ON NON-CUT SETS OF LOCALLY CONNECTED CONTINUA

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W. L. Ayres[1] and H. M. Gehman[2] have proved independently that if a locally connected continuum S contains a non-cut point p, there exists an arbitrarily small region R containing p and such that $S-R$ is connected. Our paper is concerned with certain generalizations of this theorem.

We shall consider a space S which is a locally connected continuum and contains a closed set P such that $S-P$ is connected. We show that under these hypotheses P can be enclosed in an open set R, the sum of a finite number of regions, whose complement is a locally connected continuum. We show further that if there exists a family of sets $\mathfrak{F}$ no element of which separates $S-P$, then there exist two open sets R and R' (with $R\supset R'\supset P$) of the above type and having the property that no element of $\mathfrak{F}$ contained in $S-R$ separates $S-R'$. When the elements of $\mathfrak{F}$ are single points, it is possible to choose $R'=R$; but this is not possible in the more general case.

We close by showing that if S is not separated by any element of $\mathfrak{F}$ plus any set of n points, and if Q is the sum of n sets of sufficiently small diameter and having sufficiently great mutual distances, then the set $S-Q$ has at most one component whose diameter is greater than a preassigned positive quantity, and this component is not separated by any element of $\mathfrak{F}$ at a sufficiently great distance from Q.

We recall some well known results.[3]

Let M be a locally connected continuum. Then:

(1) *M is a metric space having property S.*[4]

(2) *M is the sum of a finite number of arbitrarily small connected*

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[1] See W. L. Ayres, *On continua which are disconnected by the omission of a point and some related problems*, Monatshefte für Mathematik und Physik vol. 36 (1929) pp. 135–147. The theorem quoted here corresponds to Theorem 2 p. 149.

[2] See H. M. Gehman, *Concerning certain types of non-cut points, with an application to continuous curves*, Proc. Nat. Acad. Sci. U.S.A. vol. 14 (1928) pp. 431–433. Theorem 4 p. 432 is essentially that quoted here.

[3] See G. T. Whyburn, *Analytic topology*, Amer. Math. Soc. Colloquium Publications, vol. 28 (1942) p. 20 ff.

[4] A set is said to have property S if for any $\epsilon>0$ it can be expressed as the sum of a finite number of connected sets of diameter less than ϵ. The property was first introduced by W. Sierpinski in his paper *Sur une condition pour qu'un continu soit une courbe jordanienne*, Fund. Math. vol. 1 (1920) pp. 44–60.

subsets each having property S. Furthermore these subsets may be chosen either as open sets or as closed sets.

(3) *If $N \subset M$ and N has property S, then any set N_0 such that $N \subset N_0 \subset \overline{N}$ is locally connected.*

From the preceding results follows:

LEMMA 1. *If N is any subset of a locally connected continuum M and V is any δ-neighborhood of N, then there exists an open set U that contains N, has property S, and is such that $\overline{U}$ is the sum of a finite number of locally connected continua contained in V.*

It may clearly be supposed that every component of U contains a point of N.

Throughout this paper we shall deal with a compact metric space S, which we suppose to be a locally connected continuum. We denote by $\delta(A)$ the diameter of any set A, and by $V_\epsilon(A)$ the ϵ-neighborhood of A.

We shall require the following lemma, which was pointed out to the author by Dr. D. W. Hall. Its proof follows directly from the definitions involved.

LEMMA 2. *Let M be any locally connected subcontinuum of S. Then if T is the sum of any set of components of $S-M$, the set $K=S-T$ is a locally connected continuum.*

Using methods very similar to those of Ayres[1] and Gehman,[2] we obtain the following generalization of their theorem.

THEOREM 1. *Let P be any closed set such that $S-P$ is connected. Then for any $\epsilon>0$ there exists an open set R such that* (1) $P\subset R\subset V_\epsilon(P)$, (2) *the set R is the sum of a finite number of regions, each of which intersects P, and* (3) *$S-R$ is a locally connected continuum.*

PROOF. Let $R_1=V_\delta(P)$ and $R_2=V_{\delta'}(P)$, where $0<\delta'<\delta\leqq\epsilon$. Then, since S is locally connected, at most a finite number of components of $S-R_2$ intersect $S-R_1$; let these be $K_1, K_2, \cdots, K_n$, and choose $p_i\in K_i(R_1-R_2)$ for $i=1, \cdots, n$. Since P is closed, $S-P$ is a region, and we can find arcs $p_1p_i\subset(S-P)$ for $i=1, 2, \cdots, n$; thus we construct the connected set $C=\sum_{i=1}^{n}(K_i+p_1p_i)$.

