NULLIFYING FUNCTIONS[1]

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Introduction. A function $f(x)$ defined on the unit interval $(0, 1)$ will be called nullifying if we can find a set S of $(0, 1)$ for which $m(S)=1$, $m(\{f(x); x \varepsilon S\})=0$. Examples of homeomorphisms which are nullifying and hence termed singular are well known.[2] We shall however consider simply the nullifying property itself.

If $f(x)$ is nullifying and $\phi(x)$ is not, one might expect that $\phi(x)+f(x)$ shares with ϕ the property of being not nullifying. But this is not always true as the following example shows. Let $.\alpha_1\alpha_2 \cdots$ denote the dyadic expansion for x, that is, $x=\alpha_1/2+\alpha_2/2^2+ \cdots$ with $\alpha_i=0$ or 1. Let $v_1(x)=.\alpha_1 0\alpha_3 0 \cdots$ and $v_2(x)=.0\alpha_2 0\alpha_4 \cdots$. It is easily verified that both v_1 and v_2 are nullifying. Hence $f(x)=1-v_1$ is also nullifying. Let $\phi=x$. Then $f+\phi=1-v_1+x=1+v_2$ is also nullifying.

But this suggests the question: Does there exist a nullifying function $f(x)$ such that $f(x)+\rho x$ is nullifying for every value of ρ? We construct such a function in the present note.

Our method of proof can be summarized as follows. Considering the set $\{f(x)+\rho x; x \varepsilon (0, 1)\}$, we let $\rho=\cot\theta$. (Note $\theta\neq 0$.) If this set has measure zero, this will still be true if we multiply by $\sin\theta$ and conversely. Thus we may consider the sets $\{x\cos\theta+f(x)\sin\theta; x \varepsilon (0, 1)\}$ for each $\theta\neq 0$ between $-\pi/2$ and $\pi/2$. If we consider the line through the origin of inclination θ, we can assign a coordinate to each of its points in the usual manner with positive direction to the right or, in the case of the y axis, upwards. The set $\{x\cos\theta+f(x)\sin\theta; x \varepsilon (0, 1)\}$ is the set of coordinates of the projection onto this line of the graph of $f(x)$. Thus it suffices to find a function $f(x)$ which is such that the projection of its graph onto any line not parallel to the x axis is of measure zero. We proceed to find the graph of such a function by an intersection process on sets in the plane. This process is described in detail in what follows.

A more general question is: Given $F(x, y, \rho)$, under what circumstances can we find a function $f(x)$ such that $F(x, f(x), \rho)$ is nullifying in x for every value of ρ? It is comparatively easy to abstract the properties of $F=y+\rho x$ which are essential to the present discussion, and these will prove sufficient to obtain an answer to the question.

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[2] Cf. E. R. van Kampen and Aurel Wintner, *On a singular monotone function*, Journal of the London Mathematical Society, vol. 12 (1937), pp. 243–244. References to preceding examples are given in this paper.

However the writer hopes to obtain a more penetrating analysis of this subject soon. The writer will also consider the more general question of obtaining an $f(x)$ the substitution of which will make nullifying not only one but a set of F's.

Definition of a G_s. Let us divide evenly the unit square of the plane in squares of side $1/2^s$. The coordinates of the vertices of these squares are dyadic rational, that is, of the form $a/2^s$ for some integer a. Two squares are said to be in the same column if they have the same projection on the x axis. A set of such squares with one and only one (closed) square in each column will be termed[3] a G_s.

Notation. If l_θ is the line through the origin with inclination θ and P is any point of the plane, we shall denote by $\pi_\theta(P)$ the projection of P onto l_θ. If S is a set of points, we denote by $\pi_\theta(S)$ the projection of S onto l_θ. The linear measure of a linear set S we denote by $\mu(S)$.

LEMMA 1. *Let m be such that $0 \leqq m \leqq 1$. Let $\epsilon > 0$ and a G_s be given. Let α be such that $\pi/2 \geqq \alpha \geqq \pi/4$ and* $\cot \alpha = m$. *Then we can find a $G_{s+t} \subset G_s$ such that for $0 \leqq \theta \leqq \pi/2$,*

$$\mu(\pi_\theta(G_{s+t})) \leqq (1 + m^2)^{1/2} \sin \left| \theta - \alpha \right| + \epsilon.$$

PROOF. Let us consider a square B of G_s. Then it is possible to take a line $l^{(1)}$ of slope $-m$, such that for every x in the projection of B on the x axis, we have a point (x, y) in B and on $l^{(1)}$. Let $l^{(1)}$ intersect the left-hand side of B at P_B and the right-hand side at Q_B.

