COMPLEX METHODS IN THE THEORY OF FOURIER SERIES

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1. **General remarks.** Two new ideas which greatly influenced the theory of Fourier series in this century are the Lebesgue integral and the applications of complex functions. The original impetus due to the discoveries of Lebesgue would have been spent long ago, but for the fact that its combination with complex methods opened entirely new prospects for trigonometric series.

The essential tool of Lebesgue theory is the fact that the integral is differentiable almost everywhere and that the derivative is equal to the integrand almost always. Most of the fundamental results of the theory of trigonometric series which were based on that fact had been known, roughly, before 1920. Although some important results have been discovered since then, the progress of purely real methods in the last twenty odd years has been relatively slow and limited to isolated problems. It seems quite likely that the structure of real functions must be investigated in more detail before purely real methods can resume their progress. On the other hand, it seems that the complex variable approach to many problems of the theory is the most natural one and may even be of considerable help in the analysis of the structure of real functions.

Every trigonometric series

$$\tag{1} \tfrac{1}{2}a_0 + \sum_{\nu=1}^{\infty} (a_\nu \cos \nu\theta + b_\nu \sin \nu\theta)$$

is the real part of the power series

$$\tag{2} \tfrac{1}{2}a_0 + \sum_{\nu=1}^{\infty} (a_\nu - ib_\nu)z^\nu$$

on the unit circle $z=e^{i\theta}$. The imaginary part of the series (2) for $z=e^{i\theta}$ is the series

$$\tag{3} \sum_{\nu=1}^{\infty} (a_\nu \sin \nu\theta - b_\nu \cos \nu\theta)$$

and is called the *conjugate* of (1).

Similarly, the harmonic function

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$$(4) \qquad u(r, \theta) = \tfrac{1}{2}a_0 + \sum_{\nu=1}^{\infty} (a_\nu \cos \nu\theta + b_\nu \sin \nu\theta)r^\nu$$

associated with the series (1) is the real part of the analytic function $\phi(z)$, $z=re^{i\theta}$, defined by the series (2). The harmonic function

$$(5) \qquad v(r, \theta) = \sum_{\nu=1}^{\infty} (a_\nu \sin \nu\theta - b_\nu \cos \nu\theta)r^\nu$$

associated with the series (3) is conjugate to the function $u(r, \theta)$ and is the imaginary part of the function $\phi(z)$.

Thus the problems of trigonometric series may be treated as problems (boundary value problems) of the theory of analytic functions. By complex methods in the theory of trigonometric series we however mean something more special, namely the application of the methods of analytic functions and in particular of the fact that the latter form a field. Elementary operations performed on analytic functions lead to analytic functions, as does also the operation of taking a function of a function. Nothing like that holds for harmonic functions, since even the square of a harmonic function need not be harmonic. Thus dealing directly with analytic functions instead of with their real parts gives obvious advantage.

The complex methods in trigonometric series have been systematically developed in the last quarter century, although some isolated applications can be traced back to an earlier period. Roughly speaking, in the development of complex methods we may discern three major trends:

(a) The method of the classes H^p,
(b) The method of conformal representation,
(c) The Littlewood-Paley method,

and it is the purpose of this talk to say a few words about each of these methods. It goes without saying that in a talk like this the presentation may be only very sketchy and must be limited to a discussion of a few particular results.

2. **Classes H^p of analytic functions.** One of the important problems of the theory of trigonometric series is to establish conditions under which a given trigonometric series (1) is a Fourier series. In other words: when is there an integrable function $f(x)$ such that the coefficients a_n, b_n are given by the familiar formulas

$$(6) \qquad a_n = \frac{1}{\pi}\int_0^{2\pi} f(x) \cos nx dx, \qquad b_n = \frac{1}{\pi}\int_0^{2\pi} f(x) \sin nx dx?$$

In the theory of Fourier series we often consider besides the most general integrable functions (by "integrable" we always mean L-integrable) classes of more special functions, for example, continuous, bounded, of the Lebesgue class L^p, $p \geqq 1$, and so on, and we may ask, in addition, under what condition does f belong to one of those classes. The Fourier character of the series (1) may be easily detected by means of the harmonic function (4) associated with the series. For functions of the class L^p, $p > 1$, we have a very simple test: a necessary and sufficient condition that (1) is the Fourier series of a function f of the class L^p is that the integral

$$\int_0^{2\pi} |u(r, \theta)|^p d\theta \tag{7}$$

be bounded for $0 \leqq r < 1$. For $p = 1$ this result is no longer true: a necessary and sufficient condition for the boundedness of the integral

$$\int_0^{2\pi} |u(r, \theta)| d\theta \tag{8}$$

is that there exist a function $F(x)$, $0 \leqq x \leqq 2\pi$, of bounded variation and such that

$$a_n = \frac{1}{\pi}\int_0^{2\pi} \cos nx dF(x), \qquad b_n = \frac{1}{\pi}\int_0^{2\pi} \sin nx dF(x). \tag{9}$$