Taking $d=\rho(P, C)$, we conclude from Lemma 1 that there exists a locally connected continuum V contained in $V_d(C)$ and therefore disjoint with P. Now let R be the sum of all components of $S-V$ that contain points of P. It follows from the local connectivity of S that R is open. To prove the theorem, we must show that R satisfies the conditions (1), (2), and (3).

Obviously $P\subset R$. On the other hand, $R\subset(S-V)\subset(S-C)\subset R_1\subset V_\epsilon(P)$. Thus R satisfies (1).

To see that (2) holds, we note that $\rho(P, S-R)>0$, because R is open. It follows from Lemma 1 that there exists an open set R_3, the sum of a finite number of regions, such that $P\subset R_3\subset R$. We see that R is likewise the sum of a finite number of regions, each of which intersects P, for every component of R contains a point of P and thus a component of R_3.

Applying Lemma 2 with $M=V$ and $T=R$ shows that $S-R$ is a locally connected continuum. Consequently R satisfies (3), and the proof is complete.

The following example shows that Theorem 1 loses its validity if the requirement that P be closed is dropped.

Example. Take for S the closed rectangle in the xy-plane bounded by the lines $x=\pm 2$, $y=\pm 1$. Divide the rectangle into four rectangles by drawing the lines $x=0$, $x=\pm 1$. Denote by P' the set consisting of the two end rectangles (open or closed), the segment $x=0$, $-1\leqq y\leqq 1$, and the curve $y=\sin(1/x)$, $-1\leqq x\leqq 1$; thus P' is connected but not locally connected. We see that Theorem 1 does not hold for $P=S-P'$, for no set having the properties of R can be found corresponding to $\epsilon<1$.

Theorem 2. *Let P be any closed set such that $S-P$ is connected, and suppose that $\mathfrak{F}$ is a family of subsets of S such that $S-(P+Q)$ is connected for each $Q\in\mathfrak{F}$. Then, given $\epsilon>0$, there exist open sets R and R', each of which is the sum of a finite number of regions intersecting P, such that* (1) *$P\subset R'\subset R\subset V_\epsilon(P)$,* (2) *the sets $S-R$ and $S-R'$ are locally connected continua, and* (3) *if $Q\in\mathfrak{F}$ and $Q\subset(S-R)$, then $S-(R'+Q)$ is connected.*

Proof. We first select an open set $R\supset P$ which has the same properties as the R of Theorem 1. Next we choose another open set $R_1\supset P$, having the same properties as R, and such that $\overline{R}_1\subset R$. Then all components of $R-R_1$ have limit points in both $S-R$ and $\overline{R}_1$, since $S-R_1$ is connected and R is the sum of a finite number of regions, each intersecting P and in one of which any component of $R-R_1$ must lie. It follows from the local connectivity of S that the number of such components is finite.

Now, if any two components of $R-R_1$ lie in some component C of $R-P$, we connect them by a simple arc in C; this is possible because C is a region. We define V_1 as the sum of $S-R_1$ and all such arcs.

Clearly V_1 is connected. Moreover, if $Q\in\mathfrak{F}$ and $Q\subset(S-R)$, then V_1-Q is connected. For suppose that x and y are points of V_1-Q.

Then Q cannot separate x from $R-R_1$ in V_1. This is obvious if $x \in RV_1$; if $x \in (V_1-R)$, we see (since $R-R_1$ separates x from $S-V_1$ in S) that Q cannot separate x from $R-R_1$ in V_1 without separating them in S, contrary to hypothesis. Thus there exists a component X of $R-R_1$ such that x and X lie in the same component of V_1-Q. Similarly, there exists a component Y of $R-R_1$ such that y and Y lie in the same component of V_1-Q.

If X and Y are in the same component of $R-P$, there exists an arc in RV_1 connecting X and Y. On the other hand, if X and Y lie in different components of $R-P$, they must lie in the same component of V_1-Q. For suppose that $X \subset AB$, where A is a component of $R-P$ and B is a component of V_1-Q. Then A is closed in $R-P$, while B is closed in V_1-Q. Thus the sets $(A+B)\overline{R}_1 = A\overline{R}_1$ and $A+B-R = B-R$ are closed in $S-(P+Q)$. From the construction of V_1 we see that $A(R-R_1) \subset B(R-R_1)$, and it follows that $(A+B)(\overline{R}-R_1)$ is closed in $S-(P+Q)$. Hence $A+B$, being the sum of three sets closed in $S-(P+Q)$, is closed in $S-(P+Q)$. Now suppose $Y \not\subset A+B$. Then we can find another set $A'+B' \supset Y$ of the same form as, and disjoint with, $A+B$. In this way it follows that $S-(P+Q)$ is the sum of a finite number of disjoint sets closed in $S-(P+Q)$, which is impossible since $S-(P+Q)$ is connected. Thus X and Y, and therefore x and y, lie in the same component of V_1-Q.