Let us divide B into squares of side $1/2^{s+t}$. Now it is readily seen that for each column of smaller squares in B there is at least one square with an interior point on $l^{(1)}$. (If $m = 0$, this is not true, but then we can substitute for "interior point," "interior point or point on the upper side.") Define G_{s+t} so as to contain for each column the lowest such square.

We also take a square of side $1/2^{s+t}$, whose upper right-hand vertex is at P_B. We denote the lower left-hand end point of this square by P_B'. Similarly we take a square of side $1/2^{s+t}$ whose lower left-hand vertex is Q_B, and we denote the upper right-hand vertex by Q_B'.

We are going to consider $\pi_\theta(G_{s+t})$ and we let $<$ and $\leqq$ refer to the order of the points on l_θ, the direction of the greater being to the right of the smaller.

We shall show that if $P \in G_{s+t}$, and $0 \leqq \theta \leqq \alpha$, then $\pi_\theta(P_B') \leqq \pi_\theta(P)$.

[3] We shall ignore the logical distinction between a set of squares and the corresponding set of points. G_s denotes either a (closed) set of points or a set of squares according to the context.

For P_B' (x_B', y_B') is the lower left-hand vertex of a square whose upper right-hand vertex alone is on $l^{(1)}$. Now every square C of G_{s+t} contains an interior point which is on $l^{(1)}$ (if $m=0$, confer above). Thus if $l^{(1)\prime}$ is parallel to $l^{(1)}$ and through P_B', every point P (x, y) of G_{s+t} is above or on $l^{(1)\prime}$. Furthermore $x>x_B'$.

Let P' (x', y') be the intersection of $l^{(1)\prime}$ and the line through P parallel to the y axis. P is above P' and hence $\pi_\theta(P)\geqq\pi_\theta(P')$. On the other hand $x'=x>x_B'$. Let $\bar{l}$ denote the line through P_B' perpendicular to l_θ. The slope of $\bar{l}$ is more strongly negative than that of $l^{(1)\prime}$. It follows that for $x\geqq x_B'$, $l^{(1)\prime}$ is to the right of l and hence $\pi_\theta(P')\geqq\pi_\theta(P_B')$. This and the previous result yield $\pi_\theta(P)\geqq\pi_\theta(P_B')$.

A similar argument will show that $\pi_\theta(Q_B')\geqq\pi_\theta(P)$.

Thus $\pi_\theta(P_B')\leqq\pi_\theta(P)\leqq\pi_\theta(Q_B')$. This implies that the projection of that part of G_{s+1} which lies in B is contained in the interval $\pi_\theta(P_B')\pi_\theta(Q_B')$. Now if P_B has the coordinates (x_B, y_B), then Q_B has the coordinates $(x_B+1/2^s, y_B-m/2^s)$, P_B' has the coordinates $(x_B-1/2^{s+t}, y_B-1/2^{s+t})$, and Q_B' has the coordinates $(x_B+1/2^s+1/2^{s+t}, y_B-m/2^s+1/2^{s+t})$.

Consider $l_{\theta+\pi/2}$, the line through the origin perpendicular to l_θ. According to a formula of elementary analytic geometry, the directed distance of any point P (x, y) to $l_{\theta+\pi/2}$ is $x\cos\theta+y\sin\theta$, the directed distance being positive if (x, y) is to the right of $l_{\theta+\pi/2}$ and negative to the left. This directed distance is not changed if we project P onto l_θ, and so $x\cos\theta+y\sin\theta$ represents also the directed distance of $\pi_\theta(P)$ from $l_{\theta+\pi/2}$, or since l_θ is perpendicular to $l_{\theta+\pi/2}$, from the origin O along l_θ.

Thus $O\pi_\theta(P_B')$ has length $(x_B-1/2^{s+t})\cos\theta+(y_B-1/2^{s+t})\sin\theta$ and $O\pi_\theta(Q_B')$ has the length $(x_B+1/2^s+1/2^{s+t})\cos\theta+(y_B-m/2^s+1/2^{s+t})\sin\theta$. Hence $\pi_\theta(P_B')\pi_\theta(Q_B')$ has length

$$(1/2^s+1/2^{s+t-1})\cos\theta-(m/2^s-1/2^{s+t-1})\sin\theta$$

for $0\leqq\theta\leqq\alpha$. Thus the projection of that part of G_{s+1} which lies in B has measure not greater than

$$1/2^s(\cos\theta-m\sin\theta)+1/2^s(1/2^{t-1}(\cos\theta+\sin\theta)$$
$$<1/2^s(\cos\theta-m\sin\theta)+1/2^s(1/2^{t-2}).$$

