vol. 48 (1942) pp. 589–601) which relate tensor algebra and invariant theory are used in the consideration of the elementary problem of the three-line configuration. (Received November 23, 1942.)

16. T. L. Wade: *On conjugate tensors.*

How a contravariant (skew-symmetric) tensor V of order $n-p$ may be associated with a covariant skew-symmetric tensor U of order p, in an n-dimensional coordinate system, is well known (see Veblen and von Neumann, *Geometry of complex domains*). This standard association holds only when U is skew-symmetric. The purpose of this note is to show how a contravariant tensor V, of defined order and symmetry, can be associated with the covariant tensor U, where U is of any type $[\alpha]$ of symmetry. (Received November 23, 1942.)

17. T. L. Wade: *On the factorization of rank tensors.*

Let $C^{(i)}_{(j)}=D^{(i)}_{(j)}+E^{(i)}_{(j)}$, where $D^{(i)}_{(j)}$ and $E^{(i)}_{(j)}$ are mutually orthogonal idempotent numerical tensors. Expressions for the contravariant and covariant factors of the rank tensor (see Amer. J. Math. vol. 64 (1942) pp. 725–752) of $C^{(i)}_{(j)}$ in terms of like factors of the rank tensors of $D^{(i)}_{(j)}$ and $E^{(i)}_{(j)}$ are established in this paper. (Received November 23, 1942.)

18. T. L. Wade and R. H. Bruck: *Types of symmetries.*

This paper considers some aspects of symmetries with tensorial significance which are believed not to have appeared in the literature. (Received November 23, 1942.)

19. André Weil: *Differentiation in algebraic number-fields.*

Analogies with function-fields have long ago led E. Noether and others to the conjecture that the theory of the different in number-fields can be built upon some arithmetical analogue of differentiation. This is now done, by defining a derivation modulo an ideal $\mathfrak{a}$ in a number-field as an operator D with the following properties: (a) D maps the ring $\mathfrak{o}$ of integers in the field into the ring $\mathfrak{o}/\mathfrak{a}$; (b) $D(\alpha+\beta)=D\alpha+D\beta$; (c) if $\bar{\alpha}$, $\bar{\beta}$ are the classes of α, β mod $\mathfrak{a}$, then $D(\alpha\beta)=\bar{\alpha}\cdot D\beta+\bar{\beta}\cdot D\alpha$; D is essential if there is α in $\mathfrak{o}$, such that $D\alpha$ is not a zero-divisor in $\mathfrak{o}/\mathfrak{a}$. The different is then the least common multiple of all ideals modulo which there exists an essential derivation. This is easily extended to the relative different, to p-adic fields, and so on. (Received November 9, 1942.)

20. Alexander Wundheiler: *An algebraic definition of affine space.*

A simple set of axioms for affine geometry based on one operation $C=hAB$, where h is a real number, A, B points and C the point collinear with A and B, and such that $CA/CB=h$, is given. There are essentially five axioms, only one of them involving more than two (namely, three) points. An "affine calculus," which permits the writing of every affine theorem as an implication between formulas, arises from the mentioned operation. (Received November 20, 1942.)

Analysis

21. R. P. Agnew: *Euler transformations.*

Let $E(r)$ denote the Euler transformation $\sigma_n=\sum_{k=0}^{n}C_{nk}r^k(1-r)^{n-k}s_k$ by means of which a sequence s_n is summable to σ if $\sigma_n \to \sigma$ as $n \to \infty$. The fundamental properties of the transformations $E(r)$ are developed for the general case in which r is complex. The family of transformations $E(r)$ for which $r \neq 0$ is a consistent family even though $E(r)$ is regular only when r is real and $0 < r \leqq 1$. Relations between the methods $E(r)$ and other methods of summability are established. The members of a family of series-to-series transformations are shown to be equivalent to the transformations $E(r)$ when the parameters are suitably restricted. The open sets in which power series are summable are characterized. If $r_1, r_2, \cdots$ is a sequence of real numbers for which $r_n \to 0$ and $nr_n \to \infty$, then the transformation $\sigma_n = \sum_{k=0}^{n} C_{nk} r_n^k (1-r_n)^{n-k} s_k$ includes $E(r)$ for each $r > 0$. (Received October 13, 1942.)

22. R. P. Agnew: *On sequences with vanishing even or odd differences.*

Let $x_0, x_1, x_2, \cdots$ be a sequence of real or complex numbers, and let $d_n = \sum_{k=0}^{n} (-1)^k C_{n,k} x_k$, $n = 0, 1, 2, \cdots$, denote its sequence of differences. By use of a theorem on Euler methods of summability, a short proof of the two following theorems is obtained. If x_n if bounded and $d_n = 0$ when n is even, then $x_n = 0$ for each n. If x_n is bounded and $d_n = 0$ when n is odd, then $x_n = x_0$ for each n. (Received October 13, 1942.)

