A REMARK ON REPRESENTATIONS OF GROUPS*

TADASI NAKAYAMA

The purpose of this short note is to remark that we can state an analog of a famous theorem of Frobenius† on the induced characters of a finite group also for the representations of a general group.‡ This extension has not yet been explicitly stated, so far as I know, although it can be quite easily verified.

Let $\mathfrak{g}$ be a group, and let $\mathfrak{h}$ be a subgroup (*of a finite or infinite index*) of $\mathfrak{g}$.

DEFINITION. Let $F(x)$ and $f(\xi)$ be almost periodic (a. p.) functions (with complex numbers as values) on $\mathfrak{g}$ and $\mathfrak{h}$ respectively. Then we define the compositions of $F(x)$ and $f(\xi)$ by

$$f \times F(x) = M_{\xi\epsilon\mathfrak{h}}[f(\xi)F(\xi^{-1}x)],$$
$$F \times f(x) = M_{\xi\epsilon\mathfrak{h}}[F(x\xi^{-1})f(\xi)];$$

where $M_{\xi\epsilon\mathfrak{h}}$ means the construction of the mean with respect to a variable ξ in $\mathfrak{h}$. *Here* $f\times F(x)$ *and* $F\times f(x)$ *are a. p. functions on* $\mathfrak{g}$, *and they are linear with respect to both factors*, $f(\xi)$ *and* $F(x)$.

If $\mathfrak{h}_1$, $\mathfrak{h}_2$, $\mathfrak{h}_3$ *are three subgroups of* $\mathfrak{g}$ *such that* $\mathfrak{h}_i \subseteqq \mathfrak{h}_k$ *or* $\mathfrak{h}_i \supseteqq \mathfrak{h}_k$ *for every* i, $k=1, 2, 3$, *then*

$$(f_1 \times f_2) \times f_3 = f_1 \times (f_2 \times f_3)$$

for a. p. functions $f_1(\xi_1)$, $f_2(\xi_2)$, $f_3(\xi_3)$ *on* $\mathfrak{h}_1$, $\mathfrak{h}_2$, $\mathfrak{h}_3$ *respectively.*

(Both sides of the equality are a. p. functions on the greatest among the $\mathfrak{h}_i$.) This product we denote by $f_1\times f_2\times f_3$.

All these statements we can prove by a procedure similar to that

* Presented to the Society, October 30, 1937.

† G. Frobenius, *Ueber Relationen zwischen den Charakteren einer Gruppe und denen ihrer Untergruppen*, Berlin Sitzungsberichte, 1898; H. Weyl, *Gruppentheorie und Quantenmechanik*; J. Levitzki, *Ueber vollständig reduzible Ringe und Unterringe*, Mathematische Zeitschrift, vol. 33 (1931).

‡ J. von Neumann, *Almost periodic functions in a group*, Transactions of this Society, vol. 36 (1934). Cf. also S. Bochner and J. von Neumann, *Almost periodic functions in groups*, II, ibid., vol. 37 (1935); W. Maak, *Eine neue Definition der fastperiodischen Funktionen*, Abhandlungen aus dem Mathematischen Seminar, Hamburg, vol. 11 (1936); B. L. van der Waerden, *Gruppen von linearen Transformationen*, Ergebnisse der Mathematik, vol. 4 (1935).

By a representation of a group we understand always a bounded one in the field of complex numbers.

of von Neumann in the paper cited. Therefore we can consider the ring $\mathfrak{R}_{\mathfrak{h}}$ of a. p. functions on $\mathfrak{h}$ as a (right and left) operator-ring of the ring $\mathfrak{R}_{\mathfrak{g}}$ of a. p. functions on $\mathfrak{g}$. If $\mathfrak{M}$ and $\mathfrak{m}$ are submoduli of $\mathfrak{R}_{\mathfrak{g}}$ and $\mathfrak{R}_{\mathfrak{h}}$ respectively, then we denote by $\mathfrak{M}\times\mathfrak{m}$ the submodule of $\mathfrak{R}_{\mathfrak{g}}$ generated by the elements $(F\times f(x),\ F(x)\epsilon\mathfrak{M},\ f(\xi)\epsilon\mathfrak{m})$. We define $\mathfrak{m}\times\mathfrak{M}$ in a similar manner.

Definition. Let $\mathfrak{n}$ be a left ideal of $\mathfrak{R}_{\mathfrak{h}}$ with a finite rank with respect to the field Ω of complex numbers.* Then $\mathfrak{R}_{\mathfrak{g}}\times\mathfrak{n}$ is obviously a left ideal of $\mathfrak{R}_{\mathfrak{g}}$ (with a finite or infinite rank with respect to Ω). We call $\mathfrak{R}_{\mathfrak{g}}\times\mathfrak{n}$ *the left ideal of* $\mathfrak{R}_{\mathfrak{g}}$ *induced by* $\mathfrak{n}$.

