LAMBERT SUMMABILITY OF ORTHOGONAL SERIES

RICHARD BELLMAN

If we define Lambert summability of a series, $\sum_1^\infty a_n$, in terms of the existence of the limit

$$(1) \qquad L(a_n) = \lim_{x\to 1-0} (1-x)\sum_1^\infty \frac{na_nx^n}{1-x^n}$$

we have, by a well known theorem of Hardy-Littlewood [1],[1] that $C(a_n)\to L(a_n)\to A(a_n)$; $C(a_n)$, $A(a_n)$ are respectively the Cesàro and Abel means of the series $\sum_1^\infty a_n$.

The proof of $C(a_n)\to L(a_n)$ is elementary in nature, but the proof of $L(a_n)\to A(a_n)$ requires the prime number theorem, and conversely the theorem $L(a_n)\to A(a_n)$ implies the prime number theorem.

For that reason, it is perhaps interesting to show that for orthogonal series of functions $f(x)$, belonging to L^2, the inclusion of $L(a_n)$ between $C(a_n)$ and $A(a_n)$ follows in completely elementary fashion.

That $C(a_n)\sim A(a_n)$ for orthogonal series of L^2 is a known result of Kaczmarz [2]. Hence it is sufficient to show that $L(a_n)\to C(a_n)$. In addition, it is further known that $C(a_n)$ is equivalent to the convergence of the partial sums of the orthogonal series $s_{2^n}(\theta)=\sum_1^{2^n} a_k\phi_k(\theta)$ [3] Therefore, finally, it comes to showing that Lambert summability implies the convergence of the partial sums $s_{2^n}(\theta)$, in order to prove the theorem.

Let $f(\theta)\subset L^2(a, b)$, $a_n=\int_a^b f(\theta)\phi_n(\theta)d\theta$; where $(\phi_n(\theta))$ is an orthonormal sequence in (a, b), $s_n(\theta)=\sum_1^n a_n\phi_n(\theta)$.

Write, where x is $1-1/2^n$,

$$(2) \qquad U_n(\theta) = \sum_1^\infty ka_k\phi_k(\theta)\frac{(1-x)x^k}{1-x^k} - s_{2^n}(\theta) = T_n(\theta) + V_n(\theta)$$

where

$$(3) \qquad T_n(\theta) = \sum_1^{2^n} a_k\phi_k(\theta)\left(\frac{k(1-x)x^k}{1-x^k} - 1\right),$$

$$(4) \qquad V_n(\theta) = \sum_{2^n+1}^\infty ka_k\phi_k(\theta)\frac{(1-x)x^k}{1-x^k}\,.$$

If $\lim_{n\to\infty} U_n(\theta)=0$, the result is proven. To that end, consider the

Received by the editors June 10, 1943.

[1] Numbers in brackets refer to the references listed at the end of the paper.

series

$$\sum_{1}^{\infty} [U_n(\theta)]^2. \tag{5}$$

To prove convergence almost everywhere in θ, it is sufficient to show

$$\sum_{n=1}^{\infty} \int_a^b [U_n(\theta)]^2 d\theta < \infty. \tag{6}$$

We have

$$\sum_n \int_a^b [U_n(\theta)]^2 d\theta \leqq 2\sum_n \int_a^b [T_n(\theta)]^2 d\theta + 2\sum_n \int_a^b [V_n(\theta)]^2 d\theta. \tag{7}$$

Let us consider the convergence of each series separately.

$$\begin{aligned} \sum_n \int_a^b [T_n(\theta)]^2 d\theta &= \sum_n \int_a^b \left(\sum_{1}^{2^n} a_k\phi_k(\theta) \left(\frac{k(1-x)x^k}{1-x^k} - 1 \right) \right)^2 d\theta \\ &= \sum_n \left(\sum_{1}^{2^n} a_k^2 \left(\frac{k(1-x)x^k}{1-x^k} - 1 \right)^2 \right) \end{aligned} \tag{8}$$

where the x appearing in $\sum_1^{2^n}$ is $1-1/2^n$, $n \geqq 1$.

Now

$$1 - x^k \leqq k(1-x), \qquad 0 \leqq x \leqq 1, \tag{9}$$

$$1 - x^k \geqq 1 - \frac{k(1-x)x^k}{1-x^k} \geqq 0, \tag{10}$$

so that

$$\begin{aligned} \sum_n \int_a^b [T_n(\theta)]^2 d\theta &\leqq \sum_n \sum_1^{2^n} a_k^2 (1-x^k)^2 \leqq \sum_n \sum_1^{2^n} k^2 a_k^2 (1-x)^2 \\ &\leqq \sum_n \frac{1}{2^{2n}} \sum_1^{2^n} k^2 a_k^2 \leqq \sum_k a_k^2 k^2 \sum_{n \geqq \log_2 k} 2^{-2n} \leqq A \sum_k a_k^2 \end{aligned} \tag{11}$$

and $\sum_k a_k^2 < \infty$ since $f(x) \subset L^2(a, b)$.

Now for the second series $\sum_n \int_a^b [V_n(\theta)]^2 d\theta$:

$$\begin{aligned} \sum_n \int_a^b [V_n(\theta)]^2 d\theta &= \sum_n \int_a^b \left(\sum_{2^n+1}^{\infty} a_k\phi_k(\theta) \frac{k(1-x)x^k}{1-x^k} \right)^2 d\theta \\ &= \sum_n \left\{ \sum_{2^n+1}^{\infty} k^2 a_k^2 \frac{(1-x)^2 x^{2k}}{(1-x^k)^2} \right\} \end{aligned} \tag{12}$$

where the x appearing in $\sum_{2^n+1}^{\infty}$ is $1-1/2^n$, $n\geqq 1$.

Since $(1-2^{-n})^k$ is a decreasing function of k,

$$\begin{aligned}
&\sum_n \sum_{2^n+1}^{\infty} k^2 a_k^2 \frac{(1-x)^2 x^{2k}}{(1-x^k)^2} \\
(13)\qquad &\leqq \sum_n \frac{1}{1-(1-2^{-n})^{2^n}} \sum_{2^n+1}^{\infty} k^2 a_k^2 (1-x)^2 x^{2k} \\
&\leqq A \sum_n \sum_{2^n+1} k^2 a_k^2 (1-x)^2 x^{2k}.
\end{aligned}$$

We can majorize $k^2 \sum_1^{\infty} 2^{-2n}(1-2^{-n})^{2k}$ by the integral

$$\begin{aligned}
k^2 \int_0^{\infty} 2^{-2x}(1-2^{-x-1})^{2k} dx &= 4k^2 \int_1^{\infty} 2^{-2x}(1-2^{-x})^{k} dx \\
(14)\qquad &< 4k^2 \int_0^{\infty} 2^{-2x}(1-2^{-x})^{2k} dx \\
&= A k^2/(2k+1)(2k+2)
\end{aligned}$$

which is obviously bounded.

Therefore we have proven the convergence of the series, which implies that $\lim_{n\to\infty} U_n(\theta)=0$ almost everywhere in θ, which implies that

$$L(a_n) = \lim_{n\to\infty} s_{2^n}(\theta) \tag{15}$$

almost everywhere in θ.

This is equivalent to what we set out to prove.

References

1. Hardy and Littlewood, *On a Tauberian theorem for Lambert's series*, Proc. Lond. Math. Soc. (2) vol. 19 (1921) pp. 21–29.

2. Kaczmarz and Steinhaus, *Theorie der Orthogonalreihen*, Warsaw, 1935, Theorem 583.

3. Ibid., Theorem 585.

University of Wisconsin