REMARKS ON TRANSITIVITIES OF BETWEENNESS

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This note provides lattice theoretic interpretations of the transitivities

T_8. $abc \cdot dab \cdot xcd \cdot a \neq b \rightarrow acx$,
T_9. $abc \cdot dab \cdot xcd \cdot a \neq b \rightarrow bcx$,
T_{10}. $abc \cdot abd \cdot xbc \cdot a \neq b \cdot b \neq c \rightarrow xbd$,

introduced by Pitcher and Smiley.[1] It may be recalled that in a lattice the relation abc (b is between a and c) is said to hold if and only if

$$(a \cup b) \cap (b \cup c) = b = (a \cap b) \cup (b \cap c).$$

THEOREM 1. *If L is a lattice then its betweenness relation has one of the transitivities T_8 or T_9 if and only if L is linearly ordered.*

PROOF. It is obvious that T_8 and T_9 are satisfied if L is linearly ordered. To show that T_8 implies linear order, consider two elements $a, c \in L$. Suppose that a and c are not comparable, that is, that none of the relations $a=c$, $a<c$, $a>c$ holds. Then $a \neq a \cup c$, $c \neq a \cup c$. Moreover, we have

$$a\ a \cup c\ c \cdot a \cap c\ a\ a \cup c \cdot a \cup c\ c\ a \cap c \cdot a \neq a \cup c$$

and by T_8 this implies $a\ c\ a \cup c$ which, with $a\ a \cup c\ c$, implies $c = a \cup c$, contrary to our assumption that $c \neq a \cup c$. In the same way T_9 can be shown to imply linear order.

THEOREM 2. *If L is a lattice then its betweenness relation has the transitivity T_{10} if and only if L is linearly ordered or is composed of two linearly ordered systems with a common greatest element, I, and a common least element,* 0.

PROOF. It is easy to see that lattice betweenness in such a lattice has the transitivity T_{10}. Denote the two linearly ordered systems by L_1 and L_2. Then if, in the hypotheses of T_{10}, $b \neq 0$, $b \neq I$, $b \in L_1$, all the elements a, c, d, and x must belong to L_1 and the conclusion follows from the fact that T_{10} holds for linear order. If $b=0$ or $b=I$ in the hypotheses of T_{10} and if $a \in L_1$, then we must have $c \in L_2$, $d \in L_2$, $x \in L_1$ and the conclusion again follows.

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[1] Everett Pitcher and M. F. Smiley, *Transitivities of betweenness*, Trans. Amer. Math. Soc. vol. 52 (1942) pp. 95–114. We shall use the notations and terminology of Pitcher and Smiley.

To show that T_{10} implies that the lattice must have one of these forms, we show first that the lattice consists of a number of chains connecting 0 and I unless it is linearly ordered and then show that the number of these chains cannot be greater than two. To prove the first of these statements let us show that if a and b are two elements which are not comparable then there is a greatest element, I, and a least element, 0, and $a\cup b=I$, $a\cap b=0$. Suppose $a\cup b\neq I$, that is, that $a\cup b\geqq x$ for every $x\in L$ fails. Then there exists $u\in L$ such that $u>a\cup b$ and

$$a\ a\cup b\ u\cdot a\ a\cup b\ b\ b\cdot b\ a\cup b\ u\cdot a\neq a\cup b\cdot a\cup b\neq u\rightarrow b\ a\cup b\ b$$

which implies $a\cup b=b$ and $b\geqq a$, contrary to the assumption that a and b are not comparable. This shows that $a\cup b=I$, and $a\cap b=0$ may be shown in the same way.

To show that the number of chains is not greater than two we again suppose the contrary. Then we have three elements a, b, and c, no two of which are comparable. Then from T_{10}, since $a\cup b=a\cup c=b\cup c=I$,

$$a\ I\ b\cdot a\ I\ c\cdot c\ I\ b\cdot a\neq I\cdot I\neq b\rightarrow c\ I\ c$$

which implies $c=I$, contrary to our assumption that it was not comparable with a and b. This completes the proof of Theorem 2.

Finally we include a correction to the previously quoted paper of Pitcher and Smiley.[2] It is stated (p. 113) that "Examples of semi metric spaces are easily given in which τ_2 fails."

$$\tau_2.\quad abc\cdot adb\cdot dbc\rightarrow adc.$$

That this transitivity is, in fact, present in all semi metric spaces may be seen as follows. The hypotheses of τ_2 for betweenness in a semi metric space are

$$\delta(a, b)+\delta(b, c)=\delta(a, c), \tag{1}$$

$$\delta(a, d)+\delta(d, b)=\delta(a, b), \tag{2}$$

$$\delta(d, b)+\delta(b, c)=\delta(d, c). \tag{3}$$

Substituting from (2) in (1) and using (3) we have $\delta(a, d)+\delta(d, c)=\delta(a, c)$. A consequence of this fact is that no non-modular lattice can be supplied with a semi metric so that metric and lattice betweenness are the same.

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[2] This error was noticed by the authors and is included here at their request.