphic in terms of regions where the functions $f(s)$ and $\phi(s)$ fail to take on two values. (Received February 25, 1944.)

129. H. D. Brunk: *Transformations of Dirichlet series*. Preliminary report.

Let $f(s)$ be a function having a Dirichlet series expansion $\sum a_n e^{-\lambda_n s}$, and $F(s)$ the function whose Dirichlet series expansion is $\sum \psi(a_n)e^{-\lambda_n s}$, where $\psi(z)$ is an entire function, $\psi(z) = \sum c_k z^k$. If $f(s)$ has a nonpositive axis of absolute convergence and satisfies a condition of a boundedness type in a certain half-plane, then, under certain auxiliary conditions on the coefficients $\{a_n\}$, the sequence $\{\lambda_n\}$, and the magnitude of the coefficients $\{c_k\}$, the possible singular points of $F(s)$ are given very simply in terms of those of $f(s)$. (Received February 24, 1944.)

130. R. H. Cameron and W. T. Martin: *Transformations of Wiener integrals under a general class of linear transformations*.

In this paper the authors study the behavior of Wiener integrals under transformations of the form $y(t) = T[x](t) = x(t) + \int_0^1 K(t, s)x(s)ds + x_0(t)$. The function $x(t)$ ranges over a Wiener measurable set S in the space C of all continuous functions $x(t)$ on $0 \leq t \leq 1$ that vanish at the origin. The authors obtain the formula for changing variables from y to x in the integral $\int_{TS}^W F(y)d_W y$ for any functional $F[y]$ which is Wiener integrable over the transform TS of S by T. Finally, the authors show by evaluating $\int_C^W \exp\{-\lambda^2\int_0^1 x(s)^2 ds\} d_W x = [\operatorname{sech}\lambda]^{1/2}$ that this transformation theory may be used to calculate new integrals. (Received March 30, 1944.)

131. M. M. Day: *Cluster points of subsequences*.

Buck and Pollard (Bull. Amer. Math. Soc. vol. 49 (1943) pp. 924–931) showed that a number x is a cluster point of a sequence of real numbers $\{x_n\}$ if and only if x is a cluster point of almost every subsequence of $\{x_n\}$. If the index system of integers is replaced by a countable partially ordered system S with no terminal elements, the class C of cofinal subsets of S is a natural substitute for the class of infinite subsets of integers used in defining subsequences. It is shown that a measure can be defined in a natural way in C and that for every function g from S into a neighborhood space satisfying the first countability axiom, x is a cluster point of g if and only if x is a cluster point of $g|E$ for almost every E in C (where $g|E$ is the function equal to g defined only over E. (Received March 29, 1944.)

132. M. M. Day: *On symmetric closed convex curves*.

An elementary proof not using fixed point theorems is given for the following theorem: If C is a closed convex plane curve with a center of symmetry, there is a parallelogram P circumscribed about C such that the center of P is the center of C and moreover the midpoint of each side of P is on the curve C. (Received March 29, 1944.)

133. M. M. Day: *Inner-product spaces*.

Two characterizations are given of inner-product spaces, that is normed spaces which are linear subsets of generalized Hilbert spaces. The first of these is a weakening of the condition of von Neumann-Jordon (Ann. of Math. vol. 36 (1935) pp. 719–723); it holds for either real- or complex-linear spaces: *The normed linear space B is an inner-product space if and only if $\|b_1+b_2\|^2 + \|b_1-b_2\|^2 = 4$ whenever $\|b_1\| = \|b_2\| = 1$.* The second condition involves both elements b_i of B and β_i of B^* and is suggested by an elementary inequality that holds in all normed spaces: *The normed real-linear space B is an*

inner product space if and only if $\|\beta_1+\beta_2\| \cdot \|b_1+b_2\| + \|\beta_1-\beta_2\| \cdot \|b_1-b_2\| = 4$ *whenever* $\|\beta_i\| = \|b_i\| = \beta_i(b_i) = 1$. Whether this holds for complex linear spaces is unknown. Both sufficiency proofs begin with the real two-dimensional case; they use the theorem of the preceding abstract (*On symmetric closed convex curves*) to select an ellipse and then show that this ellipse must be the unit sphere. (Received March 29, 1944.)

