A REMARK ON THE SUM AND THE INTERSECTION OF TWO NORMAL IDEALS IN AN ALGEBRA

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Let F be a quotient field of a commutative domain of integrity o in which the usual arithmetic holds.[1] Consider an algebra $\mathfrak{A}$ with a unit element over F. Let $\mathfrak{J}_1, \mathfrak{J}_2, \mathfrak{J}_3, \mathfrak{J}_4$ be four arbitrary maximal orders in $\mathfrak{A}$ and $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}$ be three arbitrary normal ideals. We prove the following theorems.

THEOREM 1. *If* $\mathfrak{J}_1 \cap \mathfrak{J}_2 = \mathfrak{J}_3 \cap \mathfrak{J}_4$ [*or* $(\mathfrak{J}_1, \mathfrak{J}_2) = (\mathfrak{J}_3, \mathfrak{J}_4)$], *then either* $\mathfrak{J}_1 = \mathfrak{J}_3$, $\mathfrak{J}_2 = \mathfrak{J}_4$ *or* $\mathfrak{J}_1 = \mathfrak{J}_4$, $\mathfrak{J}_2 = \mathfrak{J}_3$.

THEOREM 2. *Both the left and the right orders of* $(\mathfrak{J}_1, \mathfrak{J}_2)$ *are* $\mathfrak{J}_1 \cap \mathfrak{J}_2$. *Also* $\mathfrak{J}_1 \cap \mathfrak{J}_2 \subseteqq \mathfrak{J}_3$ *if and only if* $(\mathfrak{J}_1, \mathfrak{J}_2) \supseteqq \mathfrak{J}_3$; *if this is the case the distance ideal* $\mathfrak{d}_{21}$ *of* $\mathfrak{J}_2$ *to* $\mathfrak{J}_1$ *is divisible by the distance ideal*[2] $\mathfrak{d}_{31}$ *of* $\mathfrak{J}_3$ *to* $\mathfrak{J}_1$.

THEOREM 3. *The left, say, order* $\mathfrak{o}$ *of the intersection* $\mathfrak{a} \cap \mathfrak{b}$ [*the sum* $(\mathfrak{a}, \mathfrak{b})$] *is an intersection of two suitable maximal orders.*

More precisely, if $\mathfrak{r}$ and $\mathfrak{s}$ are normal ideals such that $\mathfrak{b} = \mathfrak{r}\mathfrak{a}\mathfrak{s}$ in the sense of proper multiplication and if $\mathfrak{t}$ is the smallest two-sided ideal of the right order of $\mathfrak{a}$ which divides $\mathfrak{s}$ while $\mathfrak{t}'$ is the largest two-sided ideal of the same maximal order which is divisible by $\mathfrak{s}$, then $\mathfrak{o}$ is the intersection of the left orders of the two normal ideals $\mathfrak{a} \cap \mathfrak{r}\mathfrak{a}\mathfrak{t}$ and $\mathfrak{a} \cap \mathfrak{r}\mathfrak{a}\mathfrak{t}'$ [$(\mathfrak{a}, \mathfrak{r}\mathfrak{a}\mathfrak{t})$ and $(\mathfrak{a}, \mathfrak{r}\mathfrak{a}\mathfrak{t}')$].[3] *The left order of* $\mathfrak{a} \cap \mathfrak{b}$ *coincides with the right order of* $(\mathfrak{a}^{-1}, \mathfrak{b}^{-1})$.

THEOREM 4. $\mathfrak{a} \cap \mathfrak{b} \subseteqq \mathfrak{c}$ *implies* $(\mathfrak{a}^{-1}, \mathfrak{b}^{-1}) \supseteqq \mathfrak{c}^{-1}$ *and conversely.*

For the proof we have, according to the well known reduction, only to treat the case where F is a p-adic field $F = F_p$ and $\mathfrak{A}$ is a normal simple algebra over F. Then $\mathfrak{A}$ is a (complete) matric ring $D_r = \sum_{i,k=1}^{r} \epsilon_{ik} D$ over a division algebra D, where ϵ_{ik} is a system of matric units commutative with every element of D. D possesses a unique maximal order I, and I has a unique prime ideal P.