By Lemma 1, there exists a locally connected continuum V' containing V_1 and disjoint with P. We denote by R' the sum of all components of $S-V'$ that contain points of P. It follows from Lemmas 1 and 2, as in the proof of Theorem 1, that R' is the sum of a finite number of regions and that $S-R'$ is a locally connected continuum. Moreover, if $Q \in \mathfrak{F}$ and $Q \subset (S-R)$, we conclude, since $(S-R')-V_1 \subset R_1$, that $S-(R'+Q)$ is connected. This completes the proof.

The question naturally arises whether, under the hypotheses of Theorem 2, it is possible to find a single open set T containing P and playing the parts of both R and R' in that theorem. The following example shows that such a set T cannot in general be found.

Example. Take for S the plane set consisting of two line segments, a_0p and b_0p, together with a sequence of parallel lines $a_0b_0, a_1b_1, a_2b_2, \cdots$, where $a_0, a_1, a_2, \cdots$ and $b_0, b_1, b_2, \cdots$ are sequences of points, converging monotonically to p, on the lines a_0p and b_0p, respectively. We denote by c_n the midpoint of the segment a_nb_n for $n=0, 1, 2, \cdots$.

Now, take for P the point p and for $\mathfrak{F}$ the family of all pairs of points of S not separating $S-p$. Let R be any region containing p (but disjoint with a_1b_1). Then there exists a greatest integer $n \geqq 1$

for which $a_n b_n \subset (S-R)$. It follows that the pair of points $(c_n + a_{n-1})$ separates $S-R$. However, the set $S-(p+c_n+a_{n-1})$ is connected; hence $(c_n+a_{n-1}) \in \mathfrak{F}$. Thus R cannot be taken as T.

However, the stronger conclusion can be drawn when $\mathfrak{F}$ is a family of single points, as we now show.

THEOREM 3. *Let P be any closed set such that $S-P$ is connected. Suppose F is a set such that $S-(P+q)$ is connected for $q \in F$. Then for any $\epsilon > 0$ there exists an open set $R \supset P$, contained in $V_\epsilon(P)$ and consisting of a finite number of regions, such that $S-R$ is a locally connected continuum and $S-(R+q)$ is connected for $q \in F$.*

PROOF. By Theorem 2, there exist open sets R_1 and R_2, each consisting of a finite number of regions, such that (1) $P \subset R_2 \subset R_1 \subset V_\epsilon(P)$, (2) the sets $S-R_1$ and $S-R_2$ are locally connected continua, and (3) $S-(R_2+q)$ is connected for $q \in F(S-R_1)$. We write $V_2 = S-R_2$.

Now, for $y \in V_2 R_1 F$ we define K_y as the set consisting of y plus the component of $V_2 - y$ containing $S-R_1$. Then we let

$$V = \prod_{y \in V_2 R_1 F} K_y, \qquad R = S - V = \sum_{y \in V_2 R_1 F} (S - K_y).$$

For any $y \in V_2 R_1 F$, the set $S-K_y$ is the sum of a finite number of regions. For suppose $x \in (S-K_y)$. Since $S-y$ is a region, there exists an arc $xr \subset (S-y)$ for all $r \in R_2$. If there exists a point $x_1 \in K_y$ on xr, there exists a first such point x_2, since K_y is closed. The arc xx_2 is not contained in V_2, since x and x_2 lie in different components of $V_2 - y$; thus there exists a point $x_3 \in (S-V_2) = R_2$ on xx_2. In any case, therefore, there exists in $S-K_y$ an arc joining x to some point of R_2. It follows that $S-K_y$ is the sum of a finite number of regions, each containing at least one component of R_2. Since this is true for all $y \in V_2 R_1 F$, the same must be true of R.

The set V is an A-set[5] in V_2. For suppose otherwise. Then, since V is closed, there exists an arc xqy in V_2 spanning V. Since $q \notin V$, there exists a point $z \in V_2 R_1 F$ such that $q \notin K_z$. But $x+y \subset K_z$. Therefore z must separate both x and y from q in V_2, which is impossible. Since V is an A-set, it is a locally connected continuum.