23. I. A. Barnett: *A note on skew-symmetric kernels.*

It is proved in this note that a necessary and sufficient condition that $\Gamma(x, y; \lambda)$ be the resolvent kernel of a skew-symmetric kernel is that $\Gamma(x, y; \lambda) + \Gamma(y, x; \lambda) + 2\lambda \int_a^b \Gamma(x, t; \lambda)\, \Gamma(y, t; \lambda)\, dt$ be identically zero in x, y and λ. This is the analogue of the fact that the adjoint $C(\lambda)$ of $A - \lambda I$ satisfies the equation $\det(A - \lambda I)[C(\lambda) + C'(\lambda)] + 2\lambda C(\lambda) C'(\lambda) = 0$, where A is a skew-symmetric matrix, and $C'(\lambda)$ denotes the transpose of $C(\lambda)$ (cf. Amer. Math. Monthly vol. 49 (1942) pp. 169–170). As a consequence of the first relation, it follows that a solution of the quadratic integral equation $\phi(x, x) = \lambda \int_a^b \phi^2(x, t) dt$ is $\phi(x, y) = -\Gamma(x, y; \lambda)$ where Γ is the resolvent of an arbitrary skew-symmetric kernel. (Received November 20, 1942.)

24. E. F. Beckenbach: *The stronger form of Cauchy's integral theorem.*

According to the stronger form of Cauchy's integral theorem, if $f(z)$ is holomorphic on the interior D of a simply closed rectifiable curve C, and continuous on $D + C$, then $\int_C f(z) dz = 0$. A brief and simple though not elementary proof is given, based on considerations of conformal mapping, the length of level curves of the Green's function, and a simple property of a Stieltjes integral. (Received November 23, 1942.)

25. Henry Blumberg: *On arbitrary point transformations.*

The transformations τ considered are correspondences $\tau: y = f(x)$ which associate with each point $x: (x_1, x_2, \cdots, x_m)$ of euclidean m-space S_m a point $y: (y_1, y_2, \cdots, y_n)$ of euclidean n-space S_n. No other conditions are imposed on τ except "one-valuedness," which, too, is not an essential one. A series of theorems are proved, of descriptive and matric character, for the structure of τ. The principal results are of the latter sort, and relate to the notion of "salient point," defined as a point of non-infinitesimal approach—in a certain readily suggested sense—of the "graph" of τ in

S_{m+n}, a euclidean space of $m+n$ dimensions. Theorems are obtained, among others on the distribution, closure properties, and measurability—in a certain sense—of the set of salient points. Finally, necessary and sufficient conditions are established characterizing such a set. (Received November 19, 1942.)

26. R. F. Clippinger: *General remarks about the set of products of positive powers of n-by-n matrices and the associated manifold.* Preliminary report.

Let P be the set of products of arbitrary positive powers of the exponentials of m given n-by-n matrices, A_i, of real numbers. Let M be the corresponding manifold of points of n^2 Euclidean space. Let $\overline{P}$ and $\overline{M}$ be the closure of P and the corresponding manifold. P is a groupoid. M is connected and r-dimensional if r is the number of linearly independent matrices B_j among A_i and their alternants. If C_k is an arbitrary linear combination of the B_j, there exists a number l such that any matrix of P is a product of at most l matrices $\exp C_k$. $\overline{P}$ contains all matrices of the form $\exp A$ if A is any linear combination with positive coefficients of A_i. $\overline{P}$ is identical with the closure of the set of particular values of solutions of the differential equation $dY/dt = YB$, $Y(a)=1$ where B is a linear combination of A_i with coefficients which are arbitrary non-negative functions of class C^∞. (Received November 21, 1942.)

27. R. F. Clippinger: *Matrix products of matrix powers.* Preliminary report.

If A and B are 2-by-2 matrices with constant elements a, b, c, d, e, f, g, h and distinct characteristic roots such that A, B, (AB) and $(A(AB))$ are linearly independent and $[(bg+cf-ah-de)(e+h)+2(eh-fg)(a+d)][(bg+fc-ah-de)(a+d)+2(ad-bc)(e+h)]$ is negative, and if $\alpha, \beta, \gamma, \delta, \alpha_1, \beta_1, \alpha_2, \beta_2, \cdots$ are arbitrary non-negative numbers, then the closure of the set of matrices $\exp \alpha_1 A \exp \beta_1 B \exp \alpha_2 A \cdot \exp \beta_2 B \cdots$ is identical with the closure of the set of matrices $\exp \alpha A \exp \beta B \exp \gamma A \cdot \exp \delta B$. (Received November 21, 1942.)