Notation. If a_{ik}, $(i, k = 1, 2, \cdots, r)$, is a system of rational integers, we denote by $M(a_{ik})$ the ideal $\sum_{i,k} \epsilon_{ik} P^{a_{ik}}$ in $\mathfrak{A}$.

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[1] In the following we shall adopt the terminologies used in M. Deuring, *Algebren*, Ergebnisse der Mathematik, vol. 4, no. 1, 1935.

[2] If the algebra is a quaternion algebra, then the converse is also valid. Cf. M. Eichler, Journal für die reine und angewandte Mathematik, vol. 174 (1936), §7.

[3] Thus the intersection and the sum are no more normal ideals except for trivial cases; cf. Nakayama, Proceedings of the Imperial Academy of Japan, vol. 12 (1936).

$M(a_{ik})$ is an order if and only if $a_{ii}=0$, $a_{il}+a_{lk}\geqq a_{ik}$ for all i, k, l. On assuming this condition it is a maximal order if and only if $\sum a_{ik}=0$. By a simple calculation we then have the following lemma.[4]

LEMMA 1. *A necessary and sufficient condition that $M(a_{ik})$ be a maximal order is that there should exist r rational integers c_i such that $a_{ik}=c_k-c_i$. Every normal ideal whose left and right orders are $M(c_k-c_i)$ and $M(d_k-d_i)$ respectively has the form $P^aM(d_k-c_i)=M(d_k-c_i+a)$.*

It follows from a lemma of Chevalley[5] that a maximal order in $\mathfrak{A}$ has really the form $M(a_{ik})$ (whence the form $M(c_k-c_i)$) whenever it contains all diagonal $\epsilon_{11}, \epsilon_{22}, \cdots, \epsilon_{rr}$.

LEMMA 2. *There exists a regular element α in $\mathfrak{A}$ such that*

$$\alpha^{-1}\mathfrak{J}_1\alpha = M(0), \quad \alpha^{-1}\mathfrak{J}_2\alpha = M(c_k - c_i); \qquad c_1 \geqq c_2 \geqq \cdots \geqq c_r.$$

PROOF. There is, as is well known, a regular element β such that $\beta^{-1}\mathfrak{J}_1\beta=M(0)$. Consider the distance ideal $\mathfrak{d}_{12}=(\mathfrak{J}_2\mathfrak{J}_1)^{-1}=\mathfrak{J}_1\delta$ of $\mathfrak{J}_1$ to $\mathfrak{J}_2$. The theory of elementary divisors tells the existence of two units ξ, η in $M(0)$ such that $\gamma=\xi\beta^{-1}\delta\beta\eta$ is a diagonal matrix with diagonal elements P^{c_i}, $(c_1\geqq \cdots \geqq c_r)$, $\gamma=\sum\epsilon_{ii}P^{c_i}$, where we denote, for the sake of convenience, a prime element of the prime ideal P by the same letter P. Put $\alpha=\beta\eta$. Then this α possesses the required property: $\alpha^{-1}\mathfrak{J}_1\alpha=\eta^{-1}\beta^{-1}\mathfrak{J}_1\beta\eta=M(0)$, $\alpha^{-1}\mathfrak{J}_2\alpha=\gamma^{-1}M(0)\gamma=M(c_k-c_i)$.