Moreover, V has no cut points in F. For let $x \in V$, $y \in V$, and $q \in VF$. If $q \in (V-R_1)$, there exists an arc $xy \subset (V_2 - q)$, because q is not a cut point of the locally connected continuum V_2; since V is an A-set in V_2, the arc $xy \subset (V-q)$. If $q \in VR_1$, we have $x+y \subset K_q$ and hence there exists an arc $xy \subset (K_q - q)$; again $xy \subset (V-q)$.

[5] See Kuratowski and Whyburn, *Sur les éléments cycliques et leurs applications*, Fund. Math. vol. 16 (1930) pp. 305–331.

We have now shown that R is the sum of a finite number of regions, that $S-R$ is a locally connected continuum, and that $S-(R+q)$ is connected for any $q \in F$. Thus the proof is complete.

THEOREM 4. *Suppose that no m points separate S, and that $\mathfrak{F}$ is a family of sets such that $S-(Q+\sum_{i=1}^{m} p_i)$ is connected for any $Q \in \mathfrak{F}$ and any m points $p_1, p_2, \cdots, p_m$ of S. Then corresponding to any $\epsilon>0$ there exists a number $\delta>0$ such that if $P_1, P_2, \cdots, P_m$ are m sets contained in S, each of diameter less than δ, while $\rho(P_i, P_j)>2\epsilon$ for $0 \leqq i \leqq j \leqq m$, the set $S-\sum_{i=1}^{m} P_i$ has at most one component K of diameter greater than ϵ, and (if K exists) $K-Q$ is connected for every $Q \in \mathfrak{F}$ for which $\rho(Q, \sum_{i=1}^{m} P_i)>\epsilon$.*

PROOF. Let $\mathfrak{F}^{(1)}$ be the family of sets having as elements all sets of the type $Q+Q_1$, where $Q \in \mathfrak{F}$ and Q_1 is any set of at most $m-1$ points. Then if $F \in \mathfrak{F}^{(1)}$, the set $S-(F+p)$ is connected for every $p \in S$.

Using Theorem 2, we obtain for every point $x \in S$ two regions V_x and W_x, each of diameter less than ϵ, such that $x \in V_x \subset W_x$, while the sets $S-V_x$ and $S-W_x$ are locally connected continua, and $S-(F+V_x)$ is connected if $F \in \mathfrak{F}^{(1)}$ and $F \subset (S-W_x)$. We then choose a third region $U_x \supset x$ such that $\overline{U}_x \subset V_x$. By the Heine-Borel theorem, there exists a finite subfamily $\{U_1, \cdots, U_{n_1}\}$ of the family $\{U_x\}$ such that $S=\sum_{i=1}^{n_1} U_i$. In each set U_i $(i=1, 2, \cdots, n_1)$ we choose a point x_i for which $U_{x_i}=U_i$ and define $V_i=V_{x_i}$, $W_i=W_{x_i}$. Let $\delta_1=\min_{i=1,\cdots,n_1} \rho(U_i, S-V_i)$.

Now denote by $\mathfrak{F}^{(2)}$ the family of sets having as elements all sets of the type $Q+Q_2$, where $Q \in \mathfrak{F}$ and Q_2 is any set of at most $m-2$ points, and for $i=1, 2, \cdots, n_1$ define $\mathfrak{F}_i^{(2)}$ as the largest subfamily of $\mathfrak{F}^{(2)}$ all of whose elements are contained in $S-W_i$. Then if $F \in \mathfrak{F}_i^{(2)}$, we see that $(S-V_i)-(F+p)$ is connected for every $p \in (S-W_i)$.

Applying Theorem 2 to the locally connected continuum $S-V_i$, we obtain for every point $x \in (S-W_i)$ $(i=1, \cdots, n_1)$ three regions U_{ix}, V_{ix}, and W_{ix}, each of diameter less than ϵ, in the locally connected continuum $S-V_i$, such that $x \in U_{ix} \subset \overline{U}_{ix} \subset V_{ix} \subset W_{ix}$ and $S-(V_i+V_{ix})$ is a locally connected continuum, while $(S-V_i)-(F+V_{ix})$ is connected if $F \in \mathfrak{F}_i^{(2)}$ and $F \subset (S-W_{ix})$. Writing

$$T_i = \underset{x}{E}\,[\rho(x, W_i) \geqq \epsilon], \qquad i = 1, 2, \cdots, n_1,$$

we see that if $x \in T_i$, the set W_{ix} is contained in the interior of $S-W_i$ and is therefore a region in S. It follows by the Heine-Borel theorem that the family of regions $\{U_{ix}\}$ (for all $x \in T_i$) contains a finite subfamily $\{U_{i1}, U_{i2}, \cdots, U_{in_2(i)}\}$ such that

$$T_i \subset \sum_{j=1}^{n_2(i)} U_{ij} \subset (S - \overline{W}_i), \qquad i = 1, \cdots, n_1.$$

