COMBINATORIC INTERPRETATION OF A FORMULA FOR THE nTH DERIVATIVE OF A FUNCTION OF A FUNCTION

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Let $f(x) = F(\phi(x))$. The formula of Faa' di Bruno* states that

$$(1)\qquad \frac{d^n f}{dx^n} = \sum_{k=1}^{n} \frac{d^k F}{d\phi^k} \sum_s A_{ks} \prod_r^{(s)} \left(\frac{d^{k_r}\phi}{dx^{k_r}}\right)^{i_r},$$

where $\prod_r^{(s)}$ is taken for *a system* (s) of positive integral solutions i_r, k_r of the equations $\sum_r i_r = k$, $\sum_r k_r i_r = n$ and $\sum_s$ is taken for *all such systems*. The factor A_{ks} is equal to $n![\prod_r i_r!(k_r!)^{i_r}]^{-1}$. From a recent result of H. S. Wall† we have therefore *that the numerical factor* A_{ks} *in* (1) *is equal to the number of ways that n different objects can be placed in* $k = \sum_r i_r$ *compartments*, k_r *in each of* i_r *compartments, without regard to the order of arrangement of the compartments.*

H. S. Wall expressed the nth derivative of $f(x)$ in terms of logarithmic derivatives of $f(x)$.‡ Putting $F(\phi) = e^{\phi}$, $\phi(x) = \log f(x)$ in (1) his formula appears as a particular case of (1).

For *functions of many variables* $f(x) = F(\phi_1(x), \phi_2(x), \cdots, \phi_n(x))$ the formula of F. G. Teixeira§ states that

$$\frac{d^n f}{dx^n} = \sum_{k=1}^{n} \sum_\sigma \frac{\partial^k F}{\prod_t^{(\sigma)} \partial x_t^{a_t}} \sum_s A_{ks\sigma} \prod_{tr}^{(s)} \left(\frac{d^{k_{tr}}\phi_t}{dx_t^{k_{tr}}}\right)^{i_{tr}},$$

where $\prod_t^{(\sigma)}$ is taken for a system (σ) of nonnegative integral solutions a_t of the equation $\sum_t a_t = k$ and $\sum_\sigma$ is taken for all such systems; $\prod_{tr}^{(s)}$ is taken for a system (s) of positive integral solutions k_{tr}, i_{tr} of the equations $\sum_r i_{tr} = a_t$, $\sum_{tr} i_{tr} k_{tr} = n$ and $\sum_s$ is taken for all such systems. $A_{ks\sigma}$ is equal to $n![\prod_{tr} i_{tr}!(k_{tr}!)^{i_{tr}}]^{-1}$. $A_{ks\sigma}$ has therefore the same combinatoric meaning as A_{ks}.

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* F. Faa' di Bruno, *Sullo sviluppo delle funzioni*, Annali di Scienze Matematiche e Fisiche di Tortolini, vol. 6 (1855), pp. 479–480. For a bibliography on the subject see A. Voss, Encyclopädie der mathematischen Wissenschaften, II A 2, p. 88; E. Pascal, *Esercizi Critici di Calcolo*, Milano, 1921, 3d edition, pp. 111–112. See also L. S. Dederick, Annals of Mathematics , (2), vol. 27 (1926), pp. 385–394.

† H. S. Wall, *On the nth derivative of* $f(x)$, this Bulletin, vol. 44 (1938), pp. 395–397; see the theorem p. 395, and formula (13).

‡ Wall, loc. cit., formula (1).

§ F. G. Teixeira, *Sur les dérivées d'ordre quelconque*, Giornale di Matematica di Battaglini, vol. 18 (1880), p. 306.