LEMMA 3. *There exist two regular elements α, β in $\mathfrak{A}$ such that*

$$\alpha\mathfrak{a}\beta = M(0), \qquad \alpha\mathfrak{b}\beta = M(d_k - c_i);$$
$$c_1 \geqq c_2 \geqq \cdots \geqq c_r, \; d_1 \geqq d_2 \geqq \cdots \geqq d_r.$$

PROOF. Let $\mathfrak{J}_1'$, $\mathfrak{J}_2'$ [$\mathfrak{J}_3'$, $\mathfrak{J}_4'$] be the left and the right orders of $\mathfrak{a}$ [$\mathfrak{b}$]. According to the above lemma there exist γ, β such that $\gamma^{-1}\mathfrak{J}_1'\gamma=\beta^{-1}\mathfrak{J}_2'\beta=M(0)$, $\gamma^{-1}\mathfrak{J}_3'\gamma=M(c_k-c_i)$, $\beta^{-1}\mathfrak{J}_4'\beta=M(d_k'-d_i')$. $\gamma^{-1}\mathfrak{a}\beta$ is a two-sided ideal of $M(0)$ and has a form $P^aM(0)$. Put $\alpha=(\gamma P^a)^{-1}$. Then $\alpha\mathfrak{a}\beta=M(0)$. Moreover, $\alpha\mathfrak{b}\beta$ is of a form $M(d_k'-c_i+b)$ (Lemma 1). We put $d_k=d_k'+b$, and this completes the proof.

We note further that the left order of an ideal $M(a_{ik})$ is $M(b_{ik})$ where $b_{ik}=\max_j\,(a_{ij}-a_{kj})$.

After these preliminaries our theorems are easy to prove. In Theorem 1 we may, according to Lemma 2, assume that $\mathfrak{J}_1=M(0)$, $\mathfrak{J}_2=M(c_k-c_i)$, $(c_1\geqq \cdots \geqq c_r)$. Suppose $\mathfrak{J}_1\cap\mathfrak{J}_2=\mathfrak{J}_3\cap\mathfrak{J}_4$. Since

[4] Cf. Nakayama, Japanese Journal of Mathematics, vol. 13 (1937), p. 339.

[5] Chevalley, Abhandlungen aus dem mathematischen Seminar der Hamburgischen Universität, vol. 10 (1934), p. 87.

$\epsilon_{ii} \in \mathfrak{J}_1 \cap \mathfrak{J}_2 \subseteq \mathfrak{J}_3, \mathfrak{J}_4$, it follows that $\mathfrak{J}_3, \mathfrak{J}_4$ have the form $\mathfrak{J}_3 = M(d_k - d_i)$, $\mathfrak{J}_4 = M(f_k - f_i)$. Moreover $\max (d_k - d_i, f_k - f_i) = \max (0, c_k - c_i)$, $(i, k = 1, 2, \cdots, r)$. This implies $\max (d_k - d_i, f_k - f_i) = 0$ if $i \geqq k$, whence $d_1 \geqq \cdots \geqq d_r$ and $f_1 \geqq \cdots \geqq f_r$. On applying the same relation to $i = 1$, $k = r$, we find that either $d_1 = d_r$ or $f_1 = f_r$. In the first case we have $d_1 = \cdots = d_r$, $f_1 - f_i = c_1 - c_i$, $(i = 1, 2, \cdots, r)$, whence $\mathfrak{J}_1 = \mathfrak{J}_3$, $\mathfrak{J}_2 = \mathfrak{J}_4$. The second case gives of course $\mathfrak{J}_1 = \mathfrak{J}_4$, $\mathfrak{J}_2 = \mathfrak{J}_3$.

The assertion about the sum follows now from Theorem 2, which is in turn contained in Theorems 3 and 4.

As to Theorem 3 we notice first that if α, β are two regular elements, the ideals $\alpha\mathfrak{r}\alpha^{-1}$, $\beta^{-1}\mathfrak{s}\beta$, $\beta^{-1}\mathfrak{t}\beta$, $\beta^{-1}\mathfrak{t}'\beta$ have the same significance for $\alpha\mathfrak{a}\beta$ and $\alpha\mathfrak{b}\beta$ as the ideals $\mathfrak{r}$, $\mathfrak{s}$, $\mathfrak{t}$, $\mathfrak{t}'$ have for $\mathfrak{a}$ and $\mathfrak{b}$. Hence it is sufficient, by Lemma 3, to consider the case where

$$(1)\qquad \begin{aligned} \mathfrak{a} = M(0), \qquad \mathfrak{b} = M(d_k - c_i);\\ c_1 \geqq c_2 \geqq \cdots \geqq c_r, \ d_1 \geqq d_2 \geqq \cdots \geqq d_r. \end{aligned}$$