134. Herbert Federer: *The Gauss-Green theorem.*

A new definition of the *exterior normal* of a subset of n-space at a point of n-space is given in terms of the Lebesgue densities of the set and its complement on opposite sides of a plane through the point. This concept is used together with the $(n-1)$-dimensional *surface measure* over n-space previously introduced by the author (see *Surface Area*, I and II, abstracts 49-11-259, 260. These papers will appear in the Transactions) to prove the Gauss-Green formula under weak assumptions. In particular, the formula applies to every bounded open set in the plane, whose boundary has finite Carathéodory linear measure. The present results are less complete in the case of dimension $n>2$. As an application of the general theory it is proved that the integral "around" a bounded open set in the plane of a function, which is analytic inside the open set and continuous on its boundary, equals zero, provided the boundary has finite Carathéodory linear measure. (Received March 22, 1944.)

135. J. G. Herriot: *Nörlund summability of multiple Fourier series.*

In a previous paper (Trans. Amer. Math. Soc. vol. 52 (1942) pp. 72–94) the author showed that certain regular Nörlund methods of summability including Cesàro (C, α), $\alpha \geq 1$ but not $\alpha < 1$, applied to the square partial sums s_{nn} of the double Fourier series of an integrable periodic function of two variables, possess the localization property, and that certain other regular Nörlund methods, including (C, α), $\alpha > 0$, applied to the s_{nn}, are effective almost everywhere. The same results are now obtained replacing the s_{nn} by the triangular partial sums T_n^0 formed by adding terms the sum of whose indices does not exceed n. For regular Nörlund methods with $p_n \geq 0$ (hence including (C, α), $\alpha \geq 0$), these localization results do not admit of extension to Fourier series of dimension exceeding two. The proof is based on a theorem of Banach and Steinhaus on linear functionals. In the author's previous paper it was shown that certain restricted double Nörlund methods, including restricted (C, α, β), $\alpha, \beta \geq 1$ but not α or $\beta < 1$, applied to the double Fourier series, possess the localization property. It is now shown that for Fourier series of dimension exceeding two many restricted multiple Nörlund methods, including restricted $(C, \alpha, \beta, \cdots, \kappa)$, $\alpha, \beta, \cdots$, $\kappa \geq 0$, do not possess the localization property. (Received March 15, 1944.)

136. J. D. Hill: *Cesàro summability of sequences of 0's and 1's.*

Let $0.a_1 a_2 a_3 \cdots$ be the dyadic representation of an irrational number x between 0 and 1. Borel showed that $\lim_n n^{-1} \sum_1^n a_k = 1/2$ for almost all x. This result asserts that *almost all* sequences of 0's and 1's are summable $(C, 1)$ to the value 1/2. In this paper it is shown that *almost all sequences of* 0's *and* 1's *are summable* (C, r) *to the value* 1/2 *for each* $r > 1/2$. The proof depends on the general inclusiveness theorem of Mazur, a well known lemma from the general theory of series, and certain properties of the Rademacher functions. (Received April 1, 1944.)

137. J. D. Hill: *Nörlund methods of summability that include the Cesàro methods of all positive orders.*

H. L. Garabedian (Bull. Amer. Math. Soc. vol. 48 (1942) pp. 124–127) has re-

cently given examples of Hausdorff methods of summability which include the Cesàro methods of all positive orders. In this paper the corresponding problem for Nörlund methods $(N;\ p_k)$ is considered. By employing the general inclusiveness theorem of Mazur (Math. Zeit. vol. 28 (1928) pp. 599–611) a necessary and sufficient condition is given for the relation $(N;\ p_k) \supset (C,\ r)$. Finally, it is shown that $(N;\ \cosh k^{1/2}) \supset (C,\ r)$ for every $r > 0$. This article will appear shortly in the American Journal of Mathematics. (Received April 1, 1944.)