Of course, if $F(x)$ is absolutely continuous and $F'(x) = f(x)$, these formulas reduce to (6), but in general it is not true. The series (1) with coefficients (9) is called a *Fourier-Stieltjes series*. The series (1) is a Fourier-Stieltjes series if and only if the integral (8) is bounded for $0 \leqq r < 1$. A necessary and sufficient condition that (1) should be an ordinary Fourier series is slightly more complicated: it is

$$\lim_{r, r' \to 1} \int_0^{2\pi} | u(r, \theta) - u(r', \theta) | d\theta = 0. \tag{10}$$

(For the proofs of all these results see, for example, Evans [2][1] or Zygmund [33]; in the sequel, the latter book will be quoted TS.)

Let us now consider any function

$$\phi(z) = \sum_{\nu=0}^{\infty} c_\nu z^\nu \tag{11}$$

[1] Numbers in brackets refer to the references listed at the end of the paper.

regular for $|z|<1$. If the integral

$$\int_0^{2\pi} |\phi(re^{i\theta})|^p d\theta \tag{12}$$

(analogous to (7)) is bounded for $0 \leqq r < 1$, the function $\phi(z)$ is said to belong to the class H^p (H stands for Hardy); p is here any positive number. The case $p=2$ is of special interest since in this case the integral (12) is easily expressible in terms of the coefficients c_ν by means of the Parseval formula

$$\frac{1}{2\pi}\int_0^{2\pi} |\phi(re^{i\theta})|^2 d\theta = \sum_{\nu=0}^{\infty} |c_\nu|^2 r^{2\nu}.$$

Hence the function $\phi(z)$ belongs to the class H^2 if and only if the series $\sum |c_\nu|^2$ converges. No result of this kind holds for $p \neq 2$, and this makes H^2 a central class ("central" in more than one sense), whose properties are the easiest to study. The fact that if $\sum |c_\nu|^2$ is finite then the series

$$\sum_{\nu=0}^{\infty} c_\nu e^{i\nu\theta}$$

is the Fourier series (with respect to the system $\{e^{i\nu\theta}\}$) of a function of the class L^2 is the classical Riesz-Fischer theorem.

Of course, given any function $\phi(z)$ regular in $|z|<1$ and of the class H^p we might set

$$\phi^p(z) = \psi^2(z) \tag{13}$$

so that the boundedness of the integral (13) is equivalent to that of

$$\int_0^{2\pi} |\psi(re^{i\theta})|^2 d\theta,$$

that is to say to the fact that ψ is of the class H^2, but the formula (13) defines a function ψ regular in $|z|<1$ only if ϕ has no zeros there. If $\phi(z)$ does have zeros, the argument has to be modified slightly, and following F. Riesz [26] (TS, [6]) we may proceed as follows. Let $z_1, z_2, \cdots$ ($|z_\nu|<1$) be the zeros (counted according to their multiplicity) of the function $\phi(z) \neq 0$, of the class H^p. It may be shown that the product

$$\prod_{\nu=1}^{\infty} |z_\nu| \tag{14}$$

converges. In other words, the sum $\sum(1-|z_\nu|)$ is finite. Conversely, given any sequence, finite or infinite, of numbers z_ν, $|z_\nu|<1$, such that the product (14) converges, there is a function $B(z)$, regular and bounded in $|z|<1$ (and so, in particular belonging to every class H^p), having zeros at the points z_ν and only there. If, for example, the points z_ν are all different from the origin, the function $B(z)$ may be defined as the product ("Blaschke product")

$$B(z) = \prod_\nu \frac{z - z_\nu}{z - z_\nu^*} \cdot \frac{1}{|z_\nu|} \tag{15}$$

where $z_\nu^* = 1/\bar{z}_\nu$ is the point conjugate to z_ν with respect to the circumference $|z|=1$. (If ϕ has a zero of order k at the origin, we have to insert the factor z^k on the right of (15).) It may be easily shown that $|B(z)|<1$ for $|z|<1$. Thus the function $\phi(z)/B(z)=\psi(z)$ is regular for $|z|<1$, does not vanish there, and we have the decomposition

$$\phi(z) = B(z)\psi(z).$$

If $\phi(z)$ has no zeros we set $B(z)\equiv 1$. It is an important fact that the function ψ also belongs to H^p (more precisely, if for the function ϕ the integral (12) does not exceed a constant M for all $r<1$, the function ψ has the same property). Since we may write