Then $\mathfrak{a} \cap \mathfrak{b} = M(\max (0, d_k - c_i))$ and $\mathfrak{o} = M(a_{ik})$ with

$$\begin{aligned} a_{ik} &= \max_j (\max (0, d_j - c_i) - \max (0, d_j - c_k)) \\ &= \begin{cases} \max (0, d_1 - c_i) - \max (0, d_1 - c_k) = f_k - f_i & \text{for } i \geqq k, \\ \max (0, d_r - c_i) - \max (0, d_r - c_i) = g_k - g_i & \text{for } i \leqq k, \end{cases} \end{aligned}$$

where $f_i = -\max (0, d_1 - c_i)$, $g_i = -\max (0, d_r - c_i)$. Since $f_k - f_i \geqq$ or $\leqq g_k - g_i$ according as $i \geqq k$ or $i \leqq k$, we find that $\mathfrak{o}$ is the intersection of the two maximal orders $M(f_k - f_i)$ and $M(g_k - g_i)$. Further, if we put $\gamma = \sum \epsilon_{ii} P^{-c_i}$, $\delta = \sum \epsilon_{ii} P^{-d_i}$, then $\mathfrak{r} = \gamma P^a M(0)$ and $\mathfrak{s} = M(0) P^{-a} \delta$, whence $\mathfrak{t} = P^{a - d_r} M(0)$, $\mathfrak{t}' = P^{d_1 - a} M(0)$. From this we can easily verify the precise characterization of $\mathfrak{o}$ given in the theorem.

The part on the sum $(\mathfrak{a}, \mathfrak{b})$ can be shown by a similar computation. And indeed from that computation we obtain the last assertion in the theorem.

Finally, to prove Theorem 4 we observe again that we have only to consider the case where $\mathfrak{a}$, $\mathfrak{b}$ have the form (1). $\mathfrak{a} \cap \mathfrak{b} = M(\max (0, d_k - c_i)$, $(\mathfrak{a}^{-1}, \mathfrak{b}^{-1}) = M(\min (0, c_k - d_i))$ because $\mathfrak{b}^{-1} = M(c_k - d_i)$, and here we notice that $\max (0, d_k - c_i) = -\min (0, c_i - d_k)$. The third normal ideal $\mathfrak{c}$ can be expressed as $\mathfrak{c} = \tau^{-1} M(0) \sigma^{-1}$ with regular elements $\sigma = \sum \epsilon_{ik} s_{ik}$, $\tau = \sum \epsilon_{ik} t_{ik}$. Let $P^{c_{ik}}$ be the exact power of P which divides s_{ik}, $P^{c_{ik}} \| s_{ik}$; if $s_{ik} = 0$ we put $c_{ik} = \infty$. Let similarly $P^{d_{ik}} \| t_{ik}$. It is evident that $M(a_{ik})$, with a system of rational integers a_{ik}, contains $\mathfrak{c}^{-1} = \sigma M(0) \tau$ if and only if

$$(2)\qquad c_{ij} + d_{lk} \geqq a_{ik}, \qquad \text{for all } i, j, k, l.$$

Hence, if we show that the same condition is also necessary and sufficient in order that $M(-a_{ki}) \subseteqq \mathfrak{c}$, then we will be through. But this is also easy to see. For, $\mathfrak{c} = \tau^{-1}M(0)\sigma^{-1}$ consists of all $\eta = \sum \epsilon_{ik} y_{ik} = \tau^{-1}(\sum \epsilon_{ik} x_{ik})\sigma^{-1}$ with $x_{ik} \in I$. On taking a pair (j, l) of indices, let us consider those η such that $y_{ik} = 0$ for $(i, k) \neq (j, l)$. In other words, we consider the equation $\tau^{-1}(\sum \epsilon_{ik} x_{ik})\sigma^{-1} = \epsilon_{jl} y_{jl}$. But this is equivalent to $\sum \epsilon_{ik} x_{ik} = \tau \epsilon_{jl} y_{jl} \sigma$, or