28. R. F. Clippinger: *Mean value theorems for a certain linear matrix differential equation.* Preliminary report.

Let A and B be two, n-by-n matrices, with distinct characteristic roots, which verify the condition $AB-BA=\lambda A+\mu B$, $\mu\neq 0$. If $\rho(t)$ and $\sigma(t)$ are arbitrary, non-negative, bounded measurable functions of t on the interval $a\leqq t\leqq b$, and if $Y(b)$ is the particular value for $t=b$ of the solution of the matrix differential equation $dY(t)/dt=Y(t)[\rho(t)A+\sigma(t)B]$, $Y(a)=1$, then there exist non-negative numbers $\alpha, \beta, \gamma, \delta, \epsilon, \zeta$ such that, without changing $Y(b)$, $\rho(t)$ and $\mu(t)$ may be replaced by: (a) α and β on $a\leqq t\leqq b$; or, (unless $\mu>0$ and $\lambda>0$), (b) γ and 0 on $a\leqq t\leqq (a+b)/2$; 0 and δ on $(a+b)/2\leqq t\leqq b$; or, (unless $\mu<0$ and $\lambda<0$), (c) 0 and δ on $a\leqq t\leqq(a+b)/2$; γ and 0 on $(a+b)/2\leqq t\leqq b$. (Received November 21, 1942.)

29. Max Coral: *Solution of quasi-linear partial differential equations through a characteristic initial curve.*

A new method is presented of passing a solution of a quasi-linear first order partial differential equation in two independent variables through an initial curve C which is characteristic for the equation. It is shown that C yields a solution S of the equations of variation of the characteristic differential equations. If T is any solution of these equations of variation which is independent of S, then a one-parameter family of characteristic curves imbedding C can be found, whose variation along C is T. These characteristic curves envelop the desired solution. The present method has the advantage that the solution thus found contains the entire extent of the initial curve. (Received October 30, 1942.)

30. G. M. Ewing: *Minimizing an integral on a class of continuous curves.*

Hahn published an example in 1925 of a positive semidefinite positive quasi-regular variation problem which admits no rectifiable minimizing curve. Menger has obtained several existence theorems for problems in very general spaces which apply to the example of Hahn. In the present paper attention is confined to variation problems in euclidean spaces. The integral J is defined for a continuous curve C as g.l.b. lim inf $J(C_n)$, the greatest lower bound being on the class of sequences C_n of rectifiable continuous curves converging to C. Using δ-length based on any distance $\delta(p, q)$ satisfying the axioms of a metric space, a set of admissible curves of bounded δ-length is compact. The method involves finding a suitable metric δ and determining hypotheses under which the problem admits a minimizing sequence of rectifiable curves of bounded δ-length. Existence theorems based on several choices of δ and related lower-semicontinuity theorems are discussed. (Received October 14, 1942.)

31. N. A. Hall: *Confluence of the basic hypergeometric series.*

The general properties of the basic hypergeometric series are reviewed with special attention being given to the introduction of a satisfactory comprehensive notation and to the establishment of criteria for identifying analogues of theorems concerning generalized hypergeometric series. In particular two analogues of the confluence of generalized hypergeometric series are discussed and applied to several identities appearing in mathematical literature. Examples are given relating these confluent basic series to identities of modular functions. It is shown that the whole field of theta function, mock-theta function, modular function, and Raman Ramanujan identities may probably be expressed very elegantly and completely in terms of the basic hypergeometric series and their established relations. (Received November 19, 1942.)

32. Einar Hille and Max Zorn: *Open additive semi-groups of complex numbers.*

The main object of this paper is to determine the open, connected and additive semi-groups of the complex number plane. These sets are important as parameter manifolds of semi-groups of linear transformations which have been studied in detail by Hille. It is shown that the open semi-groups of the plane depend upon the upper-semi-continuous solutions of the inequality $f(x+y) \leqq f(x)+f(y)$, the function f being defined for all real or all positive (negative) numbers x. These sets are automatically connected and simply connected. They are also shown to be the maximal domain of existence for suitable 1-parameter semi-groups of linear transformations in appropriate Banach spaces. Many of the results about the parameter manifolds permit considerable generalizations. (Received November 28, 1942.)

33. H. K. Hughes and Cleota G. Fry: *Asymptotic developments of certain integral functions.*

The authors obtain asymptotic developments valid for value of z of large modulus, of the function $f_\alpha(z)=\sum^{\infty}_{n=0}g(n)z^n$, where $g(n)$ is of the form $h(n)\{\Gamma(\alpha n+p)\}^{-1}$; $\alpha>0$; p is any constant, real or complex. It is assumed that the function $g(w)$, where $w=x+iy$, is single-valued and analytic in the finite w-plane, and that $h(w)$ is developable in a factorial series of the form $\sum_{n=0}^{\infty}a_n\Gamma(\alpha w+p)/\Gamma(\alpha w+p+n)$. The paper may be regarded as a generalization of results already obtained by W. B. Ford, and appearing in chap. 6 of his book entitled *Asymptotic developments of functions defined by Maclaurin series* (Michigan Science Series, vol. 11, Ann Arbor, 1936). The methods employed in the paper are essentially those employed by Ford. (Received November 20, 1942.)