There are 2^s squares like B, and this implies that for $0<\theta\leqq\alpha$, the projection of G_{s+t} has measure not greater than

$$(\cos\theta-m\sin\theta)+1/2^{t-2}=(1+m^2)^{1/2}\sin(\alpha-\theta)+1/2^{t-2}.$$

A similar argument holds in the case $\alpha<\theta\leqq\pi/2$, with however the smaller added squares in different positions, and during the argument certain inequalities are reversed. The corresponding result is that, for $\alpha<\theta\leqq\pi/2$,

$$\begin{aligned}\mu(\pi_\theta(G_{s+t})) &< m \sin\theta - \cos\theta + 1/2^{t-2}\\ &= (1+m^2)^{1/2}\sin(\theta-\alpha) + 1/2^{t-2}.\end{aligned}$$

If t is taken large enough so that $1/2^{t-2}<\epsilon$, the final results of the preceding two paragraphs are sufficient to prove our lemma.

LEMMA 2. *Suppose $m>1$, and that a G_s and an $\epsilon>0$ are given. Let α be such that $\pi/4>\alpha>0$, $\cot\alpha=m$. Then we can find a $G_{s+t}\subset G_s$ such that for $0\leqq\theta<\pi/2$,*

$$\mu(\pi_\theta(G_{s+t})) \leqq (1+m^2)^{1/2}\sin|\theta-\alpha| + \epsilon.$$

PROOF. Let B be any square of G_s. Let $l^{(1)}, l^{(2)}, \cdots, l^{(k)}$ denote a set of lines of slope $-m$ with the following properties: (a) each $l^{(i)}$ contains an interior point of B; (b) if $P_B^{(i)}Q_B^{(i)}$ is the line segment $B\cdot l^{(i)}$ (B is closed), $P_B^{(1)}$ is on the left side of B, $Q_B^{(k)}$ is on the right side of B, and for $i=1, 2, \cdots, k-1$, $P_B^{(i+1)}$ is on the upper side of B, $Q_B^{(i)}$ is on the lower side of B, and $Q_B^{(i)}$ has the same x coordinate as $P_B^{(i+1)}$. The existence of such a set of lines is easily shown.

Let the coordinates of $P_B^{(i)}$ be $(x_B^{(i)}, y_B^{(i)})$, those of $Q_B^{(i)}$ be $(u_B^{(i)}, v_B^{(i)})$. From the above we see that $x_B^{(1)}=a/2^s$, $y_B^{(1)}=b/2^s+\rho/2^s$ where $0<\rho\leqq1$, $u_B^{(1)}=a/2^s+\rho/m\cdot2^s$, $u_B^{(2)}=a/2^s+(\rho+1)/m\cdot2^s$, and in general $u_B^{(i)}=a/2^s+(\rho+i-1)/2^s\cdot m$, for $i=1, 2, \cdots, k-1$, while $u_B^{(k)}=(a+1)/2^s$. From this it follows that

$$(\rho+k-2)/m\cdot2^s < 1/2^s \leqq (\rho+k-1)/m\cdot2^s$$

or $(\rho+k-2)<m<\rho+k-1$ or $k-1<m+(1-\rho)\leqq k$. Hence if $[m]$ denotes the largest integer less than or equal to m, then $k=[m]$, $[m]+1$, or $[m]+2$. Thus $k\leqq m+2$.

We next divide the square B of S_s into smaller squares of side $1/2^{s+t}$. The lines $l^{(1)}, \cdots, l^{(k)}$ have been chosen so that for each x in the projection of B on the x axis, there is at least one $l^{(i)}$ which contains a point (x, y) in B. This can be used to show that for each column of smaller squares in B, there is at least one smaller square which contains a point on some $l^{(i)}$ and interior to B. Define G_{s+t} so as to contain the lowest such square in each column.

