A TRANSFORMATION OF JONAS SURFACES

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It is well known that when an analytic surface S is referred to its asymptotic net (u, v) the homogeneous point coordinates $x^i(u, v)$ $(i=1, 2, 3, 4)$ of a generic point on S can then be normalized, so that they satisfy the differential equations,

$$\left\{\begin{aligned} x_{uu} &= \beta x_v + px, \\ x_{vv} &= \gamma x_u + qx, \end{aligned}\right. \tag{1}$$

where the coefficients β, γ, p, q satisfy the conditions of integrability,

$$\left\{\begin{aligned} &(\beta_v + 2p)_v = (\beta\gamma)_u + \beta\gamma_u, \qquad (\gamma_u + 2q)_u = (\beta\gamma)_v + \gamma\beta_v, \\ &\qquad\qquad (p_v + \beta q)_v + \beta_v q = (q_u + \gamma p)_u + \gamma_u p. \end{aligned}\right. \tag{2}$$

The conjugate net Ω of S defined by

$$Cdu^2 + Ddv^2 = 0,$$

has equal point invariants when and only when[1]

$$(\log (C/D))_{uv} - (\gamma(C/D))_v + (\beta(D/C))_u = 0. \tag{3}$$

The necessary and sufficient condition that Ω should have equal tangential invariants is obtained from (3) by replacing β, γ by $-\beta$, $-\gamma$ respectively. If Ω has equal invariants, both point and tangential, then it is a Jonas net, and S then becomes a Jonas surface.[2] For a Jonas net we have thus the following relations:

$$(\log (C/D))_{uv} = 0, \qquad (\gamma(C/D))_u - (\beta(D/C))_v = 0.$$

By a suitable transformation of asymptotic parameters, leaving the asymptotic net unaltered, the above equations reduce to

$$\beta_u = \gamma_v, \qquad C = D.$$

Hence a Jonas net on a Jonas surface S may be represented by the equation

$$du^2 - dv^2 = 0, \tag{4}$$

and the surface is characterized by

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[1] Cf. G. Fubini-E. Čech, *Geometria Proiettiva Differenziale*, vol. 1, Bologna, Zanichelli, 1927, p. 105.

[2] Cf. Fubini-Čech, ibid. p. 106.

$$\beta_u = \gamma_v. \tag{5}$$

The main object of this note is to prove the following theorem:

THEOREM. *The projection in a fixed plane of a Jonas net of a Jonas surface is a plane net with equal point invariants, and stands for the projection of the asymptotic net of another Jonas surface.*

A point P_θ with the coordinates $\theta = rx + sx_u + tx_v + lx_{uv}$ is fixed in space if

$$r = l_{uv} - l\beta\gamma, \qquad s = -l_v, \qquad t = -l_u, \tag{6}$$

where l satisfies the system of equations

$$\begin{cases} l_{uu} = -\beta l_v + (\beta_v + p)l, \\ l_{vv} = -\gamma l_u + (\gamma_u + q)l. \end{cases} \tag{7}$$

A point P_y on the straight line $P_\theta P_x$, is evidently given by $y = \lambda\theta + x$. In virtue of (1), we find by differentiation that

$$\begin{cases} y = (1 + \lambda r)x + \lambda s x_u + \lambda t x_v + \lambda l x_{uv}, \\ y_u = \lambda_u r x + (1 + \lambda_u s)x_u + \lambda_u t x_v + \lambda_u l x_{uv}, \\ y_v = \lambda_v r x + \lambda_v s x_u + (1 + \lambda_v t)x_v + \lambda_v l x_{uv}, \\ y_{uv} = \lambda_{uv} r x + \lambda_{uv} s x_u + \lambda_{uv} t x_v + (1 + \lambda_{uv} l)x_{uv}, \\ y_{uu} = (\lambda_{uu} r + p)x + \lambda_{uu} s x_u + (\lambda_{uu} t + \beta)x_v + \lambda_{uu} l x_{uv}, \\ y_{vv} = (\lambda_{vv} r + q)x + (\lambda_{vv} s + \gamma)x_u + \lambda_{vv} t x_v + \lambda_{vv} l x_{uv}. \end{cases} \tag{8}$$

In order that P_y be in a fixed plane, it is necessary and sufficient that

$$\begin{cases} y_{uu} = Ay_u + By_v + Cy, \\ y_{uv} = A'y_u + B'y_v + C'y, \\ y_{vv} = A''y_u + B''y_v + C''y. \end{cases} \tag{9}$$