34. Glenn James: *On equivalence of methods of summing divergent series.*

This paper investigates the uniqueness of sums given by the general, regular, definition $\lim_{n\to\infty}\sum_1^n s_i\alpha_i=S$, where the α_i's are positive or zero and converge uniformly to zero and $\lim_{n\to\infty}\sum_1^n\alpha_i=1$. It is shown that S is generally arbitrary. A fourth restriction is then placed upon the α_i's which, at least for most ordinary summable series, makes S unique, and enables one actually to determine S. This restriction is that the limit of the quotient of the number of changes in monotony in the sequence by n, as n increases, shall be zero. (Received October 31, 1942.)

35. H. F. S. Jonah and A. H. Smith: *Zero order summability of the series conjugate to the derived Fourier series.*

This paper proves that a theorem of Takahashi (Jap. J. Math. vol. 10 (1933) pp. 127–132) concerning the Cesàro summability of the series conjugate to the derived Fourier series remains true if the Cesàro method of summation is replaced by the zero order method of Bosanquet-Linfoot (J. London Math. Soc. vol. 6 (1931) pp. 117–126), which is weaker than the Cesàro method of any order $\alpha>0$. (Received October 29, 1942.)

36. Knox Millsaps: *Characterization of the abstract exponential function.*

After defining the abstract exponential function over normed rings, it is possible to characterize this function as the solution of a first order general differential system. The validity of the theorem is established by the use of two theorems of Kerner on the existence of a solution for such equations and the symmetry of second order Fréchet differentials. (Received October 31, 1942.)

37. C. N. Moore: *On the relationship between Nörlund means of a certain type.*

The purpose of this paper is to obtain relationships of inclusion between Nörlund means defined in terms of the coefficients of the power series developments of functions analytic within the unit circle and having a singularity of algebraic-logarithmic type at the point $z=1$. The inclusion relationship is found to depend on the relative order of infinity of the functions involved in the neighborhood of the singular point. (Received November 19, 1942.)

38. G. B. Price: *Cauchy-Stieltjes and Riemann-Stieltjes integrals.*

This note treats the equivalence of the Riemann-Stieltjes and Cauchy-Stieltjes integrals (abbreviated RS and CS integrals) and conditions for the existence and equality of the latter. The RS and the left and right CS integrals are defined as limits of the sums $\sum_1^n f(\xi_i)[g(x_i)-g(x_{i-1})]$, $x_{i-1}\leqq\xi_i\leqq x_i$, $\sum_1^n f(x_{i-1})[g(x_i)-g(x_{i-1})]$, and $\sum_1^n f(x_i)[g(x_i)-g(x_{i-1})]$. When $g(x)\equiv x$, these integrals are the Riemann and the two Cauchy integrals; Gillespie (Ann. of Math. vol. 17 (1915) pp. 61–63) proved that they are equivalent. Examples show (i) that the CS integrals may exist when the RS does not; (ii) that the two CS integrals may exist and have different values; (iii) and that one CS integral may exist when the other does not. One theorem is the following: if g is non-decreasing, if f and g have no common discontinuities on the same side, and if the left (right) CS integral exists, then the RS integral exists and has the same value, the integrals being limits in the sense of increasing refinement of subdivisions. (Received November 23, 1942.)

39. Maxwell Reade: *On a theorem of Fédoroff and Binney.*

A theorem on harmonic functions, due to Fédoroff (Rec. Math. (Mat. Sbornik) N.S. vol. 40 (1933) pp. 168–179) and Binney (Trans. Amer. Math. Soc. vol. 37 (1935) pp. 254–265), has an analogue for subharmonic functions. The result is the following theorem. If $u(x, y)$ and $v(x, y)$ are continuous in the unit circle, and if $\int_R udy-vdx\geqq 0$ and $\int_R udx+vdy=0$ for all oriented rectangles, R lying in the unit circle, then there exists a function $f(x, y)$ that is subharmonic in the unit circle for which $\partial f/\partial x=u(x, y)$, $\partial f/\partial y=v(x, y)$. The proof follows Levin (*Contributions to the calculus of variations*, Chicago, 1942, pp. 363–410). (Received October 17, 1942.)