For each i, let us consider those squares of G_{s+t} which contain points of $l^{(i)}$. An argument similar to that used in Lemma 1 will show that the projection of this part of G_{s+t} on l_θ has measure not greater than

$$(\text{length of } l^{(i)}\cdot B)\sin|\theta-\alpha| + 1/2^{s+t-2}.$$

The sum of the lengths of the $l^{(i)}\cdot B$ is $(1/2^s)(1+m^2)^{1/2}$ and there are at most $[m]+2$ of them. Thus

$$\mu(\pi_\theta(G_{s+t}\cdot B)) \leqq (1/2^s)((1 + m^2)^{1/2} \sin \left|\theta - \alpha\right| + ([m] + 2)/2^{t-2}).$$

Since there are 2^s such squares B, we have

$$\mu(\pi_\theta(G_{s+t})) \leqq (1 + m^2)^{1/2} \sin \left|\theta - \alpha\right| + ([m] + 2)/2^{t-2}.$$

We can take t sufficiently large to obtain our result.

This result is interesting only in the range $0<\theta<2\alpha$, for outside this range other methods give more effective inequalities.

The argument of Lemmas 1 and 2 can be modified to apply to the case in which m is negative, with the following result.

LEMMA 3. *Suppose an $m<0$, a G_s and an $\epsilon>0$ are given. Let α be such that* $\cot \alpha=m$ *and* $-\pi/2\leqq\alpha<0$. *Then we can find a $G_{s+t}\subset G_s$ such that for* $-\pi/2\leqq\theta<0$,

$$\mu(\pi_\theta(G_{s+t})) \leqq (1 + m^2)^{1/2} \sin \left|\theta - \alpha\right| + \epsilon.$$

LEMMA 4. *Let m_1 and m_2 be given with $m_2>m_1\geqq 0$. Let α_i be such that* $\pi/2\geqq\alpha_i>0$, $\cot \alpha_i=m_i$ for $i=1, 2$. *Consider the function*

$$F(\theta) = \min \left((1 + m_2^2)^{1/2} \sin (\theta - \alpha_2),\ (1 + m_1^2)^{1/2} \sin (\alpha_1 - \theta)\right)$$

for $\alpha_2\leqq\theta\leqq\alpha_1$. Then for this range of θ,

$$F(\theta) \leqq (m_2 - m_1)/2(m_1^2 + 1)^{1/2}.$$

PROOF. For $\alpha_2\leqq\theta\leqq\alpha_1$, $(1+m_2^2)^{1/2}\sin(\theta-\alpha_2)$ is increasing while $(1+m_1)^{1/2}\sin(\alpha_1-\theta)$ is decreasing. Also we readily see that if η is such that

$$(1 + m_2^2)^{1/2} \sin (\eta - \alpha_2) = (1 + m_1^2)^{1/2} \sin (\alpha_1 - \eta)$$

then $F(\theta)$ is equal to the first expression for $\alpha_2\leqq\theta\leqq\eta$ and to the second expression for $\eta\leqq\theta\leqq\alpha_1$. Hence $F(\eta)$ is the maximum value of $F(\theta)$ for the given range of θ.

Expanding $\sin(\eta-\alpha_2)$ and $\sin(\alpha_1-\eta)$, using the definition of α_i, collecting terms in $\sin\eta$ and $\cos\eta$ and dividing by $\cos\eta$ yields

$$\tan \eta = 2/(m_1 + m_2).$$

The value of $F(\eta)$ is then seen to be

$$(m_2 - m_1)/2(1 + (m_1 + m_2)^2/4)^{1/2} \leqq (m_2 - m_1)/2(1 + m_2^2)^{1/2}.$$

This and the result of the preceding paragraph prove the lemma.

Consider the sequence $0, 1, 0, -1, 2, 3/2, 1, 1/2, 0, -1/2, -1, -3/2, -2, 4, 15/4, \cdots, 1/4, 0, -1/4, \cdots, -15/4, -4, \cdots$. We denote the ith term in this sequence by m_i. Notice that the terms of this sequence can be grouped so that the first group contains 1, the second group 3, the third group 9, and the kth group $2^{2k-3}+1$. The maximum value in the kth group is 2^{k-2} and the difference between any two adjacent terms in this group is $1/2^{k-2}$.

Let $\{\epsilon_i\}$ denote the sequence such that if i is a subscript of the kth group of the preceding paragraph, then $\epsilon_i=1/2^k$. Let $G^{(0)}=G_0$, the unit square itself, and $G^{(i)}$ be defined as the G_{s+t}, which results when either Lemma 1, 2, or 3 (depending on m_i) is applied to $G^{(i-1)}$, m_i and ϵ_i.

LEMMA 5. *Let $G^{(i)}$ be as above. Let θ be such that $\pi/2\geqq\theta>0$. Then* $\lim_{i\to\infty}\mu(\pi_\theta(G^{(i)}))=0$.