$$\phi(z) = \psi(z) + (B(z) - 1)\psi(z) = \psi(z) + \psi_1(z)$$

say, and since $B(z)-1$ does not vanish in $|z|<1$ and is absolutely less than 2 there, we get the following decomposition theorem: *every function of the class H^p may be represented as a sum of two functions of the class H^p which have no zeros in* $|z|<1$. Since the functions of the class H^p and without zeros are reducible to functions of the class H^2, whose properties are particularly simple, those properties may be extended to classes H^p. In particular, we get the following fundamental theorem (in which by a non-tangential path we mean any continuous curve approaching a point z_0, $|z_0|=1$, from inside the unit circle and contained between two chords through z_0 of that circle).

Suppose that $\phi(z)$ is of the class H^p. Then for almost every point $e^{i\theta}$ on $|z|=1$,

$$\lim_{z\to e^{i\theta}} \phi(z) \tag{16}$$

exists and is finite provided that z approaches $e^{i\theta}$ along any non-tangential curve. Moreover, if $\phi(e^{i\theta})$ denotes the limit (16),

(17) $$\int_0^{2\pi} |\phi(re^{i\theta}) - \phi(e^{i\theta})|^p d\theta \to 0,$$

(18) $$\int_0^{2\pi} |\phi(re^{i\theta}) - \phi(r'e^{i\theta})|^p d\theta \to 0,$$

as r and r' tend to 1.

It must be added that for $p>1$ this result had been known before the decomposition theorem was proved (for then, by what was said before, the series $\sum c_\nu e^{i\nu\theta}$ is a Fourier series, and so may be studied directly by familiar methods), but in the case $0<p\leqq 1$ it brings to light some new facts.

The most important case here is that of $p=1$. Suppose that $\phi(z)$ belongs to H (that is H^1), say

(19) $$\int_0^{2\pi} |\phi(re^{i\theta})| d\theta < M$$

for $0\leqq r<1$. The function $\phi(z)$ is a (complex-valued) harmonic function. Thus (by what was said before) the series

(20) $$\sum_{\nu=0}^{\infty} c_\nu e^{i\nu\theta} = \sum_{\nu=0}^{\infty} c_\nu(\cos \nu\theta + i \sin \nu\theta)$$

is a Fourier-Stieltjes series. On the other hand, since (19) implies (18) with $p=1$, we see that the series (20) is an ordinary Fourier series. Thus it turns out that for the trigonometric series (20) which are generated by power series on the circle of convergence, the distinction between ordinary Fourier series and Fourier-Stieltjes series disappears. If we take into account the familiar fact [TS, **16**] that Fourier-Stieltjes series are obtained by differentiating formally Fourier series of functions of bounded variation we obtain the following result.

If the series

(21) $$\sum_{\nu=0}^{\infty} C_\nu e^{i\nu\theta}$$

is the Fourier series of a function of bounded variation, this function must be absolutely continuous. In particular, if $\Phi(z)$ is regular in $|z|<1$, continuous in $|z|\leqq 1$, and if $\Phi(e^{i\theta})$ is of bounded variation, then $\Phi(e^{i\theta})$ is absolutely continuous (F. and M. Riesz [27]).

This fact is of great importance for the theory of conformal mapping. Another result which may be obtained by means of the decomposition theorem, and which shows the difference that exists between ordinary trigonometric series and the series (20), is the following theorem.

If (21) *is the Fourier series of a function of bounded variation the series* $\sum|C_\nu|$ *converges* (Hardy and Littlewood [5]).

There exist such simple proofs of this result that one of them may be reproduced here (See Hardy, Littlewood and Pólya [7]). The problem reduces to showing that if the function $\phi(z)=\sum c_\nu z^\nu$ is of the class H, the series $\sum|c_\nu|/(\nu+1)$ converges. We may assume that $\phi(z)$ has no zeros in $|z|<1$, so that $\phi=\psi^2$ where $\psi(z)=\sum d_\nu z^\nu$ is of the class H^2. If we use the fact that $\sum|d_\nu|^2<+\infty$, the convergence of $\sum|c_\nu|(\nu+1)^{-1}$ follows from the fact that

$$\begin{aligned}\sum_{\nu=0}^{\infty}(\nu+1)^{-1}|c_\nu| &= \sum_{\nu=0}^{\infty}(\nu+1)^{-1}\left|\sum_{k+l=\nu}d_kd_l\right| \\ &\leqq \sum_{\nu=0}^{\infty}(\nu+1)^{-1}\sum_{k+l=\nu}|d_kd_l| \\ &= \sum_{k,l=0}^{\infty}|d_k||d_l|/(k+l+1)\end{aligned}$$

and from the familiar theorem of Hilbert asserting the convergence of the series $\sum x_kx_l/(k+l+1)$ if $\sum x_k^2$ is finite.