Substituting (8) in (9) and reducing, we obtain

$$\begin{array}{lll} A = 0, & B = \beta, & C = p, \\ A' = (\log l)_v, & B' = (\log l)_u, & C' = \beta\gamma - l_{uv}/l, \\ A'' = \gamma, & B'' = 0, & C'' = q, \end{array}$$

and the conditions for the parameter λ

$$\lambda_{uu} = \beta\lambda_v + p\lambda,$$

$$\lambda_{vv} = \gamma\lambda_u + q\lambda,$$

$$r\lambda + s\lambda_u + t\lambda_v + l\lambda_{uv} + 1 = 0.$$

Thus we have a plane net given by the equations

$$(10)\qquad \begin{cases} y_{uu} = \beta y_v + p y, \\ y_{vv} = \gamma y_u + q y, \\ y_{uv} = (\log l)_v y_u + (\log l)_u y_v + (\beta\gamma - l_{uv}/l) y. \end{cases}$$

The curves of this net are the perspectives on the fixed plane of the asymptotic curves of a Jonas surface S obtained by projecting from the centre P_θ. In order to obtain the perspectives on the same fixed plane of the Jonas net Ω, we have to use the transformation

$$\bar{u} = u - v, \qquad \bar{v} = u + v,$$

so that

$$(11)\qquad \begin{cases} y_{\bar{u}} = (y_u - y_v)/2, \\ y_{\bar{v}} = (y_u + y_v)/2, \end{cases}$$

namely,

$$(11')\qquad \begin{cases} y_u = y_{\bar{u}} + y_{\bar{v}}, \\ y_v = y_{\bar{v}} - y_{\bar{u}}. \end{cases}$$

Putting

$$\log l = \theta, \qquad \beta\gamma - l_{uv}/l = c,$$

we find after a simple calculation that

$$(12)\qquad \begin{cases} y_{\bar{u}\bar{u}} = (1/4)(\gamma - \beta + 2\theta_u - 2\theta_v) y_{\bar{u}} \\ \qquad + (1/4)(\gamma + \beta - 2\theta_v - 2\theta_u) y_{\bar{v}} + (1/4)(p + q - 2c) y, \\ y_{\bar{v}\bar{v}} = (1/4)(\gamma - \beta - 2\theta_u + 2\theta_v) y_{\bar{u}} \\ \qquad + (1/4)(\gamma + \beta + 2\theta_v + 2\theta_u) y_{\bar{v}} + (1/4)(p + q + 2c) y, \\ y_{\bar{u}\bar{v}} = -(1/4)(\beta + \gamma) y_{\bar{u}} + (1/4)(\beta - \gamma) y_{\bar{v}} + (1/4)(p - q) y, \end{cases}$$

which represent the perspectives of the Jonas net $du^2 - dv^2 = 0$. Since

$$-(\beta + \gamma)_{\bar{u}} = (\beta - \gamma)_{\bar{v}},$$

this net is of equal point invariants and therefore asymptotic. That is, it may be regarded as the perspectives of the asymptotic curves of a certain surface Q. The projective linear element of the surface Q is easily found to be

$$(\bar{\beta} d\bar{u}^3 + \bar{\gamma} d\bar{v}^3)/2 d\bar{u} d\bar{v},$$

where[3]

[3] The projective linear element of a plane net has been defined by E. Čech. Cf. Fubini-Čech, *Introduction à la géométrie projective différentielle des surfaces*, 1931, chap. 10.

$$\bar{\beta} = (\gamma + \beta - 2\theta_v - 2\theta_u)/4, \qquad \bar{\gamma} = (\gamma - \beta - 2\theta_u + 2\theta_v)/4.$$

The relation $\beta_u = \gamma_v$ gives, however, the similar relation

$$\bar{\beta}_{\bar{u}} = \bar{\gamma}_{\bar{v}}.$$

Hence Q is also a Jonas surface, which completes the proof.

As a special case of the theorem we have the result:

Any Jonas net of a Jonas surface S is perspective to the asymptotic net of another Jonas surface Q from a fixed point. Conversely, if a Jonas net of a Jonas surface is perspective to the asymptotic net of another surface Q, then Q is also a Jonas surface.

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