40. Maxwell Reade: *Remarks on a paper of Beckenbach.*

Beckenbach's Theorem 1 (Duke Math. J. vol. 8 (1941) pp. 393–400) is similar to the classic Montel-Rado theorem characterizing functions having subharmonic logarithms; therefore it admits of Saks-Kierst types of generalizations (S. Saks, Acta Univ. Szeged vol. 5 (1930–1932) pp. 187–193). A typical result of this paper is the following theorem. Let $f(t)$ be of class c^2, with $f'(t)>0$ for $-\infty<t<\infty$. If $v(x, y)$ is of class c^2 in a domain D, and if $f(\log[(x-\alpha)^2+(y-\beta)^2]+v(x, y))$ is subharmonic for each choice of the real constants α and β, then $v(x, y)$ is subharmonic in D. In addition Beckenbach's Theorem 2 is given a slightly more general form. (Received October 16, 1942.)

41. Raphael Salem: *On some singular monotonic functions which are strictly increasing.*

The paper gives simple direct constructions of some singular monotonic functions which are strictly increasing. It contains also some new indications on Minkowski's singular function, particularly concerning its modulus of continuity. (Received October 9, 1942.)

42. H. M. Schwartz: *On sequences of Stieltjes integrals.* II.

Given a sequence $g_n(x)$ $(n=1, 2, \cdots)$ of functions of bounded variation, the problem of determining sets $C(F)$ of conditions necessary and sufficient for the convergence of the sequence $\int_a^b f dg_n$ for all functions $f(x)$ of a given family F, is an extension of the problem treated by Lebesgue in his paper *Sur les intégrales singulieres* (Ann. Fac. Sci. Univ. Toulouse vol. 23 (1909) pp. 25–117) and which corresponds to the case of absolutely continuous $g_n(x)$. Considering first the case of Riemann-Stieltjes integration over a finite interval, conditions $C(F)$ were obtained earlier (Bull. Amer. Math. Soc. abstract 48-1-55) for certain families F, where F are proper subsets of the set F^* of all functions which are defined in (a, b) and are simultaneously integrable with respect to all g_n. The principal result of the present paper is the determination of a set $C(F^*)$. In the special case of absolutely continuous g_n, this result provides a solution for the Lebesgue problem in the case, not treated by Lebesgue, when F consists of the class of Riemann integrable functions. The paper is concerned also with the extension of the results to the case of an infinite interval (a, b) and to more general Stieltjes integrals of Riemann type. (Received November 20, 1942.)

43. I. M. Sheffer: *Note on a linear transformation of "analytic" type.*

A linear problem concerning analytic functions is often expressed in terms of the power series coefficients of the functions, thus leading the problem to a system of linear equations in the coefficients: (1) $A_n[X] \equiv \sum_{k=0}^{\infty} a_{nk}x_k = c_n$, $n = 0, 1, \cdots$, where X represents the sequence $\{x_n\}$. Let lim sup $|x_n|^{1/n}$ be called the type of sequence $\{x_n\}$. As a first step in the analysis of transformation (1) examine the following question (together with some of its consequences): Under what conditions will sequences $\{x_n\}$ of given type transform into sequences of prescribed type under (1)? A typical result is the following: In order that $\{A_n[X]\}$ be of type at most h for every $\{x_n\}$ of type at most r it is necessary and sufficient that: (i) If r_n is the radius of convergence of $A_n(t) = \sum_{k=0}^{\infty} a_{nk}t^k$, and $r^* =$ g.l.b. $\{r_n\}$, then $r^* > r$. (ii) Let $A_n^*(t) = \sum_{k=0}^{\infty} |a_{nk}| t^k$. For every $\epsilon > 0$ there corresponds an r' in $r < r' < r^*$ such that $\{A_n^*(r')\}$ is of type at most $h + \epsilon$. (Received November 24, 1942.)

44. W. S. Snyder: *Non-parametric surfaces and inscribed polyhedra.* Preliminary report.

Let $z = f(x, y)$ define a continuous surface over an oriented rectangle of the xy-plane. Kempisty (*Sur la methode triangulaire* $\cdots$, Bull. Soc. Math. France vol. 64 (1936)) and the author have studied certain functions of rectangles associated with the surface and have derived sufficient conditions that the Burkill integrals of the functions exist and equal the Lebesgue area of the surface. This paper undertakes the study of the simple Burkill integrals (Burkill, *Functions of intervals*, Proc. London Math. Soc. (2) vol. 22 (1924)) of these functions. Necessary conditions that the simple Burkill integrals of these functions exist and equal the Lebesgue area of the surface are derived, and somewhat stronger conditions are shown to be sufficient for this result to hold. (Received November 23, 1942.)