The following theorem due to Hardy and Littlewood [6; TS, p. 249] throws additional light on the behavior of the functions of the class H^p.

Suppose that $\phi(z)$ *is of the class* H^p, *and let*

$$\phi^*(\theta) = \sup_{0\leqq r<1}|\phi(re^{i\theta})|.$$

Then $\phi^*(\theta)$ *is of the class* L^p, *and*

$$\int_0^{2\pi}\{\phi^*(\theta)\}^p d\theta \leqq C_p\int_0^{2\pi}|\phi(e^{i\theta})|^p d\theta,$$

where C_p *depends on* p *only.*

Hence every function of the class H^p has a majorant which depends on the argument θ only and which is of the class L^p. An analogous result holds for harmonic functions, and is an important tool in many problems of the theory of Fourier series. A result similar to the theorem just stated holds if instead of $\phi^*(\theta)$ we consider the upper bound of $\phi(z)$ in a sector with vertex at the point $e^{i\theta}$ and directed to the interior of the unit circle.

Let $\phi(z)=u+iv$ be a function regular for $|z|<1$. It is natural to ask for the relation between the behavior of the two integrals

$$\int_0^{2\pi} |u(re^{i\theta})|^p d\theta, \qquad \int_0^{2\pi} |v(re^{i\theta})|^p d\theta.$$

The fundamental theorem of M. Riesz [**28**; TS, p. 147] asserts that, *if* $p>1$, *these two integrals are simultaneously bounded or simultaneously unbounded*. Thus, *the series conjugate to the Fourier series of a function of the class* L^p *is also the Fourier series of a function of the class* L^p. The result is not true if $p=1$, that is to say for functions merely integrable.

How the method of the complex variable may be applied to problems of real functions is illustrated by the following example which is by now quite familiar, but deserves mention here. Let $f(\theta)$ be a function of period 2π, and let (3) be the series conjugate to the Fourier series of $f(\theta)$. If $f(\theta)$ is sufficiently regular, for example if it has a continuous derivative, or even if $f(\theta)$ only satisfies a Lipschitz condition of positive order, the series (3) converges to the sum

$$\tilde{f}(\theta) = -\frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta+t)\tfrac{1}{2}\cot \tfrac{1}{2}t\,dt = -\frac{1}{\pi}\lim_{\epsilon\to+0}\left(\int_{-\pi}^{-\epsilon}+\int_{\epsilon}^{\pi}\right),$$

which is called the function *conjugate* to $f(\theta)$. Under the conditions imposed on f, the integral converges, even absolutely. The problem is whether this integral converges, at least almost everywhere, for the most general function f integrable L? That this is so for f continuous was shown already by Fatou [**3**] and even that rather special result is far from obvious. For the most general integrable f it is easy to show that almost everywhere the existence of $\tilde{f}(\theta)$ is equivalent to the existence of the radial limit of the function $\tilde{f}(r, \theta)$ conjugate to the Poisson integral of f. If we use rather elementary facts from the theory of the complex variable, we may easily show that the radial limit of $\tilde{f}(r, \theta)$ exists almost everywhere, and this proves that the integral $\tilde{f}(\theta)$ exists almost everywhere (Privaloff [**24**], Plessner [**19**]; TS, p. 145).

Although there exist purely real proofs of the existence of $\tilde{f}(\theta)$ (Besicovitch [**1**], Titchmarsh [**30**], Marcinkiewicz [**14**]), the one sketched above is the simplest, and seems to lead to the roots of the matter.

In the case that $f(\theta)$ is the characteristic function of a measurable set E, the existence of the function $\tilde{f}(\theta)$ has a certain geometric significance. It shows that in the neighborhood of almost every point of

E that set has a certain symmetry of structure (for otherwise the integral defining $\tilde{f}(\theta)$ would be divergent). This property is not easily deducible from the familiar properties of measurable sets (for example, from the theorem on the points of density), for otherwise we should have a simple real function proof of the existence of $\tilde{f}(\theta)$.

