CONNECTED AND DISCONNECTED PLANE SETS AND THE FUNCTIONAL EQUATION $f(x)+f(y)=f(x+y)$

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Cauchy discovered before 1821 that a function satisfying the equation

$$f(x)+f(y)=f(x+y)$$

is either continuous or totally discontinuous.[1] After Hamel showed the existence of a discontinuous function satisfying the equation,[2] many mathematicians have concerned themselves with problems arising from the study of such functions.[3] However the following question seems to have gone unanswered: Since the plane image of such a function (the graph of $y=f(x)$) must either be connected or be totally disconnected, must the function be continuous if its image is connected? The answer is *no*.[4] The utility of this answer is at once apparent. For if $f(x)$ is totally discontinuous, its image obviously contains neither a continuum nor (in view of Darboux's work) a bounded connected subset even if the image itself is connected. As a matter of fact, if $f(x)$ is discontinuous but its image is connected, then the image, its complement, or some simple modification thereof, serves to illustrate rather easily many of the strange and non-intuitive properties of connected sets now illustrated by numerous complicated examples scattered through the literature. Thus this class of sets is a useful tool in studying connectedness and disconnectedness. A few illustrations are given, particularly in connection with

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[1] *Cours d'Analyse de l'École Royale Polytechnique*, part 1, *Analyse Algébrique*, 1921. This is Volume 3 of the 2d Series of Cauchy's *Complete Works* published by Gauthier-Villars et Fils, Paris, 1897, p. 99. Darboux in his paper, *Sur la composition des forces en statique*, Bulletin des Sciences Mathématiques, vol. 9 (1875), p. 281, showed (using Cauchy's methods) that if $f(x)$ is bounded in some interval, then $f(x)$ is continuous and of the form Ax.

[2] G. Hamel, *Eine Basis aller Zahlen und die unstetigen Lösungen der Funktionalgleichung*: $f(x+y)=f(x)+f(y)$, Mathematische Annalen, vol. 60 (1905), pp. 459–462.

[3] See in particular the early volumes of Fundamenta Mathematicae.

[4] It is odd that Sierpinski overlooked this, since about the time he published his papers on this subject he also published in Volume 1 of Fundamenta Mathematicae an example of a connected punctiform subset of the plane. And at this time he raised with Mazurkiewicz the question of the existence in the plane of a connected set containing no bounded connected subset.

linearly ordered metric spaces. However, such sets are of no use in connection with inner limiting sets.[5]

Convention. Throughout this paper f is used to denote a single-valued real function of a variable, whose range is the set of all real numbers, such that if x and y are real numbers (distinct or not) then $f(x)+f(y)=f(x+y)$. The graph of the equation $y=f(x)$ in a cartesian plane E will be denoted by I_f and called the *image* of f (in E). A *vertical line* in E will be understood to mean the graph of an equation of the form $x=a$, where a is a real constant.

1. **Preliminary theorems.** The following two properties are easily established:[6] (1) $f(rx)=rf(x)$ if r is zero or rational (positive or negative) and (2) if three vertices of a parallelogram in E belong to I_f, then the fourth vertex also belongs to I_f.

THEOREM 0. *For each f, I_f is either connected or totally disconnected.*

THEOREM 1. *If f is discontinuous, then I_f is dense in E.*

THEOREM 2. *Suppose that f is discontinuous. In order for I_f to be connected, it is necessary and sufficient that I_f intersect every continuum in E not lying wholly in a vertical line.*

PROOF. Suppose that I_f is connected and that M is a continuum in E not lying wholly in a vertical line. Then M contains a compact subcontinuum M_1 containing two points P_1 and P_2 which lie in distinct vertical lines. Let D denote the connected domain of E lying between these two vertical lines. It follows from Theorem 1 that if M_1 contains a domain, then I_f contains a point of M_1. On the other hand, if M_1 contains no domain, then $D-D\cdot M_1$ has more than one component. Again by Theorem 1, the segment of I_f between (but not including) P_1 and P_2 contains a point of every component of $D-D\cdot M_1$ and hence contains a point of M_1. Therefore the condition is necessary.