45. W. C. Strodt: *Analytic solutions of nonlinear difference equations.* Preliminary report.

A new approach to difference equations is outlined, and applied to a large class of nonlinear difference equations. Special solutions are obtained as uniform limits of analytic solutions of q-difference equations, equations which yield more readily to power series treatment. The given equation $P[x, y(x), y(x+\omega_1), \cdots, y(x+\omega_n)]=0$ (where the ω_k are arbitrary positive numbers, and P is a polynomial in the $y(x+\omega_k)$, analytic in x) is replaced by the approximating functional equation $P[x, y(x), y(q_1x+\omega_1), \cdots, y(q_nx+\omega_n)]=0$, where $q_i=1-\omega_i\eta$, η being any small positive number. Each transformation $x\rightarrow q_ix+\omega_i$ has $1/\eta$ for fixed point. A translation of origin to this point transforms this functional equation into the q-difference equation $P[u+1/\eta, z(u), \cdots, z(q_nu)]=0$, where $u=x-1/\eta$, $z(u)\equiv y(u+1/\eta)$. This equation has for solution a unique power series $z(u; \eta)$ in u whose convergence is shown by the calculus of limits. The uniform limit of $z(x-1/\eta; \eta)$, as η assumes a suitable sequence of values tending to 0, is an analytic solution of the given difference equation. There is evidence that the solution generalizes to nonlinear equations Nörlund's "principal solution." The procedure applies formally to essentially every difference equation; a preliminary transformation $y(x)=x^sY(x)$ is sometimes needed. (Received October 16, 1942.)

46. Otto Szász: *On Abel and Lebesgue summability.*

The main results of this paper are: If the real numbers a_n satisfy the condition (*) $\sum_n^{2n}(|a_\nu|-a_\nu)=O(1)$ as $n\rightarrow\infty$, then Abel summability of the series $\sum_1^\infty a_n$ implies its Lebesgue summability; more explicitly the series $\sum a_n(\sin nt/n)=F(t)$ converges uniformly and $\lim_{t\downarrow 0}t^{-1}F(t)=s$ exists. At the same time the series $\sum a_n$ may diverge; in fact a series $\sum a_n$ is constructed which is Abel summable and satisfies (*), while $\limsup |a_n|=1$, so that $\sum a_n$ diverges. On the other hand it is proved that if (*) holds and if $\sum a_n$ converges, then the series $\sum a_n(\sin nt/nt)=t^{-1}F(t)$ converges uniformly in the interval $0<t<\pi$, that is, $\sum a_n$ is Lebesgue summable. It is to be noted that convergence alone does not imply Lebesgue summability. The method is similar to the one used in a previous paper: Amer. J. Math. vol. 64 (1942) pp. 575–591. (Received October 20, 1942.)

47. W. C. Taylor: *Asymptotic behavior of the Abel sums of the Laguerre expansion.*

For the Laguerre expansion $f(x)\sim\sum c_\nu L_\nu^{(\alpha)}(x)$ the behavior of the Abel sum $A(x,w)=\sum c_\nu L_\nu^{(\alpha)}(x)w^\nu$ is studied. Conditions are obtained which insure that $A(x, w)-f(x)$ is $o(1)$ or $o(f(x))$ under various limiting processes involving $x\rightarrow\infty$ and/or $w\rightarrow 1$. (Received November 21, 1942.)

48. W. J. Trjitzinsky: *Singular nonlinear integral equations.*

The author studies problems (1) $\phi(x)+\int K(x, t)f(t, \phi(t))dt=0$, (2) $\phi(x)+\int\Gamma(x, y, \phi(y))dy=0$ (ϕ is the unknown; the integrals are definite), where $\Gamma(x, y, z)=\int K(x, t)k(t, y, z)dt$. The functions $f(t, u)$, $k(t, y, u)$ are subject to Lipschitz conditions in u. Past investigations were under the supposition that the kernel $K(x, t)$ is regular in the sense that the characteristic values of $K(x, t)$ form a discrete set, Fredholm theory being applicable. The author proceeds with more general kernels, not regular in the above sense, belonging to L_2 in each of the variables separately; such kernels are singular. Among other results a number of existence theorems for (1), (2) are established. These problems are of importance in themselves as well as in many phases of mathematical physics; also, many nonlinear differential problems with linear boundary conditions are reducible to the problem (1). This theory naturally depends in an essential manner on the theory of singular kernels as previously developed by Carleman and later by the present author. (Received November 20, 1942.)

49. S. M. Ulam: *On equivalence of functions.*

Two functions, $f(x)$ and $g(x)$ defined on $0 \leqq x \leqq 1$ are said to be equivalent, if there exists a one-to-one transformation of the interval $h(x)$ such that $f(x) = hgh^{-1}(x)$. This notion of equivalence is investigated under restriction of $f(x)$ and $g(x)$ to various classes of functions. Among others, it is proved that if $f(x)$ is a one-to-one analytic function, it is equivalent to a Baire function of finite class. Necessary and sufficient conditions are given for equivalence of two continuous functions—and an enumeration of possible equivalence types of such functions is obtained. (Received November 23, 1942.)