138. H. K. Hughes: *The asymptotic expansions of entire functions defined by Maclaurin series.*

Asymptotic expansions of the entire function $f(z) = \sum_{n=0}^{\infty} g(n) z^n$ (radius of convergence $= \infty$), valid for large values of $|z|$, are obtained under the asumption that the function $g(w)$, where $w = x + iy$, can be represented asymptotically in the form $g(w) \sim \sigma^w \sum_{n=0}^{\infty} c_n [\Gamma^{-1}(\alpha n + t + n)]$, where σ and α are positive, while the c's and t are any real or complex constants. Let $Z_\mu = (\sigma z e^{2\pi i \mu})^{(1/\alpha)}$, $Y_\mu = \bar{\alpha}' Z_\mu^{1-t} e^{Z_\mu} \sum_{n=0}^{\infty} c_n (Z_\mu)^{-n}$, and $-\pi < \arg z \leq \pi$. Then it is shown that $f(z) \sim \sum (Y_\mu) - \sum_{n=1}^{\infty} g(-n) z^{-n}$, the first summation being taken over those integral values of μ which satisfy $|\arg z + 2\pi\mu| \leq \pi\alpha/2$. The proof is based on recent theorems of W. B. Ford (*The asymptotic developments of functions defined by MacLaurin series*, University of Michigan Studies, Scientific series, vol. 11, 1936, p. 30) and C. V. Newsom (Amer. J. Math. vol. 60 (1938) pp. 561–572), wherein the function $f(z)$ is expressed in a form containing a definite integral. It consists largely of finding the asymptotic expansions of this integral. (Received February 21, 1944.)

139. Dunham Jackson: *Orthonormal polynomials on loci of the second degree.*

In a paper presented to the Society some years ago (abstract 45-5-192) but not published, the author discussed the boundedness of orthonormal polynomials in two variables on certain elliptic and hyperbolic loci. The treatment was based on theorems of Peebles concerning the preservation of the property of boundedness of orthonormal polynomials or trigonometric sums in a single variable on transition from one weight function to another. The approach is simplified now by the use of Korous's theorem on the bounds of orthonormal polynomials in place of the theorems of Peebles, and the scope of the results is extended in various ways, by the consideration of a greater variety of domains of orthogonality and especially by the use of affine linear transformations in the plane of the variables. (Received February 14, 1944.)

140. Herman Kober: *Approximation by integral functions in the complex domain.*

Let $F(z)$ be analytic and bounded in a sector of opening θ, $0 < \theta < 2\pi$, and uniformly continuous on the boundary. Then there exist integral functions $g_\alpha(z)$ such that $|g_\alpha(z)| \leq A \exp [(\alpha |z|)^\rho]$, $\rho = \pi/(2\pi - \theta)$, and $\lim_{\alpha \to \infty} g_\alpha(z) = F(z)$ uniformly in the sector. Here ρ is the least possible order of the approximations and their types can stay bounded if and only if $F(z)$ itself is an integral function of order ρ and finite type. More general domains than sectors can be handled. Sharper results are possible when the sector is a half-plane and here ordinary boundedness may be replaced by boundedness in the mean of order p with corresponding interpretation of convergence. Extensions to unbounded functions are possible but offer greater difficulties. The author also develops the theory of functions holomorphic and bounded in the mean

of order p in a strip and proves that such a function may be approximated uniformly in mean of order p by integral functions of order one. (Received March 17, 1944.)