One might argue that the study of the conjugate functions is strictly speaking outside the scope of the theory of Fourier series. But this is not so, and there seems to be a close relation between the behavior of the partial sums $S_n(\theta)$ of a Fourier series and certain conjugate functions. Let us consider instead of $S_n(\theta)$ the modified partial sums $S_n^*(\theta)$ differing from $S_n(\theta)$ in that only half of the last term is taken. Hence

$$S_n^*(\theta) = \tfrac{1}{2}a_0 + \sum_{\nu=1}^{n-1} (a_\nu \cos \nu\theta + b_\nu \sin \nu\theta) + \tfrac{1}{2}(a_n \cos n\theta + b_n \sin n\theta).$$

For $S_n^*(\theta)$ we have a formula similar to the classical Dirichlet formula,

$$S_n^*(\theta) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta + t)\frac{\sin nt}{2 \tan (t/2)}dt$$

(TS, p. 21) and it may be written formally

$$\begin{aligned} S_n^*(\theta) &= \frac{\cos n\theta}{\pi}\int_{-\pi}^{\pi} f(\theta + t) \sin n(\theta + t)\tfrac{1}{2} \cot (t/2) \\ &\quad - \frac{\sin n\theta}{\pi}\int_{-\pi}^{\pi} f(\theta + t) \cos n(\theta + t)\tfrac{1}{2} \cot (t/2)dt \\ &= - g_n(\theta) \cos n\theta + h_n(\theta) \sin n\theta, \end{aligned}$$

say, where $g_n(\theta)$ and $h_n(\theta)$ denote functions conjugate to $f(\theta) \sin n\theta$ and $f(\theta) \cos n\theta$. From this formula and from the slightly strengthened form of the theorem of M. Riesz just stated, the following fact (also due to M. Riesz) follows easily: For every function $f(\theta)$ of the class L^p, $p>1$,

$$\int_0^{2\pi} | f(\theta) - S_n(\theta) |^p d\theta \to 0 \qquad \text{as } n \to + \infty.$$

It is a curious fact that in order to prove this result from the theory of Fourier series (in the narrow sense of the word) we have to use properties of conjugate functions. No other proof seems to have been discovered so far.

The theory of the functions H^p has an analogue for functions regular in a half-plane. The latter theory (developed mainly by Hille and Tamarkin [**8**]) has its applications mostly in the domain of Fourier integrals, and for this reason we omit its discussion here. Also to the domain of Fourier integrals belong the complex methods developed by Paley and Wiener [**18**].

3. **The method of conformal representation.** We first recall familiar facts. Let Δ be any domain in the ζ-plane limited, say, by a simple Jordan curve Γ. There is a function $\zeta = h(z)$ defined and regular in the unit circle

$$(D) \qquad |z| < 1$$

and mapping D conformally onto Δ. The function $h(z)$ may be extended continuously to the closed domain $D+C$, where C is the circumference $|z| = 1$, and gives a one-one correspondence between $D+C$ and $\Delta+\Gamma$. If we add some normalizing conditions, the function $h(z)$ is unique.

Let us assume from now on that the curve Γ is rectifiable. For point sets situated on a rectifiable curve we have, of course, a theory of measure analogous to that of Lebesgue. In particular, we may speak of sets of measure 0, of "almost everywhere," and so on.

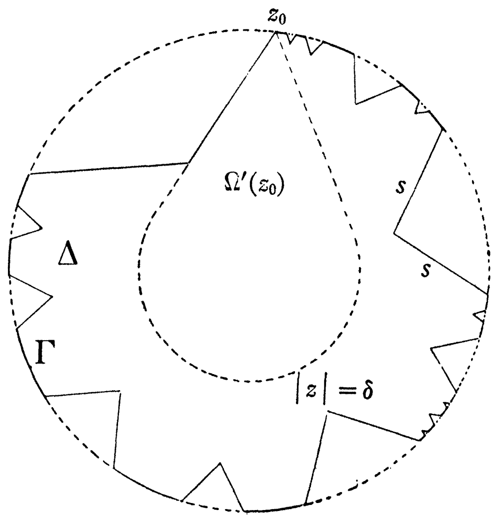
In our case, since Γ is of finite length, the function $\phi(z)$ must be of bounded variation on the circumference C. This means, as we know, that $\phi(z)$ must be absolutely continuous on C. In particular, it transforms every set of measure 0 on C into a set of measure 0 on Γ. The converse is also true, the sets of measure 0 on Γ correspond to sets of measure 0 on C. In other words, in our case, the sets of measure 0 on the boundaries are invariants of conformal mapping. This result was first proved by F. and M. Riesz [**27**].

There is another property, obtained by Privaloff [**24**], of the mapping function $h(z)$: the mapping is conformal at almost every point of the boundary. This is to be understood as follows. If we exclude a certain point set of measure 0 on C, each of the remaining points z_0 has the property that if C' and C'' are any two paths leading from the interior of D to the point z_0 and making angle α, the images Γ', Γ'' of C', C'' make angle α at the point $\zeta_0 = \phi(z_0)$.