To see that the condition is sufficient, suppose that I_f is the sum of two mutually separate sets H and K. Let D denote a component of $E-\overline{K}$, let B denote a point of K, and let ω denote the point at infinity. The outer boundary (in $E+\omega$) of D with respect to B is a compact

[5] For constructing connected inner limiting sets see Theorem 118 on p. 309 of R. L. Moore's *Foundations of Point Set Theory*, American Mathematical Society Colloquium Publications, vol. 13, New York, 1932, and Theorem 3 of E. W. Miller's *Some theorems on continua*, this Bulletin, vol. 46 (1940), pp. 150–157. Moore has inadvertently omitted the stipulation that the elements of the postulated collection be mutually exclusive.

[6] See the works referred to in Footnote 2.

continuum M lying in $E+\omega$ and containing no point of I_f.[7] Hence $M-\omega$ is a continuum[8] in E which separates D from B in E. Consequently either $M-\omega$ is an entire vertical line or $M-\omega$ is not a subset of a vertical line. In either case $M-\omega$ contains a point of I_f, which is a contradiction.

THEOREM 3. *There exists a function f such that f is discontinuous and I_f is not connected.*

PROOF. Let $\alpha, \beta, \gamma, \cdots$ denote a Hamel basis for the real numbers.[9] Hence every real number x can be expressed *uniquely* in the form $x=a\alpha+b\beta+c\gamma+\cdots$ where the numbers $a, b, c, \cdots$ are either zero or rational and at most a finite number of them are different from zero. Hamel has shown that f may be arbitrarily defined for each of the numbers $\alpha, \beta, \gamma, \cdots$ provided that if $x=a\alpha+b\beta+c\gamma+\cdots$ then $f(x)=af(\alpha)+bf(\beta)+cf(\gamma)+\cdots$. So let $f(\alpha)=1$ and $f(\beta)=f(\gamma)=f(\delta)=\cdots=0$. Since $a, b, c, \cdots$ are rational, it follows that for each real number x, $f(x)$ is either rational or zero. Hence I_f is totally disconnected.

THEOREM 4. *There exists a function f such that I_f intersects every perfect set in E not lying in the sum of a countable collection of vertical lines.*

PROOF. Since the collection of all perfect sets in E not lying in the sum of a countable collection of vertical lines is of power c (the power of the continuum),[10] there exists a well ordering Γ of this collection such that the number of elements of Γ preceding an element of Γ is less than c. Let (x_1, y_1) denote a point of the first element of Γ such that $x_1\neq 0$. Define $f(x_1)$ to be y_1; and if $x=r_1x_1$, where r_1 is zero or rational, define $f(x)$ to be $r_1f(x_1)$. Each element of Γ must contain points of c distinct vertical lines, and $f(x)$ is so far defined for less than c values of x. So let (x_2, y_2) denote a point of the second element of Γ such that $x_2\neq 0$ and $f(x_2)$ is not defined. Define $f(x_2)$ to be y_2; and if $x=r_1x_1+r_2x_2$, where r_1 and r_2 are zero or rational, define $f(x)$ to be $r_1f(x_1)+r_2f(x_2)$. In general, this process may be continued c times this way: If γ is an element of Γ such that $f(x)$ has been explicitly defined (as already indicated) by some point of each set of Γ preceding γ in Γ,

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[7] See p. 193, Theorem 23, of R. L. Moore's *Foundations of Point Set Theory*, loc. cit.

[8] Ibid., p. 195, Theorem 25.

[9] G. Hamel, loc. cit.