141. L. H. Loomis: *Abstract congruence and the uniqueness of Haar measure.*

The author proves simultaneously—and hence without use of the axiom of choice—the existence and uniqueness of Haar measure in any locally compact metric space satisfying the following weak combinatorial congruence axiom: if S_1 and S_2 are compact spheres with the same radius and if S_1 can be covered by N open spheres of radius x then so can S_2. A Haar measure is a Lebesgue measure in which "congruent sets" (compact spheres with the same radius, in this case) have equal measures. For intuitive simplicity the theorem is stated and proved in the metric case, but the proof is formulated so as to hold without modification in a suitably restricted uniform structure. The theory of Haar measure in locally compact groups is an immediate application. (Received February 8, 1944.)

142. A. N. Lowan and H. E. Salzer: *Coefficients for complex interpolation within a square grid.*

When interpolation by Taylor series up to terms in $(\Delta z)^4$ is adequate for the desired accuracy, the value of $f(z_0+\Delta z)$ may be obtained with the aid of a complex cubic approximation polynomial derived from the general Lagrange-Hermite interpolation formula. If z_{-1}, z_0, z_1, and z_2 denote points in the z-plane at the corners of a square of length h whose sides are parallel to the axes, then $f(z_0+\Delta z)=f(z_0+Ph)$ $=L_{-1}(P)f(z_{-1})+L_0(P)f(z_0)+L_1(P)f(z_1)+L_2(P)f(z_2)$, where the L's are cubic polynomials in $P=p+iq$. The present table facilitates interpolation in the complex plane, whenever $\Delta z=(p+iq)h$ where p and q range from 0 to 1 at intervals of 0.1. A similar table has been prepared for the case when a complex quadratic polynomial suffices to approximate the function and is based on the three points z_{-1}, z_0, and z_1. It should be noted that for purposes of systematic and extensive subtabulation, a method based on the use of real Lagrangian interpolation coefficients is more economical than the method here discussed. (Received March 10, 1944.)

143. G. W. Mackey: *On infinite dimensional linear spaces.*

A general description of the contents of this paper and statements of some of the theorems which it contains will be found in Proc. Nat. Acad. Sci. U.S.A. vol. 29 (1943) pp. 216–221. (Received March 28, 1944.)

144. A. M. Peiser: *On the average sum of the real roots of a random algebraic equation.*

Let $F\equiv\sum_{j=0}^n x_j t^j=0$ be an algebraic equation with real coefficients, and let $t_1,\cdots,t_k$ denote the real roots of $F=0$ in $a\leq t\leq b$. It is shown that if $\phi(t)$ is continuous, $a\leq t\leq b$, then $\sum_{j=1}^k\phi(t_j)=\lim_{\epsilon\to0}(1/2\epsilon)\int_a^b\phi(t)\psi_\epsilon(F)|\partial F/\partial t|dt$, where $\psi_\epsilon(x)=1$ if $|x|<\epsilon$ and $\psi_\epsilon(x)=0$ otherwise. If $x_0,\cdots,x_n$ are independent standard normally distributed random variables, and if $\phi(t)=|t|$, then the average value (mathematical expectation $=$m.e.) of the sum of the absolute values of the roots of $F=0$ in $a\leq t\leq b$ is given by $S_n(a,\ b)=$m.e. $\{\sum_{j=1}^k|t_j|\}=\lim_{\epsilon\to0}(1/2\epsilon)\int_a^b|t|$ m.e. $\{\psi_\epsilon(F)|\partial F/\partial t|\}dt$ $=(1/\pi)\int_a^b|t|A_n(t)dt$, where $A_n(t)$ is a concrete positive function of t. In particular, for $a>1$, $S_n(-a,\ a)\sim(2/\pi)$ log n. Kac (Bull. Amer. Math. Soc. vol. 49 (1943) pp. 314–320) has shown that the average number of real roots of $F=0$ in any interval

containing 1 and -1 is asymptotically equal to $(2/\pi) \log n$. Thus, the present result makes even more apparent the fact already noticed by Kac that the real roots of random equations show a strong tendency to cluster (on the average) around 1 and -1. Finally it is shown that for $a < 1$, $\lim_{n\to\infty} S_n(-a, a) = -(1/\pi) \log (1-a^2)$. (Received March 24, 1944.)

