AN ARITHMETICAL IDENTITY FOR THE FORM $ab-c^2$

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1. **Introduction.** The number of solutions in positive integers of the equation $n=xy+yz+zx$, n a positive integer, has been investigated Liouville,[1] Bell,[2] and Mordell.[3] Mordell, who was the first to obtain complete results, gave a strictly arithmetical treatment, while Bell made use of formulae which he obtained by paraphrasing theta-function identities. Although the latter considered only the case of n prime, his methods were extended later to the general case.[4]

Making use of other formulae derived by the method of paraphrase, Bell[5] has also solved the problem of representations in the forms $xy+yz+2zx$, $xy+2yz+2zx$. As he has pointed out, a feature of the method is the handling of the two forms simultaneously.

In this paper we derive by elementary methods a simple identity which on specialization not only yields complete results for representations of n in the forms

$$xy + yz + zx, \qquad xy + 2yz + 2zx, \qquad xy + yz + 2zx,$$

but as in Bell's paper,[5] handles the latter two forms at the same time.

2. **Fundamental identity.** Let $f(a, b, c)$ be a function, uniform and finite for all integer triples (a, b, c), but otherwise (so far) completely arbitrary. If the summation sign refers to the sum over all those integer solutions (a, b, c) of $n=ab-c^2$ subject to the restrictions indicated under it, we then have

$$\sum_{a,b>c>0} f(a, b, c) = \sum_{a>b>c>0} f(a, b, c) + \sum_{b>a>c>0} f(a, b, c) + \sum_{a=b>c>0} f(a, b, c). \tag{1}$$

Imposing on $f(a, b, c)$ the condition

$$f(a, b, c) = f(b, a, c) \tag{2}$$

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[1] Journal de Mathématiques, (2), vol. 7 (1862), p. 44.

[2] E. T. Bell, *Class numbers and the form* $yz+zx+xy$, Tôhoku Mathematical Journal, vol. 19 (1921), pp. 105–116.

[3] L. J. Mordell, *On the number of solutions in positive integers of the equation* $yz+zx+xy=n$, American Journal of Mathematics, vol. 45 (1923), pp. 1–4.

[4] W. H. Gage, *Representations in the form* $xy+yz+zx$, American Journal of Mathematics, vol. 51 (1929), pp. 345–348.

[5] E. T. Bell, *Numbers of representations of integers in a certain triad of ternary quadratic forms*, Transactions of this Society, vol. 32 (1930), pp. 1–5.

we may write

$$\sum_{a,b>c>0} f(a, b, c) = 2 \sum_{a>b>c>0} f(a, b, c) + \sum_{a=b>c>0} f(a, b, c)$$

$$(3) \qquad = 2\left[\sum_{a>b>2c>0} f(a, b, c) + \sum_{a>b>c>0,\, b<2c} f(a, b, c) + \sum_{a>b=2c>0} f(a, b, c)\right] + \sum_{a=b>c>0} f(a, b, c).$$

Note that $n=ab-c^2=a(a+b-2c)-(a-c)^2$, so that if we replace (a, b, c) by $(a+b-2c, a, a-c)$ in the second summation of the right member of (3) the conditions $a>b>c>0$, $b<2c$ become $a, b>2c>0$. Hence, if $f(a, b, c)$ satisfies (2), we have

$$\sum_{a,b>c>0} f(a, b, c) = 2 \sum_{a>b>2c>0} f(a, b, c) + 2 \sum_{a,b>2c>0} f(a + b - 2c, a, a - c)$$

$$(4) \qquad + 2 \sum_{d\equiv\delta\,(\mathrm{mod}\,2),\, d>3\delta>0} f\left(\frac{d+\delta}{2}, 2\delta, \delta\right) + \sum_{d\equiv\delta\,(\mathrm{mod}\,2),\, d>\delta>0} f\left(\frac{d+\delta}{2}, \frac{d+\delta}{2}, \frac{d-\delta}{2}\right)$$

where the last two summations refer to all integer solutions (d, δ) of $n=d\delta$ subject to the given conditions.