[10] Sierpinski's *Introduction to General Topology*, translated by C. Cecilia Krieger, The University of Toronto Press, Toronto, 1934, p. 63.

then let (x_γ, y_γ) denote a point of γ such that $x_\gamma \neq 0$ and $f(x_\gamma)$ has not been defined. Define $f(x_\gamma)$ to be y_γ and if $x = r_1x_1 + r_2x_2 + \cdots$, where not more than a finite number of the rational numbers $r_1, r_2, r_3, \cdots$ are different from zero, define $f(x)$ to be $r_1f(x_1) + r_2f(x_2) + \cdots$. By arguments similar to those of Hamel, it may be shown that this process defines a single-valued function f. Evidently I_f intersects every perfect set in E not lying in a countable number of vertical lines.

THEOREM 5. *There exists a function f such that f is discontinuous but I_f is connected.*[11]

Theorem 5 follows from Theorems 2 and 4.

THEOREM 6. *If f is discontinuous, I_f is punctiform.*[12]

PROOF. If I_f contained a nondegenerate continuum, then I_f would contain a bounded nondegenerate continuum and f would, therefore, be continuous for some value of x. But if f were continuous for some value of x, f would be continuous for *all* values of x, which is contrary to hypothesis.

THEOREM 7. *Suppose that G is a collection of subsets of E such that every vertical translation in E of a set of G produces a set which also belongs to G. If I_f contains a point of every element of G, then I_f does not contain an element of G.*

Theorem 7 may be easily proved by an indirect argument.

THEOREM 8. *If the subset M of E is punctiform, then $E-M$ is connected and locally connected.*[13]

2. **Properties of I_f when f is discontinuous but I_f is connected.** Let f be discontinuous, let I_f be connected, and for simplicity let I denote I_f. The following properties of I follow almost immediately from the preceding theorems and the elementary properties of f.

Notation. The symbol ω will be used to denote the point at infinity. If M is a point set and b is a real number, $M+b$ denotes the point set obtained by adding b to the ordinate of each point of M, the abscissa remaining unchanged.

[11] If f is discontinuous and I_f is connected, by Theorem 2, I_f must intersect every *continuum* not lying in the sum of a countable collection of vertical lines. But I_f need *not* intersect every *perfect set* not lying in the sum of a countable collection of vertical lines. I hope to include such an example in a paper pertaining more specifically to function theory and the Hamel basis.

[12] A set is said to be *punctiform* if it contains no nondegenerate continuum.

[13] See the argument on pp. 236 and 237 of Knaster and Kuratowski's, *Sur les ensembles connexes*, Fundamenta Mathematicae, vol. 2 (1921), pp. 206–255.

PROPERTY 1. *Both I and $E-I$ are connected and, hence, neither separates the plane.*

PROPERTY 2. (1) *I contains no nondegenerate continuum,*[14] *and* (2) *$E-I$ contains no continuum not lying in a vertical line.*

PROPERTY 3. *The set I contains no bounded (nondegenerate) connected subset.*[15]

PROPERTY 4. *Let I^+ denote the set of all points of I with positive ordinates. Although I^+ is totally disconnected and every quasi-component of I^+ is degenerate, I^+ is quasi-connected.*[16]

PROPERTY 5. *Let L denote a non-vertical line in E. Then $I-I\cdot L$ is totally disconnected and every quasi-component of $I-I\cdot L$ is degenerate but $(I-I\cdot L)+\omega$ is biconnected.*[17]

PROPERTY 6. *Let H denote an interval of I. Then H is punctiform, connected, and irreducible between its end points.*[18]

Remark. By Theorems 4 and 7, f exists so that I need not contain a perfect set. If this were the case, the sets $(I-I\cdot L)+\omega$ and H in Properties 5 and 6 respectively would contain no perfect subset of E or $E+\omega$.[19]

PROPERTY 7. *Let K denote $\sum(I+r)$, where r ranges (vertically) over the set of rational numbers. Then both K and $E-K$ are punctiform, connected and locally connected sets.*[20]

Remark. If, as is shown to be possible by Theorems 4 and 7, I contains no perfect set, it follows from Theorem 7 that the set K in

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[14] Cf. Sierpinski, *Sur un ensemble punctiforme connexe*, Fundamenta Mathematicae, vol. 1 (1920), pp. 7–10.