As is well known, there is an idempotent element $c(\xi)$ in $\mathfrak{R}_{\mathfrak{h}}$ such that $\mathfrak{n}=\mathfrak{R}_{\mathfrak{h}}\times c$; it is $f\times c=f$ for every $f(\xi)$ in $\mathfrak{n}$. If $F(x)\epsilon\mathfrak{R}_{\mathfrak{g}}$ and $f(\xi)\epsilon\mathfrak{n}$, then we have $F\times f=F\times(f\times c)=(F\times f)\times c\epsilon\mathfrak{R}_{\mathfrak{g}}\times c$. Therefore $\mathfrak{R}_{\mathfrak{g}}\times\mathfrak{n}\subseteq\mathfrak{R}_{\mathfrak{g}}\times c$, and this implies $\mathfrak{R}_{\mathfrak{g}}\times\mathfrak{n}=\mathfrak{R}_{\mathfrak{g}}\times c$.

The ideal $\mathfrak{R}_{\mathfrak{g}}\times\mathfrak{n}$ *consists of all functions* $G(x)\epsilon\mathfrak{R}_{\mathfrak{g}}$ *such that for every* $x\epsilon\mathfrak{g}$ *the function* $G(x\xi)$ *of* $\xi\epsilon\mathfrak{h}$ *lies in* $\mathfrak{n}$.

Let $G(x)$ have the property stated above. Then

$$M_{\eta\epsilon\mathfrak{h}}[G(x\xi\eta^{-1})c(\eta)] = G(x\xi);$$

in particular,

$$G\times c(x) = M_{\eta\epsilon\mathfrak{h}}[G(x\eta^{-1})c(\eta)] = G(x),$$

that is, $G(x)\epsilon\mathfrak{R}_{\mathfrak{g}}\times c=\mathfrak{R}_{\mathfrak{g}}\times\mathfrak{n}$.

The other half of the statement is obvious.

Now we have the following theorem:

Theorem. *Let* $\mathfrak{n}$ *be a minimal left ideal of* $\mathfrak{R}_{\mathfrak{h}}$, *and* $\mathfrak{d}$ *the irreducible representation of* $\mathfrak{h}$ *defined by* $\mathfrak{n}$. *Let* $\mathfrak{D}$ *be an irreducible representation of* $\mathfrak{g}$, *and* $\mathfrak{S}$ *the two-sided ideal of* $\mathfrak{R}_{\mathfrak{g}}$ *belonging to* $\mathfrak{D}$.† *We denote by* $\mathfrak{D}(\mathfrak{h})$ *a representation of* $\mathfrak{h}$ *formed by the matrices in* $\mathfrak{D}$ *which correspond to the elements of* $\mathfrak{h}$. *If the number of the irreducible constituents of* $\mathfrak{D}(\mathfrak{h})$ *equivalent to* $\mathfrak{d}$ *is* g, *then the representation of* $\mathfrak{g}$ *defined by the* $\mathfrak{S}$-*component* $\mathfrak{S}\times\mathfrak{N}=\mathfrak{S}\times\mathfrak{n}$ *of the induced left ideal* $\mathfrak{N}=\mathfrak{R}_{\mathfrak{g}}\times\mathfrak{n}$ *consists of just* g *irreducible constituents (equivalent to* $\mathfrak{D}$).

Proof.‡ Let $E(x)$ denote the principal unit of $\mathfrak{S}$, and put

* Note that a submodule of $\mathfrak{R}_{\mathfrak{h}}$ ($\mathfrak{R}_{\mathfrak{g}}$) with a finite rank with respect to Ω is a left ideal if and only if it is an $\mathfrak{h}$-($\mathfrak{g}$-)left-module; where we define the multiplication of $\alpha\epsilon\mathfrak{h}$ and $f(\xi)\epsilon\mathfrak{R}_{\mathfrak{h}}$, ($a\epsilon\mathfrak{g}$ and $F(x)\epsilon\mathfrak{R}_{\mathfrak{g}}$), by $\alpha\cdot f(\xi)=f(\alpha^{-1}\xi)$, $(a\cdot F(x)=F(a^{-1}x))$. Then we have $(\alpha\cdot f)\times F=\alpha\cdot(f\times F)$, $(a\cdot F)\times f=a\cdot(F\times f)$, and so on.

† Linear aggregates of the matric elements of $\mathfrak{D}$ form a two sided ideal of $\mathfrak{R}_{\mathfrak{g}}$, which is isomorphic to a matric ring.

‡ The following proof is only a slight modification of the proof in Weyl, loc. cit.

$\tilde{\mathfrak{R}} = \mathfrak{S} \times \mathfrak{R} = E \times \mathfrak{R} (= \mathfrak{R} \times \mathfrak{S} = \mathfrak{R} \times E)$. In the case $\tilde{\mathfrak{R}} = 0$ the theorem is obvious. We assume therefore $\tilde{\mathfrak{R}} \neq 0$, that is, $c \times E \neq 0$.