In order to see how the above results can be applied, let us sketch the proof of a theorem of Privaloff [**23**]. Suppose that a function $\phi(z) = u+iv$ is regular in $|z| < 1$, and that there is a point set E on the circumference $|z| = 1$ with the following property: for every $z_0 = e^{i\theta_0}$ belonging to E there is a circular sector $\Omega(z_0)$ with vertex at z_0, lying except for z_0 entirely inside C, and such that the real part $u(z)$ of

$\phi(z)$ is bounded in $\Omega(z_0)$. Then, *the function* $\phi(z)$ *has a nontangential limit at almost every point* z_0 *of* E.

A special case of this result worth a separate statement is: *if the real part* $u(z)$ *of a regular function* $\phi(z)$, $|z|<1$, *has a nontangential limit at every point* z_0 *of a set* E *of the circumference* $|z|=1$, *the imaginary part* $v(z)$ *has the same property at almost every point of* E.



Let us assume, to fix the ideas, that $\Omega(z_0)$ is symmetrical with respect to the radius $(0, z_0)$. Without loss of generality we may also assume that the function $u(z)$ is bounded *uniformly* in all the $\Omega(z_0)$, $z_0 \in E$. For the upper bound $M(z_0)$ of u in $\Omega(z_0)$ is a function of z_0 and is finite on E, so it follows that if we reject from E a subset of arbitrarily small measure, $M(z_0)$ will be bounded on the rest of E. By a similar argument we may assume that the angles of the sectors $\Omega(z_0)$ are the same, or even that all these sectors are congruent. Let $|z|=\delta$ be the circle tangent to the rectilinear sides (or their continuations) of the sectors $\Omega(z_0)$. Without loss of generality, we may replace each $\Omega(z_0)$ by the domain $\Omega'(z_0)$ limited by the two tangents from z_0 to $|z|=\delta$, and by the more distant arc of the circle. One more remark: we may assume that E is a perfect set.

Let now Δ denote the sum of all the domains $\Omega'(z_0)$ for $z_0 \in E$; Δ has a starlike shape (see the figure). It is connected. The points of its frontier Γ, which is a simple Jordan curve, may belong either (a) to the set E, (b) to a number of arcs (nonexistent on the figure) of the circle $|z| = \delta$, (c) to the denumerable set of segments s belonging to the boundaries of some $\Omega'(z_0)$. Since the regions $\Omega'(z_0)$ are all congruent, it is clear that the total length of the segments s does not exceed a fixed multiple (depending only on the shape of the Ω's) of the total length of the intervals contiguous to E. Thus the curve Γ is rectifiable.

The function $u(z)$ is bounded in Δ. Hence, if $z=h(w)$ maps Δ conformally onto the unit circle $|w| < 1$, the function

$$u(z) = u(h(w)) = u_1(w)$$

is harmonic in the circle $|w| < 1$. It is the real part of the function

$$\phi(h(w)) = \phi_1(w)$$

regular in $|w| < 1$. The real part of $\phi_1(w)$ being bounded, the latter function is of the class H^2, and so has a nontangential limit at almost every point of the circumference $|w| = 1$. If we go back from $\phi_1(w)$ to the function $\phi(z)$, and take into account the results of F. and M. Riesz and of Privaloff mentioned above, we see that $\phi(z)$ has a nontangential limit at almost every point of Γ, in particular almost everywhere in E. This completes the proof.

Without changing the idea of the proof we may generalize this result considerably (see Plessner [21]). We prefer, however, to give a different application.

Let $\phi(z)$ be a function regular in $|z| < 1$ and of the class H^2, and let $\Omega'(z) = \Omega'(\theta)$ have the same meaning as before. Lusin proved that then the integral

$$I(\theta) = \iint_{\Omega'(\theta)} |\phi'(z)|^2 d\omega \tag{22}$$

($d\omega$ an element of area) is finite for almost every θ. This integral represents the area of the domain (generally non-schlicht) obtained from $\Omega'(\theta)$ by the mapping $w = \phi(z)$. By an argument similar to the above we may show (see Marcinkiewicz and Zygmund [15]) that, *if $\phi(z)$ is any function regular in $|z| < 1$, and if it has a nontangential limit at every point $e^{i\theta}$ of a set E of positive measure, then the integral* (22) *is finite almost everywhere in E*. Recently, Spencer [29] proved a converse of the above result, namely that *the finiteness of the integral $I(\theta)$ in a set of θ of positive measure implies the existence of the nontangential limit at almost every point of that set.* Thus we see that almost everywhere the existence of the nontangential limit is equivalent to the finiteness of a certain area.