PROOF. Let $\epsilon>0$ be given. Take k such that $\epsilon/2>1/2^{k-1}$ and $2^{k-2}>\cot\theta$. Then we can find an m_i in the kth group such that $m_i>0$ and $m_i\geqq\cot\theta>m_{i+1}$.

Now since $G^{(i)}\supset G^{(i+1)}$, we have $\pi_\theta(G^{(i)})\supset\pi_\theta(G^{(i+1)})$ and $\mu(\pi_\theta(G^{(i)}))\geqq\mu(\pi_\theta(G^{(i+1)}))$. Thus $\mu(\pi_\theta(G^{(i+1)}))$ is subject to the inequality which Lemmas 1 and 2 impose upon $\mu(\pi_\theta(G^{(i)}))$. Thus

$$\mu(\pi_\theta(G^{(i+1)}))\leqq\min\left[(1+m_i^2)^{1/2}\sin\left|\theta-\alpha_i\right|,\ (1+m_{i+1}^2)^{1/2}\cdot\sin\left|\theta-\alpha_{i+1}\right|\right]+\epsilon_i,\qquad(\cot\alpha_i=m_i).$$

Since $\alpha_i\leqq\theta\leqq\alpha_{i+1}$, Lemma 4 yields

$$\mu(\pi_\theta(G^{(i+1)}))\leqq(m_i-m_{i+1})/2(1+m_{i+1}^2)^{1/2}+\epsilon_i<1/2^{k-1}+1/2^k<\epsilon.$$

Also if $j\geqq i+1$, $G^{(j)}\subset G^{(i+1)}$, and we obtain

$$0<\mu(\pi_\theta(G^{(j)}))\leqq\mu(\pi_\theta(G^{(i+1)}))<\epsilon.$$

Thus we have shown that given an $\epsilon>0$, we can find an i such that for $j>i$ this equation holds. The lemma is now proved.

LEMMA 6. *Let $G^{(i)}$ be as in Lemma 5. Let $G=\prod G^{(i)}$. Then G is a non-empty closed set such that if $0<\theta\leqq\pi/2$, then $\mu(\pi_\theta(G))=0$.*

PROOF. Since the $G^{(i)}$'s form a decreasing sequence of closed sets, their intersection is a non-empty closed set. Since $G\subset G^{(i)}$, $0\leqq\mu(\pi_\theta(G))\leqq\mu(\pi_\theta(G^{(i)}))$. Thus Lemma 5 now implies $\mu(\pi_\theta(G))=0$.

Results similar to those of Lemmas 5 and 6 hold for $0>\theta>-\pi/2$. The method of obtaining them should be clear from the preceding discussion and we merely state the final result as follows.

LEMMA 7. *If G is as in Lemma 6, then for $0>\theta>-\pi/2$, $\mu(\pi_\theta(G))=0$.*

LEMMA 8. *G is the graph of a function $y=f(x)$.*

PROOF. For $0 \leqq a \leqq 1$, let p_a denote the line $x = a$. Then the sets $p_a \cdot G^{(i)}$ form a decreasing sequence of closed intervals on p_a, with one and only one point (a, b) in common. Thus $p_a \cdot G$ consists of one and only one point (a, b) and b is a function of a.

Now, as we have pointed out in the proof of Lemma 1, for any point P (x, y), $x \cos \theta + y \sin \theta$ is the directed distance of $\pi_\theta(P)$ along l_θ from the origin. Using Lemmas 6, 7, and 8, we obtain that for $\pi/2 \geqq \theta > 0$ or $0 > \theta > -\pi/2$,

$$\begin{aligned} 0 = \mu(\pi_\theta(G)) &= \mu(\{x \cos \theta + y \sin \theta;\ (x, y) \varepsilon G\}) \\ &= \mu(\{x \cos \theta + f(x) \sin \theta;\ 0 \leqq x \leqq 1\}) \\ &= |\sin \theta| \, \mu(\{x \cot \theta + f(x);\ 0 \leqq x \leqq 1\}). \end{aligned}$$

Letting $\rho = \cot \theta$ we obtain that for every value of ρ,

$$\mu(\{f(x) + \rho x;\ 0 \leqq x \leqq 1\}) = 0.$$

Since $f(x)$ is the limit of step functions, it is measurable, and we have proved the following theorem.

THEOREM. *There exists a measurable function* $f(x)$ *defined for* $0 \leqq x \leqq 1$, *such that for every value of* ρ, $f(x) + \rho x$ *is nullifying.*

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