We write for simplicity $n_2 = \max_{i=1,\cdots,n_1} [n_2(i)]$, and by making repetitions if necessary we obtain n_1 families, each containing n_2 regions and having the above properties. Regions V_{ij} and W_{ij} are then selected for $i=1, \cdots, n_1$, $j=1, \cdots, n_2$ as in the preceding case. We let

$$\delta_2 = \min_{i=1,\cdots,n_1; j=1,\cdots,n_2} \rho(U_{ij}, S - V_{ij}).$$

We proceed by induction as follows. Suppose that for some $k<m$ we have found three sets of regions $\{U_{i_1 i_2 \cdots i_k}\}$, $\{V_{i_1 i_2 \cdots i_k}\}$, and $\{W_{i_1 i_2 \cdots i_k}\}$ (where $i_j=1, 2, \cdots, n_j$; $j=1, 2, \cdots, k$), having the following properties:

(1) $\overline{U}_{i_1 \cdots i_k} \subset V_{i_1 \cdots i_k} \subset W_{i_1 \cdots i_k}$;

(2) $\delta(W_{i_1 \cdots i_k}) < \epsilon$;

(3) $S - \sum_{j=1}^{k} V_{i_1 \cdots i_j}$ is a locally connected continuum;

(4) $T_{i_1 i_2 \cdots i_{k-1}} \subset \sum_{i_k=1}^{n_k} U_{i_1 \cdots i_k} \subset (S - \sum_{j=1}^{k-1} \overline{W}_{i_1 \cdots i_j})$,

where

$$T_{i_1 i_2 \cdots i_{k-1}} = \underset{x}{E} \left[\rho(x, W_{i_1 \cdots i_j}) \geqq \epsilon \quad \text{for} \quad j = 1, 2, \cdots, k-1\right];$$

(5) if $(Q + \sum_{i=1}^{m-k} q_i) \subset (S - \sum_{j=1}^{k} W_{i_1 \cdots i_j})$, where $Q \in \mathfrak{F}$ and $q_i \in S$ for $i=1, 2, \cdots, m-k$, then the set $(S - \sum_{j=1}^{k} V_{i_1 \cdots i_j}) - (Q + \sum_{i=1}^{m-k} q_i)$ is connected.

In order to take the next step, we define $\mathfrak{F}^{(k+1)}$ as the family of sets having as elements all sets of the type $Q + Q_{k+1}$, where $Q \in \mathfrak{F}$ and Q_{k+1} is any set of at most $m-(k+1)$ points. Then we denote by $\mathfrak{F}^{(k+1)}_{i_1 \cdots i_k}$ $(i_j=1, 2, \cdots, n_j; j=1, 2, \cdots, k)$ the largest subfamily of $\mathfrak{F}^{(k+1)}$ all of whose elements are contained in $S - \sum_{j=1}^{k} W_{i_1 \cdots i_j}$. It follows from (5) that $(S - \sum_{j=1}^{k} V_{i_1 \cdots i_j}) - (F + p)$ is connected for all $F \in \mathfrak{F}^{(k+1)}_{i_1 \cdots i_k}$ and $p \in (S - \sum_{j=1}^{k} W_{i_1 \cdots i_j})$.

Applying Theorem 2 to the locally connected continuum $S - \sum_{j=1}^{k} V_{i_1 \cdots i_j}$, we obtain for any point $x \in (S - \sum_{j=1}^{k} W_{i_1 \cdots i_j})$ three regions $U_{i_1 \cdots i_k x}$, $V_{i_1 \cdots i_k x}$, and $W_{i_1 \cdots i_k x}$, each of diameter less than ϵ, in the locally connected continuum $S - \sum_{j=1}^{k} V_{i_1 \cdots i_j}$, such that

$$x \in U_{i_1 \cdots i_k x} \subset \overline{U}_{i_1 \cdots i_k x} \subset V_{i_1 \cdots i_k x} \subset W_{i_1 \cdots i_k x}$$

