ON REARRANGEMENTS OF SERIES[1]

RALPH PALMER AGNEW

1. **Introduction.** Let E denote the metric space in which a point x is a permutation $x_1, x_2, x_3, \cdots$ of the positive integers and the distance (x, y) between two points $x \equiv \{x_1, x_2, \cdots\}$ and $y \equiv \{y_1, y_2, \cdots\}$ of E is given by the Fréchet formula

$$(x, y) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{|x_n - y_n|}{1 + |x_n - y_n|}.$$

The space E is of the second category (Theorem 2).

Let $c_1 + c_2 + \cdots$ be a convergent series of real terms for which $\sum |c_n| = \infty$. To simplify typography, we write $c(n)$ for c_n. To each $x \in E$ corresponds a rearrangement $c(x_1) + c(x_2) + \cdots$ of the series $\sum c_n$. By a well known theorem of Riemann, $x \in E$ exists such that $c(x_1) + c(x_2) + \cdots$ converges to a preassigned number, diverges to $+\infty$ or to $-\infty$, or oscillates in a preassigned fashion.

The set A of $x \in E$ for which $c(x_1) + c(x_2) + \cdots$ converges is therefore a proper subset of E, and M. Kac has proposed the problem of determining whether $E - A$ is of the second category. The following theorem shows not only that A is of the first category (and hence that $E - A$ is of the second category) but also that the set of $x \in E$ for which the series $c(x_1) + c(x_2) + \cdots$ has unilaterally bounded partial sums is of the first category.

THEOREM 1. *For each $x \in E$ except those belonging to a set of the first category,*

$$\liminf_{N \to \infty} \sum_{n=1}^{N} c(x_n) = -\infty, \qquad \limsup_{N \to \infty} \sum_{n=1}^{N} c(x_n) = \infty.$$

2. **Proof of Theorem 1.** The fact that the "coordinates" x_n and y_n of two points x and y of E are integers implies roughly that, if N is large, then $x_n = y_n$ for $n = 1, 2, \cdots, N$ if and only if (x, y) is near 0. To make this precise, let $x \in E$, $r > 0$, and let $S(x, r)$ denote the set of points y such that $(x, y) < r$, so that $S(x, r)$ is an open sphere with center at x and radius r. It is easy to show that if x and y are two points of E such that $y \in S(x, 2^{-N-1})$ then $x_n = y_n$ when $n = 1, 2, \cdots, N$; and that if x and y are such that $x_n = y_n$ when $n = 1, 2, \cdots, N$ then $y \in S(x, 2^{-N})$.

[1] Presented to the Society, October 28, 1939.

To prove Theorem 1, let B denote the set of $x \varepsilon E$ for which

$$\limsup_{N\to\infty} \sum_{n=1}^{N} c(x_n) < \infty;$$

and, for each $h>0$, let B_h denote the set of $x \varepsilon E$ for which

$$\operatorname*{l.u.b.}_{N=1,2,3,\cdots} \sum_{n=1}^{N} c(x_n) < h.$$

Then

$$B = B_1 + B_2 + B_3 + \cdots.$$

We show that B is the first category by showing that B_h is nondense for each $h>0$. Suppose $h>0$ exists such that the closure $\overline{B}_h$ of B_h contains a sphere S' with center at $x' \equiv \{x_1', x_2', \cdots\}$ and radius $r>0$. Choose m so great that $2^{-m-1}+2^{-m-2}+\cdots<r/2$. Let $x_n''=x_n'$ when $1\leqq n\leqq m$; and define x_n'' for $n>m$ in such a way that $\sum c(x_n'')$ diverges to $+\infty$. Then $(x', x'')<r/2$ so that $x'' \varepsilon S'$. Choose an index q such that

$$c(x_1'') + c(x_2'') + \cdots + c(x_q'') > h,$$

and then choose $\delta>0$ such that $x_k=y_k$ for $k=1, 2, \cdots, q$ whenever $x, y \varepsilon E$ and $(x, y)<\delta$.

If x is a point within the sphere S'' with center at x'' and radius δ (that is, if $(x, x'')<\delta$), then $c(x_1)+c(x_2)+\cdots+c(x_q)>h$ and x is not in B_h. Thus B_h contains no point of S'' and consequently $\overline{B}_h$ does not contain x''. This contradicts the assumption that $\overline{B}_h$ contains S', and hence proves that B_h is nondense and B is of the first category. Similar considerations show that the set C of $x \varepsilon E$ for which $c(x_1)+\cdots+c(x_N)$ has inferior limit greater than $-\infty$ is of the first category. Since the union of two sets B and C of the first category is itself of the first category, Theorem 1 is established.

If $z_1+z_2+\cdots$ is a convergent series of complex terms for which $\sum|z_n|=\infty$, it is easy to apply our theorem to the series of real and imaginary parts of z_n to show that the set of $x \varepsilon E$ for which $z(x_1)+z(x_2)+\cdots$ has bounded partial sums is a set of the first category.

3. **The space** E. In this section we obtain some properties of E and prove the following result.

THEOREM 2. *The space E is of the second category at each of its points.*

That the space E is not complete was pointed out to the author by Professor L. M. Graves. In fact if $x^{(n)}$ is the point

$$x^{(n)} \equiv \{2, 3, \cdots, n-1, n, 1, n+1, n+2, \cdots\}$$

of E, then $x^{(n)}$ is a Cauchy sequence in E which does not converge to a point of E. If $\mathcal{E}$ is the space in which a point is a sequence of positive integers not necessarily a permutation of all positive integers, and the distance between two points of $\mathcal{E}$ is given by the Fréchet formula, then $\mathcal{E}$ is complete and E is a subspace of $\mathcal{E}$. It is easy to show that the closure of E in $\mathcal{E}$ is the space $\mathcal{E}_1$ in which a point is a sequence of positive integers containing each positive integer *at most once*, and hence that $\mathcal{E}_1$ is the least complete subspace of $\mathcal{E}$ which contains E. For example, $\{2, 4, 6, 8, \cdots\}$ is a point of $\mathcal{E}_1$ which is not a point of E.

If $\mathcal{E}_x\{x_n = k\}$ denotes, for each n, $k = 1, 2, \cdots$, the set of all $x \in \mathcal{E}$ for which $x_n = k$, then

$$\mathcal{E}_2 \equiv \prod_{k=1}^{\infty} \sum_{n=1}^{\infty} \mathcal{E}_x\{x_n = k\}$$

is the subset of $\mathcal{E}$ in which a point is a sequence containing each positive integer *at least once*. Since $\mathcal{E}_x\{x_n = k\}$ is an open subset of $\mathcal{E}$ for each n, $k = 1, 2, \cdots$, $\mathcal{E}_2$ is the intersection of a countable set of open sets (that is, $\mathcal{E}_2$ is a G_δ) in $\mathcal{E}$. Since $\mathcal{E}_1$ is a closed subset of $\mathcal{E}$ and $E = \mathcal{E}_1\mathcal{E}_2$, it follows that E is a G_δ in the complete space $\mathcal{E}$.

Therefore, by a fundamental theorem whose proof is an easy extension of the familiar proof that a complete metric space is of the second category, E is of the second category at each of its points and Theorem 2 is proved.

CORNELL UNIVERSITY