Tables of the Moments of Inertia and Section Modulus of Ordinary Angles, Channels and Bulb Angles with Certain Plate Combinations. New York, Works Projects Administration, 1941. 13+197 pp. $1.25.

These tables are a result of a suggestion of the U. S. Bureau of Marine Inspection and Navigation, which is now a part of the Navy Department. The most important entries in these tables are the moments of inertia and the section moduli of rectangular areas with adjacent L-, Γ- or C-shaped areas (called, respectively, "angles" for the L- and Γ-shape and "channels" for the C-shape, the word "bulb angle" being reserved for an L-shaped area with a lateral circular swelling at the top). Both moments of inertia and section moduli are taken with respect to the neutral axis of the whole area parallel to the larger side of the rectangle. These entries were calculated by means of the parallel axis theorem, and the tables of the U. S. Steel Corporation were used for the moments of inertia and for the position of the centroid of the "angles" and "channels." The dimensions of the areas (or "sections" as they are technically called) which we find in these tables correspond to the commercial types, and—as one would expect—the tabular entries are not appropriate for difference checking. However, the accuracy of the values was insured through independent computing by two groups of workers, and was checked by graphing the entries. The calculations are carried out to two decimal places for small entries and are limited to integer digits for large entries (the units being in.^{4} and in.^{3}). The arrangement of the tables is simple and handy, and the photo-offset reproduction gives them a very clear appearance. This publication will certainly be appreciated by many engineers.

I. OPATOWSKI

The Mathematical Papers of Sir William Rowan Hamilton. Vol. II. *Dynamics.* Edited for the Royal Irish Academy by A. W. Conway and A. J. McConnell. (Cunningham Memoir, no. 14.) Cambridge University Press; New York, Macmillan, 1940. 15+656 pp.

This volume contains Hamilton's work on dynamics and some investigations in optics. The period covered is, with two or three minor exceptions, that from 1833 to 1839. Except for a few short abstracts of papers read by Hamilton, most of the material presented has never before been published, and consists in the main, of a transcript of Hamilton's original work books and some scientific correspondence. The editors have added a few footnotes and some short explanatory material at the end of the volume.

The material is divided into three parts. A short introduction sketches the plan of the work and the nature of the problems investigated. The two hundred and eighty pages of Part I are devoted to dynamics and correspondence with Lubbock. Part II, consisting of some hundred pages, is devoted to the calculus of principal relations. Part III concerns itself with optical investigations and correspondence, and comprises some hundred and seventy pages. Also included in this volume are a thirty-two page appendix, contributed by the editors, and a very complete index.

The underlying idea in Part I is the determination of a single function whose partial derivatives will yield complete information about the trajectories. Hamilton uses at first the action function $\int 2Tdt$ and applies it to the three body problem. In order to obviate the bothersome transformations that arise in considerations of the time, he perceives the advantage of using the principal function $S=\int Ldt$ and indicates other alternative functions. It is in these investigations that the notion of varying action, the so-called canonical equations, and the Hamilton-Jacobi equation first come in as observations on the special dynamical situations treated, though, curiously enough, Hamilton makes no use of the canonical equations.

However, Hamilton requires that S satisfy a system of two partial differential equations, namely, the usual Hamilton-Jacobi equation and the corresponding equation where the differentiating variables are the initial coordinates and initial time. Now, at least when the force potential is independent of the time, the second equation is automatically satisfied whenever the first is, and Hamilton's lack of realization of this fact, quite obviously prevents him from making full application of the method. It is of course well known that it was only with Jacobi's recognition of this defect that the Hamiltonian method established its fundamental significance. (Apart from methodological novelty, Hamilton's dynamical and astronomical applications succeed only in verifying previously known computations.) Nevertheless in special cases Hamilton obtains an explicit principal function S and indicates an interesting method for this purpose. A complete integral $V(\cdots x_i \cdots ; \cdots a_i \cdots , t)$ of the Hamilton-Jacobi equation is a principal function if

$$V(\cdots \overset{0}{x_i} \cdots ; \cdots a_i \cdots ; t_0) = 0 \tag{1a}$$

and

$$L_{t\to t_0} \frac{V}{t-t_0} = L_{t\to t_0}\left(L\left(\cdots a_i \cdots , \cdots \frac{x_i - a_i}{t-t_0} \cdots ; t_0\right)\right) \tag{1b}$$

$$\text{essentially } L_{t\to t_0}\left(\frac{d}{dt}V - L\right) = 0.$$

Hence suppose

$$V = \sum V_i \tag{2}$$

where V is homogeneous of degree i in $x_i - a_i$, $t - t_0$. We may substitute (2) in the H.J. equation and expand $H(\cdots x_i \cdots, \partial V/\partial x_i \cdots, t)$ about x_i^0, t_0. On equating the linear and the quadratic terms, and so on, separately to zero, there results an infinite system of equations. The equation corresponding to terms of order n contains only V_j, $i \leqq j \leqq n$. On making use of (1) it is not difficult to show that the succeeding V_n's may be calculated step by step in terms of V_1. There are also some developments of perturbation and approximation methods for the determination of S.