50. Wolfgang Wasow: *On a boundary layer problem for a certain linear partial differential equation.*

Let $u(x, y, \lambda)$ be the function satisfying the differential equation $(1/\lambda)\Delta u + u_x = f(x, y)$ in the interior of a convex domain B and assuming prescribed boundary values, not depending on λ, on the boundary C of B. As $\lambda \rightarrow \infty$ $(\lambda > 0)$ the differential equation reduces to the limiting differential equation $u_x = f(x, y)$. In the paper the author proves that $v(x, y) = \lim_{\lambda \rightarrow \infty} u(x, y, \lambda)$ exists and satisfies the limiting differential equation. But in general $v(x, y)$ will satisfy the prescribed boundary condition only on a certain arc C_1 of C. Along the remaining arc C_2 of C the convergence of $u(x, y, \lambda)$ is non-uniform and gives rise to a boundary layer phenomenon. The proof is based on certain estimates for Green's function and uses an idea suggested in a paper by E. Rothe. (See E. Rothe, *Asymptotic solution of a boundary value problem*, Iowa State College Journal for Science vol. 13 (1939).) (Received November 23, 1942.)

51. Alexander Weinstein: *On a general variational method for the determination of eigenvalues.*

Let $U(v)$ and $H(v)$ be two positive definite quadratic functionals and let P be the eigenvalue problem relative to the variational problem $U(v)/H(v) = \text{Min.}$ under certain boundary conditions BC which express the vanishing of v, dv/dn, and so on, on certain parts of the boundary. Let P_0 be the eigenvalue problem obtained from P by cancelling a certain number of the boundary conditions BC. The eigenvalues in P_0 are obviously not greater than those in P. It can be shown that the eigenvalues in P can be expressed in terms of the solutions of P_0. This result is obtained by linking P_0 with P by a chain of intermediate problems $P_1, P_2, \cdots$ defined in a similar way as those used in particular cases by the author (Mémorial des Sciences Mathématiques, no. 88). (Received November 16, 1942.)

52. J. E. Wilkins: *A class of functions in the calculus of variations for multiple integrals in parametric form.*

In order that a multiple integral with integrand $f(y^1, \cdots, y^n, p_1^1, \cdots, p_m^n)$, where $p_a^i = \partial y^i/\partial t_a$, be independent of the parametric representation $y^i = y^i(t)$ of the variety for which it is computed, it is necessary and sufficient that the equation $f(y, pA) = |A| f(y, p)$ hold for every m-rowed square matrix A with positive determinant. It was proved in the author's doctoral dissertation that there exist constants $N^{\beta;\epsilon}_{\alpha;\gamma}$ such that the equation, $p^{\iota}_{\alpha}f^{\beta\epsilon}_{\iota\kappa} = M^{\beta;\epsilon}_{\alpha;\gamma} f^{\gamma}_{\kappa}$ holds for every function f satisfying the above relation. In this equation $\iota = (i_1, \cdots, i_g)$, $\alpha = (a_1, \cdots, a_g)$, $\beta = (b_1, \cdots, b_g)$, $\epsilon = (e_1, \cdots, e_h)$, $\kappa = (k_1, \cdots, k_h)$, $\gamma = (c_1, \cdots, c_s)$, p^{ι}_{α} stands for the product of the corresponding p's, and f^{γ}_{κ} stands for the derivative of f with respect to the corresponding p's. It is the purpose of this paper to give an explicit definition of the constants $M^{\beta;\epsilon}_{\alpha;\gamma}$ in terms of a certain set of permutations. On the basis of this definition certain properties of the M's are readily deduced. (Received October 27, 1942.)

53. J. E. Wilkins: *A note on the Weierstrass condition for multiple integrals in the calculus of variations.*

For the purpose of obtaining a sufficiency theorem for multiple integral problems in the calculus of variations, Carathéodory has introduced an E-function $E_c(x, y, p, P)$ different from the usual E-function E_0. It is proved here that $E_c = E_0$ in case $P - p$ has rank one. Consequently, a necessary condition for a minimum is that $E_c \geqq 0$ for all P such that $P - p$ has rank one. But an example is given to show that it is not true that $E_c \geqq 0$ for all P is necessary for a minimum. (Received October 30, 1942.)

54. J. E. Wilkins: *Definitely self-conjugate adjoint integral equations.*

This paper contains a new definition for definitely self-conjugate adjoint integral equations which is substantially weaker than that originally given by Reid. For such systems it is found that all of the expansion theorems obtained by Reid remain valid, but that there need not now be a denumerable infinity of characteristic values. Moreover it is shown that the index of a characteristic value is equal to its multiplicity as a zero of the Fredholm determinant. (Received November 25, 1946.)