The method of conformal representation, as presented above, seems to have originated with Golubeff [4]. If we are to use it, we must know the behavior of the function in certain sectorial domains from which we build up a new domain whose boundary has "many" points in common with the boundary of the domain in which the function is defined. It would not work if we wanted, for example, to prove that the existence of the radial limit of $u(z)$ along a set of positive measure of radii implies the existence of the radial limit of $v(z)$ along almost every radius of the set, and the problem itself is open (u and v here are the real and the imaginary part of a function regular in $|z|<1$). In other words, we do not know whether the Abel summability of a trigonometric series does or does not imply (almost everywhere) the Abel summability of the conjugate series.

For some other methods of summability, in particular for ordinary convergence, the problem is solved. A very important step in this direction was first made by Kuttner [12], who showed that *if a Fourier series converges in a set of positive measure, the conjugate series converges almost everywhere in that set.* This result was later on extended to the most general trigonometric series, and to the Cesàro summability. The existing proofs (see Plessner [22], Marcinkiewicz and Zygmund [16, 17]) are quite difficult and are essentially based on complex methods.

The problem of the convergence of power series (or trigonometric series) leads naturally to the problem of the distribution of the partial sums of divergent series. It turns out that under certain conditions the distribution of those partial sums displays a very simple geometric character.

We shall say that a sequence of points $\{s_n\}$ is of *circular structure* with respect to the point s, if the derived set of the sequence $\{s_n\}$ consists of a certain number (finite, denumerable, or nondenumerable, but in any event a closed set) of circles with center at the point s. If for the sake of simplicity we confine our attention to power series

$$\sum_{\nu=0}^{\infty} c_\nu z^\nu = \phi(z) \tag{23}$$

with bounded coefficients, we have the following result (Marcinkiewicz and Zygmund [17]).

Suppose that

$$\lim_{r\to 1} \phi(re^{i\theta}) = s(\theta)$$

exists for every θ of a set E of positive measure. Then, for almost every θ of E, the sequence $\{s_\nu(e^{i\theta})\}$ of the partial sums of the series (23) *is of circular structure with respect to the point $s(\theta)$. In particular, if the coefficients c_ν tend to* 0, *then for almost every $\theta \in E$ the derived set of the sequence $\{s_\nu(e^{i\theta})\}$ is a circle (finite or infinite) with center at the point $s(\theta)$.*

Quite recently new applications of conformal mapping to the theory of trigonometric series were obtained by F. Wolf. He combined conformal mapping with certain extensions of the Phragmèn-Lindelöf principle. I would like to mention here one of the important results he obtained.

The problem whether a given function can be represented by more than one trigonometric series (not necessarily a Fourier series) may be reduced, by subtracting these series, to the following problem: can a trigonometric series

$$\frac{1}{2}a_0 + \sum_{\nu=1}^{\infty} (a_\nu \cos \nu\theta + b_\nu \sin \nu\theta) \tag{24}$$

which does not vanish identically represent zero? This "problem of uniqueness" of trigonometric series has various aspects. It is a classical fact that in the case of convergent series it admits of a positive solution: if the series (24) converges to 0 at every point, the coefficients a_ν and b_ν all vanish. The next step is to consider summable series. The case of Abel summability is of particular importance since it means studying the radial behavior of the harmonic function

$$u(r, \theta) = \frac{1}{2}a_0 + \sum_{\nu=1}^{\infty} (a_\nu \cos \nu\theta + b_\nu \sin \nu\theta)r^\nu \tag{25}$$

associated with the series (24). Rajchman [**25**] proved that if the function (25) tends to 0 along every radius and if in addition $|a_\nu| + |b_\nu| \to 0$, the series (24) vanishes identically. Verblunsky [**31**] replaced the last condition by $|a_\nu| + |b_\nu| = O(\nu)$. If we consider only radial limits, we cannot go much further since there are trigonometric series with coefficients $O(\nu)$, summable A everywhere, and yet not vanishing identically: the series

$$\sum_{\nu=1}^{\infty} \nu \sin \nu\theta$$

is an instance in point. Thus we must impose new restrictions if we wish to obtain positive results. Wolf [**32**] proved that:

If (i) $u(r, \theta)$ *is harmonic for* $r<1$, (ii) *for every* θ_0, $u(r, \theta)\to 0$ *as* $(r, \theta)\to(1, \theta_0)$ *along any nontangential path,* (iii) $|u(r, \theta)| \leqq \exp A/(1-r)^m$, *where* A *and* m *are independent of* r *and* θ, *then the series* (24) *vanishes identically.*