3. **Specialization.** In (4) let $f(a, b, c)=1$, and in the left member replace (a, b, c) by $(x+z, y+z, z)$. Then, if N denotes the number of integer representations of n in the form stated after it, we get

$$\begin{aligned} N(n &= xy + yz + zx;\ x, y, z > 0) \\ &= 6N(n = ab - c^2;\ a > b > 2c > 0) \\ (5) \qquad &\quad + N(n = d\delta;\ d \equiv \delta \ (\mathrm{mod}\ 2),\ d > \delta > 0) \\ &\quad + 2N(n = d\delta;\ d \equiv \delta \ (\mathrm{mod}\ 2),\ d > 3\delta > 0) \\ &\quad + 2N(n = d\delta;\ d \equiv \delta (\mathrm{mod}\ 2),\ 3\delta > d > \delta > 0). \end{aligned}$$

Let $G(n)$ denote the number of classes of binary quadratic forms of determinant $-n$, subject to all the conventions of H. J. S. Smith's *Report on the Theory of Numbers*.[6]

[6] *Mathematical Papers*, vol. 1. See also Dickson's *History of the Theory of Numbers*, vol. 3, pp. 108–109.

$$
\begin{aligned}
G(n) = {} & N(n = ab - c^2;\ a > b > 2|c| > 0) \\
& + N(n = ab - c^2;\ a > b > 0,\ c = 0) \\
& + N(n = ab - c^2;\ a = b > 2c > 0) \\
& + N(n = ab - c^2;\ a > b = 2c > 0) \\
& + (1/2)N(n = ab - c^2;\ a = b > 0,\ c = 0) \\
& + (1/3)N(n = ab - c^2;\ a = b = 2c > 0).
\end{aligned}
$$

Hence we have

$$
\begin{aligned}
& 2N(n = ab - c^2;\ a > b > 2c > 0) \\
& \quad = G(n) - N(n = d\delta; d > \delta > 0) - (1/2)N(n = d^2; d > 0) \\
& \quad\quad - N(n = d\delta;\ d \equiv \delta \pmod 2,\ d > 3\delta > 0) \\
& \quad\quad - (1/3)N(n = 3d^2;\ d > 0) \\
& \quad\quad - N(n = d\delta;\ d \equiv \delta \pmod 2,\ 3\delta > d > \delta > 0).
\end{aligned} \tag{6}
$$

From (5) it therefore follows that

$$
N(n = xy + yz + zx;\ x, y, z > 0) = 3\left[G(n) - (1/2)\zeta(n)\right], \tag{7}
$$

where $\zeta(n)$ is the number of divisors of n.

On choosing $f(a, b, c) = 1$ if a, b are not both even, $f(a, b, c) = 0$ if $a \equiv b \equiv 0 \pmod 2$, we likewise obtain the result

$$
\begin{aligned}
& N(n = 2xy + 2yz + 4zx;\ x, y, z > 0,\ y \equiv 1 \pmod 2) \\
& \quad + N(n = xy + 2yz + 2zx;\ x, y, z > 0,\ x \equiv y \equiv 1 \pmod 2) \\
& \quad = F(n) - (1/2)\zeta'(n) - (1/2)\zeta'(n/2),
\end{aligned} \tag{8}
$$

where $F(n)$ is the number of uneven classes of binary quadratic forms of determinant $-n$, $\zeta'(n)$ is the number of odd divisors of n, and $\zeta'(n/2) = 0$ for n odd.

If, in (8), we replace n by $2^{\alpha+1}m$, $2m$, m, respectively, $m \equiv 1 \pmod 2$, we get Bell's theorems[5] 1, 2, 3 from which the complete results for $xy + yz + 2zx$, $xy + 2yz + 2zx$ follow.

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