$$x_{ik} = t_{ij} y_{jl} s_{lk}, \qquad i, k = 1, 2, \cdots, r. \tag{3}$$

Suppose now $M(-a_{ki}) \subseteqq \mathfrak{c}$. Then (3) with $y_{jl} = P^{-a_{lj}}$ must have a solution $x_{ik} \in I$. Hence $0 \leqq d_{ij} - a_{lj} + c_{lk}$ (for all i, k). Since (j, l) was an arbitrary pair of indices, we have thus established (2). Assume conversely (2). Then obviously $x_{ik} = t_{ij} P^{-a_{lj}} s_{lk} \in I$ whence $\epsilon_{jl} P^{-a_{lj}} \in \mathfrak{c}$ and $M(-a_{ki}) \subseteqq \mathfrak{c}$.

A second proof of the last part of Theorem 3 is as follows: We observe first that every ideal $\mathfrak{m}$ in $\mathfrak{A}$ is additively generated by regular elements contained in $\mathfrak{m}$.[6] For, if $\xi \in \mathfrak{m}$ we take a scalar element a ($\in F$) in $\mathfrak{m}$ different from all the characteristic roots of the matrix which represents ξ in a faithful representation of $\mathfrak{A}$. Then $\xi - a$ ($\in \mathfrak{m}$) is evidently a regular element and $\xi = (\xi - a) + a$. Now, let α be any regular element from the left order of $\mathfrak{a} \cap \mathfrak{b}$; $\alpha(\mathfrak{a} \cap \mathfrak{b}) \subseteqq \mathfrak{a} \cap \mathfrak{b}$. Since $\alpha\mathfrak{a}$ and $\alpha\mathfrak{b}$ are normal ideals, we have, from Theorem 4, $(\mathfrak{a}^{-1}\alpha^{-1}, \mathfrak{b}^{-1}\alpha^{-1}) \supseteqq \mathfrak{a}^{-1}, \mathfrak{b}^{-1}$ whence $(\mathfrak{a}^{-1}, \mathfrak{b}^{-1})\alpha^{-1} \supseteqq (\mathfrak{a}^{-1}, \mathfrak{b}^{-1})$, $(\mathfrak{a}^{-1}, \mathfrak{b}^{-1}) \supseteqq (\mathfrak{a}^{-1}, \mathfrak{b}^{-1})\alpha$. This shows that the left order of $\mathfrak{a} \cap \mathfrak{b}$ is contained in the right order of $(\mathfrak{a}^{-1}, \mathfrak{b}^{-1})$. But the converse can be seen in quite a similar manner.

Remark. The structure of the residue class algebra $\mathfrak{J}_1 \cap \mathfrak{J}_2/p(\mathfrak{J}_1 \cap \mathfrak{J}_2)$ is easy to analyze, but perhaps does not deserve a detailed discussion. We merely note that the algebra is not symmetric, in fact is not weakly symmetric,[7] except for the trivial case $(\mathfrak{J}_1)_p = (\mathfrak{J}_2)_p$; this remark may be of some interest in view of a recent paper by R. Brauer.[8]

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[6] We exclude here the trivial case of a finite underlying field F.

[7] See Brauer-Nesbitt, Proceedings of the National Academy of Sciences, vol. 23 (1937); Nakayama-Nesbitt, Annals of Mathematics, (2), vol. 39 (1938).

[8] Brauer, *On modular and p-adic representations of algebras*, Proceedings of the National Academy of Sciences, vol. 25 (1939).