145. Maxwell Reade: *On functions of class PL.*

If $f(x, y)$ is continuous in a domain D, then $f(x, y)$ is said to be of class PL in D provided $f(x, y) \geqq 0$ and $\log f(x, y)$ is subharmonic in D. Generalized Laplacean operators are introduced for functions of class PL in order to characterize those functions in terms of the operators. Since $f(x, y)$ of class PL implies that $f(x, y)$ is subharmonic too, there exist generalized mass distributions μ_1 and μ_2 associated with $f(x, y)$ and $\log f(x, y)$ respectively; relations between μ_1 and μ_2 are given. A typical result is the following theorem. If $f(x, y)$ is continuous and of class PL in a domain D, and if μ_1 is the associated generalized mass distribution for $f(x, y)$, then $\lim_{r\to 0} \{ [L(f; P_0, r)]^2 - A(f^2; P_0, r) \} / (r^2/4) = f(P_0) D_s \mu_1(P_0) - f_x^2(P_0) - f_y^2(P_0)$ for almost all points P_0 in D. Here $L(f; P_0, r)$ denotes the peripheral mean and $A(f; P_0, r)$ the areal mean of $f(x, y)$ for a circle with center at P_0 and radius r. (Received March 20, 1944.)

146. Maxwell Reade: *Two applications of generalized Laplaceans.*

Generalized Laplaceans are used to lessen some of the differentiability requirements in (a) the Kierst-Saks problem (S. Saks, Acta Univ. Szeged. vol. 5 (1930–1932) pp. 187–193) and (b) the author's extension of the isoperimetric inequality for functions having subharmonic logarithms (Maxwell Reade, Bull. Amer. Math. Soc. vol. 49 (1943) pp. 894–897). (Received March 22, 1944.)

147. L. I. Schoenfeld: *A transformation formula in the theory of partitions.*

Let $f(x) = \prod_{m=1}^{\infty}(1-x^{m^\kappa})^{-1}$ where κ is a positive integer. Then $f(x) = \sum_{m=0}^{\infty} p(m)x^m$, $p(0) = 1$, $|x| < 1$, so that $f(x)$ is the generating function of $p(m)$, the number of unrestricted partitions of m into κth powers. A shorter and simpler proof of Wright's Theorem 4 (Acta Math. vol. 63 (1934) pp. 143–191) is given; this theorem gives a transformation formula for $f(x)$ which exhibits its behavior in the neighborhood of its singularities at the rational points of the unit circle. The proof given here makes use of a Mellin transform technique of Rademacher (J. Reine Angew. Math. vol. 167 (1931) pp. 312–336). (Received March 16, 1944.)

148. C. F. Stephens: *Nonlinear difference equations analytic in a parameter.*

The author considers the system of difference equations (1) $y_i(x+1) = \sum_j^n b_{ij}(x)y_j(x) + f_i(y_1(x), \cdots, y_n(x); p(x); x)$, where the f_i are analytic in the $y_i(x)$ and $p(x)$, but only continuous in x in the neighborhood of $(x = \infty, y_1(x) = 0, \cdots, y_n(x) = 0, p(x) = 0)$. The parameter $p(x)$ is periodic of period 1, f_i contain no linear terms in $y_i(x)$ and $p(x)$, and $f_i(0, \cdots, 0; 0; x) \equiv 0$. The purpose of the paper is an investigation of those solutions of (1) which are analytic in $p(x)$ and continuous in x. It is shown that such solutions exist. Past methods made the associated system of linear equations $y_i(x+1) = \sum_j^n b_{ij}(x)y_j(x)$ fundamental and cannot in general be applied to the case where $b_{ij}(x) \equiv 0$. The author arrives at generalizations in two ways. (a) The f_i are taken to be continuous in x and not analytic as is usually required. (b) A new method

of treating difference equations is established and can be applied when $b_{ij}(x) \equiv 0$, and also when $p(x)$ is replaced by 0 in (1). The method makes use of analytic implicit functions and a matrix transformation, and the case where $b_{ij}(x) \neq 0$ is made to depend upon the case $b_{ij}(x) \equiv 0$. (Received February 1, 1944.)