The most ambitious appendix, quoted from a paper published by the editors, is devoted to the determination of the principal function from the knowledge of a complete integral of the H.J. equation. This is actually of slight importance, however, for it is well known that *any* complete integral determines the trajectories. Indeed (1a) suggests writing $S = V - V_0$, $V_0 = V(\cdots x_i^0 \cdots, \cdots a_i \cdots, t_0)$, and then the modification of the usual procedure replacing $b_i = \partial V/\partial a_i$ by $\partial V_0/\partial a_i = \partial V/\partial a_i$ for the elimination of the a_i's yields the principal function $S(\cdots x_i \cdots, \cdots x_i^0 \cdots, t, t_0)$.

Also in Part I is an ingenious derivation of the angular dependence of the resultant of two forces. The problem is reduced to the determination of an additive function f on the real numbers. Hamilton's argument to show $f(x)$ is linear and actually x overlooks the Hamel functions, but can readily be made rigorous.

The extensions of the dynamical methods, utilizing a principal function or relation, to arbitrary first and higher order partial differential and Mongean equations and systems, constitute the calculus of principal relations taken up in Part II. The following is typical. Consider

$$f_\sigma\left(x_1, \cdots \frac{d^{k_i}x^i}{dt^{k_i}}; \cdots; x_i \cdots \frac{d^{k_i}x_i}{dt^{k_i}}, \cdots; \cdots\right) = 0, \tag{3}$$
$$\sigma = 1, \cdots m;\ i = 1, \cdots n.$$

We associate with (3) the auxiliary equations of the calculus of variations. For instance with $\sigma = 1$, $k_i = k$ we have for a suitable choice of λ

$$(4)\qquad \lambda\frac{\partial f}{\partial x_i}-\frac{d}{dt}\lambda\frac{\partial f}{\partial \dot{x}_i}+\cdots+(-1)^k\frac{d^k}{dt^k}\lambda\frac{\partial f}{\partial x_i^k}=0,\qquad x_i^k=\frac{d^kx_i}{dt^k}\cdot$$

Consider now

$$(5)\qquad 0=\int_{t_0}^{t}\left(\delta f-\frac{df}{dt}\delta t\right)dt=I.$$

In view of (4) we may express the right side of (5) as a sum of terms of the form

$$(6)\qquad \left.\left\{(-1)^\alpha\frac{d^\alpha}{dt^\alpha}\left(\lambda\frac{\partial f}{\partial x_i^{\beta+1}}\right)\frac{d^{\beta-\alpha}}{dt^{\beta-\alpha}}(\delta x_i-\dot{x}_i\delta t)\right\}\right|_{t_0}^{t}.$$

If $F(\cdots;x_i\cdots d^{k-1}x_i/dt^{k-1};\ \cdots;\ \cdots;a_i\cdots a_i^{(k-1)};\ \cdots t,t_0)=0$ is the principal relation, then $\delta F=0$ is a consequence of (5) and hence

$$(7)\qquad \delta F=\mu I=0,$$

the central relation of this calculus. On equating the coefficients of the independent variations in (7) to 0 making use of (4), there arise auxiliary equations denoted by (8) for the partial derivatives of F with respect to $x_i^{(j)}$ and $x_i^{0,(j)}$ involving λ and μ also. When $k=1$ elimination of λ, μ, $\dot{x}_i$, $\dot{x}_i^0$ from these equations and (1) yields a differential equation for F and the equations (8) lead essentially to the bicharacteristics. For $k>1$ Hamilton indicates merely how the principal relation may be obtained. Thus to the equations (4) may be added a set of mn further equations obtained by replacing the left side of (4) by ith derivatives ($i=1,\cdots,n$) with respect to t. Then by successive eliminations we arrive at a single relation of the form specified above. Hamilton apparently did not attempt to determine the partial differential equation satisfied by F for $k>1$ and indeed the utility of the principal relation for higher order partial and Mongean equations is somewhat obscure. It would seem of interest to pursue these investigations.

The title assigned by the editors to Hamilton's treatment of higher order partial differential equations implies use of the principal relation but this is not clear from the text. He is led to relations obviously connected with the characteristic and bicharacteristic equations usually ascribed to Cauchy, Backlund, Goursat and Beudon and there seems to be some contact with Hamburger's investigations. However apart from the first order systems the results are incomplete and no account is taken of the elementary distinction between the elliptic, hyperbolic and parabolic cases.