55. Fumio Yagi: *On a certain Stieltjes integral equation.*

Let $g(z)$ and $p(\lambda)$ be two given functions which satisfy certain norm conditions. It is known (Cameron-Martin, *Infinite linear difference equations with arbitrary real spans and first degree coefficients*, to be published) that the integral equation $\int_{-\infty}^{\infty} f(z-\lambda)dp(\lambda) = g(z)$ has a unique analytic solution in $a < \text{Im } z < b$ of the same norm as $g(z)$, provided that $P(w) = \int_{-\infty}^{\infty} e^{w}\lambda dp(\lambda)$ is non-vanishing in a strip $c < \text{Re } w < d$. In this paper the author allows the function $P(w)$ to have a finite number of zeros in $c < \text{Re } w < d$ and shows that there exists an analytic solution $f(z)$ in $a < \text{Im } z < b$ of the same norm as $g(z)$. Moreover it is proved that every other solution of the required type is the form $f(z) + \sum_{n=1}^{N}\sum_{m=1}^{a_n} A_{nm} e^{-w_n z} z^{m-1}$ where A_{nm} are constants, w_n $(n = 1, 2, \cdots, N)$ are the zeros of $P(w)$, and a_n is the order of zero at w_n. (Received November 20, 1942.)

56. J. W. T. Youngs: *On the additivity of the Lebesgue area.*

Suppose a surface S is given in parametric form by the continuous triple $x = x(u, v)$, $y = y(u, v)$, $z = z(u, v)$ with (u, v) in the unit square Q. Each division of Q into two rectangles R_1 and R_2 by a line yields a decomposition of S into two surfaces S_1 and S_2. This paper provides a direct proof for the theorem that, if the continuous triple carries the dividing line into a single point, then the Lebesgue area is additive:

$L(S)=L(S_1)+L(S_2)$ (Rado and Reichelderfer, *On a stretching process for surfaces*, Amer. J. Math. vol. 61 (1939)). (Received November 28, 1942.)

57. Antoni Zygmund: *A property of the zeros of Legendre polynomials.*

Suppose that $n<m$ are positive integers and that a polynomial $\phi(x)$ of degree n does not exceed M in absolute value at the zeros of the Legendre polynomial $P_m(x)$. Then $|\phi(x)| \leqq A(\delta)M$ for $-1\leqq x\leqq +1$, where $A(\delta)$ depends only on the number δ defined by the equations $m/n=1+\delta$. Similar results hold for the integrals $\int_{-1}^{+1}|\phi(x)|^r dx$ with $r\geqq 1$. (Received November 23, 1942.)

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58. Stefan Bergman: *A formula for the stream function of compressible fluid flow.*

Let $\mathfrak{q}=ve^{i\theta}$ denote the velocity vector. A flow, $\mathcal{J}$ is said to be of the type D_n, if the boundary of the domain, B, in which $\mathcal{J}$ is defined consists of $2n$ segments S_K such that along each S_{2K}, $K=1, 2, \cdots, n$, $\theta=\theta_K$ is constant and along each S_{2K-1}, v is constant. (S_{2K} are segments of straight lines, S_{2K-1} are so-called "free boundaries.") The image of B in the logarithmic plane (see *Notes on hodograph method in the theory of compressible fluid*, publication of Brown University, p. 6) is a polygonal domain. In the case of an incompressible fluid the stream function of $\mathcal{J}$ can be represented as a closed expression with n parameters. The author considers subsonic flows, $\mathcal{C}$, of compressible fluid. Using certain linear operators (see above mentioned *Notes*, §§6 and 10, and Trans. Amer. Math. Soc. vol. 53 (1943) pp. 130–155) he derives a similar explicit formula with n parameters for the flows $\mathcal{C}$ of "nearly type D_n," that is to say, for flows whose boundaries consist of $2n$ segments along which θ or v assume nearly constant values. The angles θ_K may be prescribed. (Received November 21, 1942.)

59. R. M. Foster: *On the average resistance of an electrical network.*

In an electrical network composed of two-terminal resistance elements, let J_i designate the total resistance measured across the terminals of an element, the internal resistance of this element being r_i, and let S_i be the driving-point resistance measured in the branch containing this element r_i. It is shown in this paper that, for any network configuration whatsoever, $\sum J_i/r_i=R$ and $\sum r_iS_i=N$ (the summation being extended over all the elements of the network), with $R=V-P$ and $N=E-V+P$, where E is the number of elements, V the number of vertices, and P the number of separate, unconnected parts of the configuration. The average values of the ratios J_i/r_i and r_i/S_i are thus R/E and N/E, respectively. If all the elements of the network have the same internal resistance r, and if there is complete symmetry among the elements so that the resistance measured across any one element is necessarily equal to that across any other element, then $J_i=rR/E$ and $S_i=rE/N$. These results are extended to generalized impedances, and to infinite networks. (Received November 23, 1942.)

60. A. H. Fox: *Integral representation of the flow of a compressible fluid around a cylinder.*

The steady irrotational two-dimensional flow of a compressible fluid may be approximated by the flow of a hypothetical incompressible fluid in which the pressure is