[15] Cf. Mazurkiewicz, *Sur l'existence d'un ensemble plan connexe ne contenant aucun sous-ensemble connexe, borné*, Fundamenta Mathematicae, vol. 2 (1921), pp. 96–103.

[16] Cf. §3 of Mazurkiewicz, *Sur les ensembles quasi-connexes*, Fundamenta Mathematicae, vol. 2 (1921), pp. 201–205.

[17] Cf. R. L. Wilder, *A point set which has no true quasi-components which becomes connected upon the addition of a single point*, this Bulletin, vol. 33 (1927), pp. 423–427. Cf. Example α of §5 of Knaster and Kuratowski, *Sur les ensembles connexes*, loc. cit.

[18] Cf., ibid., §5, Example β.

[19] Cf., ibid., §5, Examples γ and δ. For a function whose image is a punctiform, connected, inner limiting (G_δ) set, see page 306 of Kuratowski and Sierpinski, *Les fonctions de classe 1 et ensembles connexes punctiformes*, Fundamenta Mathematicae, vol. 3 (1922), pp. 303–313.

[20] Cf. R. L. Moore, *A connected and regular point set which contains no arc*, this Bulletin, vol. 32 (1926), pp. 331–332; R. L. Wilder, *A connected and regular point set which has no subcontinuum*, Transactions of this Society, vol. 29 (1927), pp. 332–340.

Property 7 contains no perfect set.[21] Furthermore, since the set of real numbers contains a subset R such that every perfect subset of the real numbers contains a number in R and a number not in R, it is clear that if I contains no perfect subset of E and the range of r is R (instead of the rational numbers), then the sets K and $E-K$ in Property 7 are both connected and locally connected but neither contains a perfect subset of E.[22]

PROPERTY 8. *Let S denote a space whose points are the points of I and in which "limit point" has the same meaning that it does in E. Then* (1) *S is metric, connected, convex, and separable, but contains no compact nondegenerate continuum*; (2) *S is linearly ordered and continuous with respect to this order*; (3) *S is the sum of countably many totally disconnected, arbitrarily small domains*; (4) *if F denotes the points of S belonging to a circle in E and M denotes the points of S which are on or inside this circle in E, then although M contains no nondegenerate quasi-component, M is not the sum of two nonvacuous mutually separate sets one of which contains F*; (5) *S contains a totally disconnected closed set of which not every point is a limit point of its complement.*

Property 8 gives rise to a number of questions. Particularly, is every point of a totally disconnected closed subset of a connected and linearly ordered *complete* metric space a limit point of its complement?

3. **Plane geometry.** If one defines a line (in the cartesian plane) to be the set of all points (x, y) satisfying an equation of either the form $x=a$ (where a is a constant) or the form $y=f(x)+mx+b$ (where m and b are constants which may be different for different lines, but f is the same function for all lines of this type and I_f is connected), then one gets a curious approximation to euclidean plane geometry. In this geometry translation would be a rigid motion but rotation would not. Also in this geometry a triangle would *cut* the plane but would not *separate* the plane.

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[21] Cf. Knaster and Kuratowski, *A connected and connected im kleinen point set which contains no perfect set*, this Bulletin, vol. 33 (1927), pp. 106–109.

[22] Cf. Mazurkiewicz, *Sur la décomposition d'un domaine en deux sous-ensembles punctiformes*, Fundamenta Mathematicae, vol. 3 (1922), pp. 65–75. It follows from Theorem 8 that Bernstein's (Berichte der Saechsischen Akademie der Wissenschaften, Leipzig, vol. 60 (1908), pp. 325–338) decomposition of the plane into two mutually exclusive sets such that every perfect set contains a point of each of them, as random as this process is, always produces two connected and locally connected sets. This fact is certainly known to many mathematicians and has probably been pointed out elsewhere.