and $S - \sum_{j=1}^{k} V_{i_1 \cdots i_j} - V_{i_1 \cdots i_k x}$ is a locally connected continuum, while $(S - \sum_{j=1}^{k} V_{i_1 \cdots i_j}) - (F + V_{i_1 \cdots i_k x})$ is connected if $F \in \mathfrak{F}^{(k+1)}_{i_1 \cdots i_k}$ and $F \subset (S - W_{i_1 \cdots i_k x})$. Then, defining $T_{i_1 \cdots i_k}$ as in (4), we see as before that $U_{i_1 \cdots i_k x}$ is a region in S; using the Heine-Borel theorem, we deduce the existence of a finite family of regions $\{U_{i_1\cdots i_k i_{k+1}}\}$ $(i_{k+1}=1, 2, \cdots, n_{k+1})$ such that

$$T_{i_1\cdots i_k} \subset \sum_{i_{k+1}=1}^{n_{k+1}} U_{i_1\cdots i_{k+1}} \subset \left(S - \sum_{j=1}^{k} \overline{W}_{i_1\cdots i_j}\right).$$

Selecting families of regions $\{V_{i_1\cdots i_{k+1}}\}$ and $\{W_{i_1\cdots i_{k+1}}\}$ as before, we obtain three sets of regions for which (1)–(5) hold with k replaced by $k+1$.

We carry out this construction for $k=1, 2, \cdots, m$, and let

$$\delta_k = \min_{i_j=1,\cdots,n_j;\, j=1,\cdots,k} \rho(U_{i_1\cdots i_k}, S - V_{i_1\cdots i_k}), \qquad k = 1, \cdots, m.$$

We shall now show that the theorem holds with $\delta = \min_{k=1,\cdots,m} \delta_k$. Consider any family of sets $\{P_1, P_2, \cdots, P_m\}$ satisfying the conditions of the theorem. Since $S=\sum_{i=1}^{n_1} U_i$, there exists a positive integer $i_1 \leqq n_1$ such that $P_1 U_{i_1} \neq 0$; then since $\delta(P_1) < \delta \leqq \delta_1$, we have $P_1 \subset V_{i_1}$. Since $\rho(P_1, P_2) > 2\epsilon$, it is clear that $P_2 \subset T_{i_1}$, and hence there exists a positive integer $i_2 \leqq n_2$ such that $P_2 U_{i_1 i_2} \neq 0$; it follows that $P_2 \subset V_{i_1 i_2}$. Now suppose that for $j=1, 2, \cdots, k<m$ there exist numbers $i_j \leqq n_j$ such that $P_j \subset V_{i_1\cdots i_j}$. Since $\rho(P_j, P_{k+1}) > 2\epsilon$ for $j=1, \cdots, k$, we see that $P_{k+1} \subset T_{i_1\cdots i_k}$; thus $P_{k+1} U_{i_1\cdots i_k i_{k+1}} \neq 0$ for some $i_{k+1} \leqq n_{k+1}$, whence $P_{k+1} \subset V_{i_1\cdots i_k i_{k+1}}$. Proceeding in this way, we find positive integers $i_j \leqq n_j$ such that $P_j \subset V_{i_1\cdots i_j}$ for $j=1, 2, \cdots, m$.

We conclude from property (5) above that $S-\sum_{j=1}^{m} V_{i_1\cdots i_j}$ is connected, and hence must be contained in a single component K of $S-\sum_{i=1}^{m} P_i$. Any other component of $S-\sum_{i=1}^{m} P_i$ must therefore be contained in one of the regions $V_{i_1\cdots i_j}$; thus the diameter of such a component must be less than ϵ.

Finally, suppose that $Q \in \mathfrak{F}$ and $\rho(Q, \sum_{i=1}^{m} P_i) > \epsilon$. Then by (2) above, $Q \subset (S-\sum_{j=1}^{m} W_{i_1\cdots i_j})$; by (5), $(S-\sum_{j=1}^{m} V_{i_1\cdots i_j} - Q)$ is connected. It follows that $K-Q$ is connected.

Remark. If no n $(>m)$ points separate S, we may take $\mathfrak{F}$ as the family of all sets of $n-m$ points; then, under the above hypotheses, the component K of $S-P$ (where $P=\sum_{i=1}^{m} P_i$) is not separated by any set of $n-m$ points $q_1, q_2, \cdots, q_{n-m}$ such that $\rho(\sum_{i=1}^{n-m} q_i, P) > \epsilon$.

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