If we denote the degree of the representation $\mathfrak{d}$ by r, then the two-sided ideal $\mathfrak{s}$ of $\mathfrak{R}_{\mathfrak{h}}$ belonging to $\mathfrak{d}$ is a direct sum of r minimal left ideals operator-isomorphic to $\mathfrak{n}$:

$$\mathfrak{s} = \mathfrak{n}^{(1)} + \mathfrak{n}^{(2)} + \cdots + \mathfrak{n}^{(r)}, \qquad \mathfrak{n}^{(i)} \cong \mathfrak{n}.$$

Put $\mathfrak{R}^{(i)} = \mathfrak{R}_{\mathfrak{g}} \times \mathfrak{n}^{(i)}$ and $\tilde{\mathfrak{R}}^{(i)} = \mathfrak{S} \times \mathfrak{R}^{(i)} = \mathfrak{S} \times \mathfrak{n}^{(i)}$. It is easy to see that $\mathfrak{R}_{\mathfrak{g}} \times \mathfrak{s}$ is the direct sum $\mathfrak{R}^{(1)} + \mathfrak{R}^{(2)} + \cdots + \mathfrak{R}^{(r)}$, and also that $\mathfrak{S} \times \mathfrak{R}_{\mathfrak{g}} \times \mathfrak{s} = \mathfrak{S} \times \mathfrak{s} = \tilde{\mathfrak{R}}^{(1)} + \tilde{\mathfrak{R}}^{(2)} + \cdots + \tilde{\mathfrak{R}}^{(r)}$. Now suppose that $\tilde{\mathfrak{R}}$ is a direct sum of h minimal left ideals. (Our purpose is to show $h = g$.) Then each of $\tilde{\mathfrak{R}}^{(i)}$ has the same property: $\tilde{\mathfrak{R}}^{(i)} = \mathfrak{L}_1^{(i)} + \mathfrak{L}_2^{(i)} + \cdots + \mathfrak{L}_h^{(i)}$, for it is operator-isomorphic to $\tilde{\mathfrak{R}}$.

Let $e(\xi)$ be the principal unit of $\mathfrak{s}$. We have $\mathfrak{S} \times \mathfrak{s} = \mathfrak{R}_{\mathfrak{g}} \times (e \times E)$ and

$$e \times E = E \times e \times E = E \times e, \qquad (e \times E)^2 = E \times e \times E = e \times E.$$

Moreover

$$\mathfrak{S} = \mathfrak{S} \times \mathfrak{s} + \mathfrak{L}^* = \mathfrak{L}_1^{(1)} + \mathfrak{L}_2^{(1)} + \cdots + \mathfrak{L}_h^{(r)} + \mathfrak{L}^*$$

for a suitably chosen left ideal $\mathfrak{L}^*$ of $\mathfrak{S}$. From this decomposition we see in the usual manner that $(e \times E) \times \mathfrak{R}_{\mathfrak{g}} \times (e \times E)$ is a matric ring of degree rh (over Ω):

$$(e \times E) \times \mathfrak{R}_{\mathfrak{g}} \times (e \times E) = \sum_{\substack{k,i=1,\cdots,r \\ \mu,\lambda=1,\cdots,h}} C_{\mu\lambda}^{(k)(i)} \Omega,$$

$$(e \times E) \times \mathfrak{L}_\lambda^{(i)} = \sum_{k,\mu} C_{\mu\lambda}^{(k)(i)} \Omega,$$

($C_{\mu\lambda}^{(k)(i)}(x)$ being matric units). Here $(e \times E) \times \mathfrak{L}_\lambda^{(i)} = e \times (E \times \mathfrak{L}_\lambda^{(i)}) = e \times \mathfrak{L}_\lambda^{(i)}$, and therefore $(e \times \mathfrak{L}_\lambda^{(i)} : \Omega) = rh$.

On the other hand $\mathfrak{L}_\lambda^{(i)}$ is, considered as an $\mathfrak{h}$-left-module, completely reducible, for it defines a representation of $\mathfrak{h}$ equivalent to $\mathfrak{D}(\mathfrak{h})$. Let $\mathfrak{L}_\lambda^{(i)} = \mathfrak{M}_1 + \mathfrak{M}_2 + \cdots + \mathfrak{M}_l$ be its decomposition into simple (minimal) submoduli. According to our assumption just g of $\mathfrak{M}_j$ are operator-isomorphic to $\mathfrak{n}$ with respect to $\mathfrak{h}$, and therefore also with respect to $\mathfrak{R}_{\mathfrak{h}}$.† This implies $(e \times \mathfrak{L}_\lambda^{(i)} : \Omega) = rg$.

Comparing with the above result, we obtain $h = g$.

THE INSTITUTE FOR ADVANCED STUDY

† A submodule, with a finite rank, of $\mathfrak{R}_{\mathfrak{g}}$ is an $\mathfrak{R}_{\mathfrak{h}}$-left-module if and only if it is an $\mathfrak{h}$-left-module. Two such moduli are operator-isomorphic with respect to $\mathfrak{R}_{\mathfrak{h}}$ if and only if they are so with respect to $\mathfrak{h}$. The same holds for the isomorphism between such a module and a left ideal of $\mathfrak{R}_{\mathfrak{h}}$.