4. **The Littlewood-Paley method.** This method (see Littlewood and Paley [**13**]) cannot be easily described without going into technical details, and by necessity I shall have to be rather brief here. Let me start with some of the results they achieved. Suppose that $f(\theta)$ is an integrable function of period 2π, and let $S_n(\theta)$ denote the nth partial sum of the Fourier series of $f(\theta)$. It is a familiar fact that $S_n(\theta)$ may diverge at some points even if $f(\theta)$ is a continuous function. Whether there exists a continuous function f such that $S_n(\theta)$ diverges everywhere, or at least almost everywhere, is still an open problem. For f merely integrable, the divergence of $S_n(\theta)$ may actually occur everywhere, as shown by Kolmogoroff [**9**; TS, p. 175]. It is a curious fact that so far it has not been possible to construct a similar example for functions of any class L^p, $p>1$, and the problem seems to be much more difficult there.

Of course, from a certain point of view the function $f(\theta)$ is adequately represented by the Fejér means of its Fourier series, but the behavior of the partial sums S_n is of considerable intrinsic interest. For functions of the class L^p, $p>1$, Paley and Littlewood proved the following result (the special case $p=2$ was solved earlier by Kolmogoroff [**10**; TS, p. 257]).

Whatever the sequence of positive integers $n_1, n_2, \cdots$ *satisfying an inequality*

$$n_{k+1}/n_k > q > 1,$$

the partial sums $S_{n_k}(\theta)$ *converge to* $f(\theta)$ *at almost every point.*

The result is false for $p=1$ (although it holds for power series of the class H; see Zygmund [**34**]). Its significance consists in the fact that the sequence n_k is independent of the function f.

Another result which Littlewood and Paley proved concerns the *convergence factors* of Fourier series. It is a familiar fact due to Hardy that if (24) is the Fourier series of an integrable function, the series

$$\sum_{\nu=2}^{\infty} (a_\nu \cos \nu\theta + b_\nu \sin \nu\theta)/\log \nu$$

converges for almost every θ. If f is of integrable square, even the series

$$(26)\qquad \sum_{\nu=2}^{\infty} (a_\nu \cos \nu\theta + b_\nu \sin \nu\theta)/(\log \nu)^{1/2}$$

converges almost everywhere (Kolmogoroff and Seliverstoff [**11**], Plessner [**20**]; TS, p. 252). Littlewood and Paley proved the intermediate result: if $f \in L^p$, $1 < p < 2$, the series

$$\sum_{\nu=2}^{\infty} (a_\nu \cos \nu\theta + b_\nu \sin \nu\theta)/(\log \nu)^{1/p}$$

converges almost everywhere. Nothing is known about the case $p > 2$. In particular, about convergence factors for continuous functions we know no more than for functions of the class L^2, that is to say that the series (26) is convergent almost everywhere. The latter fact is equivalent to saying that the partial sums $S_n(\theta)$ of the Fourier series satisfy the relation

$$(27)\qquad S_n(\theta) = o(\log n)^{1/2}$$

at almost every point. It may be that the estimate (27) is the best result for the Fourier series of continuous functions.

The Littlewood-Paley theory is based on the use of the function $g(\theta)$ defined by the formula

$$g(\theta) = \left(\int_0^1 (1 - r) \mid \phi'(re^{i\theta}) \mid^2 dr \right)^{1/2}$$

where $\phi(z)$ is any function regular in $|z| < 1$. The function $g(\theta)$ has no simple geometric interpretation, but has some connection with the integral $I(\theta)$ (see (22)) whose geometric significance and relation to the existence of the nontangential limit was already mentioned. Let

$$s(\theta) = I^{1/2}(\theta),$$

so that $s(\theta)$ is of dimension 1. It may be shown that $s(\theta)$ is, effectively, a majorant of $g(\theta)$; more precisely, $g(\theta) \leqq C_\alpha s(\theta)$ where C_α depends only on the angle α. The results obtained by Littlewood and Paley for the function $g(\theta)$ when interpreted in terms of the function $s(\theta)$ may be stated as follows: *If* $\phi(z) \in H^p$, $p > 1$, *then*

$$\int_0^{2\pi} s^p(\theta) d\theta \leqq A_{p,\alpha} \int_0^{2\pi} \mid \phi(e^{i\theta}) \mid^p d\theta,$$

$$\int_0^{2\pi} \mid \phi(e^{i\theta}) \mid^p d\theta \leqq B_{p,\alpha} \int_0^{2\pi} s^p(\theta) d\theta.$$

Here $A_{p,\alpha}$ and $B_{p,\alpha}$ depend on p and α only; in addition, for the validity of the second inequality we have to assume that $\phi(0)=0$. Thus the integrability of $s(\theta)$ imitates that of $|\phi(e^{i\theta})|$.

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