149. Otto Szász: *On uniform convergence of trigonometric series.*

This paper contains generalizations of some theorems due to Chaundy and Jolliffe, to Hardy, and to the author. The following are some of the results. The trigonometric series $\sum_{1}^{\infty} b_n \sin nt$ is uniformly convergent if $\sum_{n}^{2n} |b_\nu - b_{\nu+1}| = O(n^{-1})$ and if the sequence nb_n is Abel summable to zero. The power series $\sum_{1}^{\infty} c_n z^n$ is uniformly convergent in $|z| \leq 1$, if $\sum_{n}^{2n} |c_\nu - c_{\nu+1}| = O(n^{-1})$ and if $\sum c_n$ is Abel summable. The essential part of the proof concerns the point $z = 1$, that is $t = 0$; a device of the Tauberian type is employed. (Received March 18, 1944.)

150. F. A. Valentine: *Contractions in non-euclidean spaces.*

Let $f(x)$ be a function mapping a set S in a metric space M into a set S' in a metric space M', and suppose a contraction of the type $\|f(x_1), f(x_2)\|' \leq \|x_1, x_2\|$ holds in S and S'. The existence of an extension of the range of definition of such a function so as to preserve a contraction depends upon M and M'. In this article the author shows *the extension exists when* $M = M'$ *is the n-dimensional hyperbolic* space. The proof used is applied to a metric space which includes both the hyperbolic and the spherical cases. Hence a unification of results is also obtained. (Received February 21, 1944.)

151. S. E. Warschawski: *On conformal mapping of nearly circular regions.*

Generalizing results of L. Bieberbach (Sitzungsberichte, Berliner Akademie, 1923) and of A. R. Marchenko (Bull. Acad. Sci. U.S.S.R. 1935), the author proves the following theorem: Let R be a simply connected region with the properties: (i) R contains the origin $w = 0$ and its boundary lies in the ring $1 \leq |w| \leq 1 + \epsilon$, ϵ being a fixed positive number; (ii) there exists a number $\eta \geq \epsilon$ such that any two points P_1 and P_2 of R of distance less than ϵ can be connected in R by an arc of *diameter* less than η. If $w = f(z)$ maps the circle $|z| < 1$ conformally onto R ($f(0) = 0$, $f'(0) > 0$) then, for all $|z| < 1$, $|f(z) - z| \leq B\epsilon \log (1/\epsilon) + 4\eta$, where B is an absolute constant. Analogous results for the derivatives of the mapping function, such as the following, are established. Let C be a simple closed curve $\rho = \rho(\phi)$, $0 \leq \phi \leq 2\pi$ (ρ, ϕ polar coordinates), such that $1 \leq \rho(\phi) \leq 1 + \epsilon$, $|\rho'/\rho| \leq \epsilon$, and that $|\rho'(\phi_2)/\rho(\phi_2) - \rho'(\phi_1)/\rho(\phi_1)| \leq \epsilon |\phi_2 - \phi_1|$, $0 < \epsilon < 1$. If $f(z)$ (normalized as above) maps $|z| < 1$ onto the interior of R, then, for $|z| \leq 1$, $(A(1+\epsilon^2)^{1/2})^{-1} \leq |zf'(z)/f(z)| \leq A(1+\epsilon^2)^{1/2}$ and $|f'(z) - 1| \leq 5(A\epsilon + A - 1)$, where $A = 4^\epsilon e^{\epsilon^2}$. (Received April 1, 1944.)

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152. Wilfred Kaplan and Max Dresden: *The mechanism of the condensation of gases.*

The criterion previously formulated (see abstract 49-5-158) for the condensation of a gas: namely, that condensation occurs at energy zero, when the topological structure of the energy surface changes, is further explored. It leads to a qualitative picture