The material on optics is perhaps of greatest interest. The propagation of disturbances may be taken up from two viewpoints. The first considers a compatible wave train already existing in the entire medium. Hamilton's interest is in the second viewpoint, namely that of the spreading of an originally localized disturbance. The typical problem is that of a collection of equi-spaced particles arranged over the entire x-axis and it is supposed that each particle acts only on its two adjacent neighbors.

The mathematical equivalent is the equation in mixed differences

$$y_{tt} - a^2 \frac{\Delta^2}{1+\Delta} y = 0, \qquad \Delta y(x, t) = y(x+1, t) - y(x, t)$$

subject to the initial conditions

$$y(x, 0) = 0 \text{ or } \eta(1 - \cos 2\nu x)$$
$$y_t(x, 0) = 0 \text{ or } -2a\eta \sin \nu \sin 2\nu x$$

for $x<0$ and $x \geqq 0$, respectively. If instead $y(x, 0)$ and $y_t(x, 0)$ vanish identically outside a finite interval, the conclusions are essentially unaffected. In treating these problems Hamilton makes incidental use of tools of modern appearance. For instance he uses the Fourier integral with the Sommerfeld weighting factor, the Abel summability of the Laplace integral, the Heaviside operator p^{-1}, and various operational manipulations, the Riemann-Lebesgue lemma, the method of stationary phase in conjunction with repeated use of the Dirichlet discontinuous integral (and incidentally obtains the leading term in the asymptotic expansion of the Bessel's function). If a, η are 1, then the nub of the analysis is the investigation of the properties of

$$\frac{1}{2\pi}(\sin \nu)^2 \int_0^\pi \frac{\sin(2x\theta - 2t\sin\theta)}{\sin\theta(\cos\theta - \cos\nu)} d\theta.$$

Assuming t large and positive, the results may be roughly summarized as follows. Let M and N be certain sufficiently large constants. Then for $x \geqq t+(1/2)Mt^{1/3}$, y is sensibly 0. As x decreases y takes on a pure displacement whose value is $(1/3)/(\cos \nu/2)^2$ for $x=t$ and rises to $(\cos \nu/2)^2$ for I: $t\cos \nu+(1/2)N(tsin\nu)^{1/2} \leqq x \leqq t-(1/2)M_1t^{1/3}$. The displacement of amount $(1/3)(\cos \nu)^2$ travels with unit velocity. As x decreases still further an oscillatory disturbance sets in. Thus in the range II: $-t+(1/2)Mt^{1/3} \leqq x \leqq t\cos \nu-(1/2)N(t\sin \nu)^{1/2}$, $y=(1-\cos \phi-(\sin \nu/2)^2$, $\phi=2\nu x-2t \sin \nu$. This is a copy of the initial disturbance shifted to a new mean position. In the transition region linking I and II the disturbance is a shifted, uniformly attenuated copy of the original. With further decrease in x, the effect is merely to change the mean position of the vibrations and for III: $x \leqq -t-(1/2)Mt^{1/3}$, $y=\cos \phi$. Hamilton adopts the view that it is the region of oscillatory disturbance that is significant. He chooses the value $x=t \cos \nu$ in the transition region T between I and II for the wave front. Here $y=(1/2)(1-\cos \phi+\cos \nu)$ and $\cos \nu$ is a dispersive (group) velocity. This is not altogether capricious, for with increasing t the velocity throughout T approaches $\cos \nu$, and the ratio of the size of T to the time approaches zero.

It has not been realized that the modern work on the transient phenomena in the propagation of waves in continuous media by Sommerfeld, Brillouin, and Colby is anticipated in these researches. (Even the reference to the published abstract of Hamilton's work, in Havelock's well known Cambridge tract, overlooks this fact.) There is moreover a marked parallelism in the methods, the stationary phase calculations being superseded by the more delicate saddle point evaluations. Actually, the adoption of a discrete rather than a continuous medium has certain advantages in indicating more intuitively the nature and genesis of the conclusions. Hamilton extends his work to two-dimensional and three-dimensional problems using the now familiar Cauchy-Fourier integral methods, and gives the explicit general formula for the group velocity.

The editors have obviously presented a most felicitous and careful rendering of Hamilton's work. However, it would seem desirable in collections of hitherto unpublished material to give a succinct summary of the *details* of (a) methods and (b) results in modern terminology and, wherever possible, (c) some indication of the present status of the field covered. The editors have made an attempt in this direction for (a) and (b), particularly as regards the dynamical papers, but the appendices and footnotes, could, with profit, be expanded considerably. The viewpoint and incomplete character of many of the investigations may well attract the attention of mathematicians ordinarily unconcerned with the historical development of mathematics.

D. G. BOURGIN