ON HIGHER DERIVATIVES OF ORTHOGONAL POLYNOMIALS*

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1. *Introduction.* Let $\{\phi_n(x)\}$ be a set of orthogonal polynomials in a finite interval (a, b) with the integrable (L) weight function† $p(x)$, that is,

$$\int_a^b p(x)\phi_n(x)\phi_m(x)dx = 0, \qquad (n \neq m),$$

$$p(x) \geqq 0, \quad \int_a^b p(x)dx > 0, \quad \phi_n(x) = x^n + a_{n,n-1}x^{n-1} + \cdots + a_{n0}.$$

It has been shown‡ that if the first derivatives $\{\phi_n'(x)\}$ also form a set of orthogonal polynomials, then the original set are Jacobi polynomials. The purpose here is to show that if the rth derivatives $\{\phi_n{}^r(x)\}$ form an orthogonal set, then again $\{\phi_n(x)\}$ is a set of Jacobi polynomials. The proof is based on the following lemma.§

LEMMA. *Let $Q(x)$ be non-negative in the (finite or infinite) interval (c, d), and such that the constants β defined by the formula*

$$\beta_k = \int_c^d Q(x)x^k dx, \qquad (k = 0, 1, \cdots),$$

exist, and for a certain positive integer r

$$\int_c^d Q(x)\phi_n(x)G_{n-r-1}(x)dx = 0, \qquad (n = r + 1, r + 2, \cdots),$$

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* Presented to the Society, April 11, 1936.

† There is no restriction in assuming (as we do) that if α, β are any two numbers, $a<\alpha<b<\beta$, then $p(x)\neq 0$ almost everywhere in (a, α); $p(x)\neq 0$, almost everywhere in (β, b).

‡ W. Hahn, *Über die Jacobischen Polynome und zwei verwandte Polynomklassen*, Mathematische Zeitschrift, vol. 39 (1935), pp. 634–638. H. L. Krall, *On derivatives of orthogonal polynomials*, this Bulletin, vol. 42 (1936), pp. 423–428.

§ See Krall, loc. cit.

where $G_n(x)$ is an arbitrary polynomial of degree $\leqq n$. Then almost everywhere

$$Q(x) = \begin{cases} P_r(x)p(x) & \text{in } (a, b), \\ 0 & \text{elsewhere}, \end{cases}$$

where $P_r(x)$ is a polynomial of degree $\leqq r$.

2. *Identity of the Intervals of Orthogonality.* The $\{\phi_n(x)\}$ satisfy the recurrence relations

$$\phi_{n+2}(x) = (x - c_{n+2})\phi_{n+1}(x) - \lambda_{n+2}\phi_n(x), \quad (c_{n+2}, \lambda_{n+2}, \text{ constants}).$$

Differentiating this r times, we obtain

$$\begin{aligned} \phi'_{n+2}(x) &= (x - c_{n+2})\phi'_{n+1}(x) - \lambda_{n+2}\phi'_n(x) + \phi_{n+1}(x), \\ \phi''_{n+2}(x) &= (x - c_{n+2})\phi''_{n+1}(x) - \lambda_{n+2}\phi''_n(x) + 2\phi'_{n+1}(x), \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ \overset{r}{\phi}_{n+2}(x) &= (x - c_{n+2})\overset{r}{\phi}_{n+1}(x) - \lambda_{n+2}\overset{r}{\phi}_n(x) + r\overset{r-1}{\phi}_{n+1}(x). \end{aligned} \tag{1}$$

Let $q(x)$ be the weight function of the orthogonal set $\{\phi_n{}^r(x)\}$ in the interval (c, d). If we multiply the last equation of (1) by $q(x)G_{n-r-1}(x)$ and integrate, we get

$$\int_c^d q(x)\overset{r-1}{\phi}_{n+1}(x)G_{n-r-1}(x)dx = 0, \text{ or } \int_c^d q(x)\overset{r-1}{\phi}_n(x)G_{n-r-2}(x)dx = 0.$$

In this way we obtain successively

$$\int_c^d q(x)\overset{r-1}{\phi}_n(x)G_{n-r-2}(x)dx = 0, \quad \int_c^d q(x)\overset{r-2}{\phi}_n(x)G_{n-r-3}(x)dx = 0, \quad \cdots, \quad \int_c^d q(x)\phi_n(x)G_{n-2r-1}(x)dx = 0. \tag{2}$$

The lemma can be applied to the last equation, whence

$$q(x) = P_{2r}(x)p(x), \qquad (a, b) \equiv (c, d).$$

3. *Existence of $q^r(x)$.* Consider the function

$$S(x) = \int_a^x dx_1 \int_a^{x_1} dx_2 \cdots \int_a^{x_{r-1}} (f_r t^r + f_{r-1}t^{r-1} + \cdots + f_0)p(t)dt$$

$$= \frac{1}{(r-1)!}\int_a^x (x-t)^{r-1}(f_r t^r + \cdots + f_0)p(t)dt,$$

where the constants $f_r, f_{r-1}, \cdots, f_0$ are determined so that

$$S(b) = S'(b) = \cdots = S^{r-1}(b) = 0, \qquad \int_a^b S(x)dx = \int_a^b q(x)dx.$$

From the definition of $S(x)$, it follows that $S(a) = S'(a) = \cdots = S^{r-1}(a) = 0$, and $S^r(x) = (f_r x^r + \cdots + f_0)p(x)$. Successive integration by parts gives

$$\begin{aligned}\int_a^b S(x)\overset{r}{\phi_n}(x)dx &= \int_a^b S'(x)\overset{r-1}{\phi_n}(x)dx = \cdots \\ &= \int_a^b S^r(x)\phi_n(x)dx = 0, \qquad (n \geqq r+1).\end{aligned} \tag{3}$$

Thus

$$\int_a^b S(x)\overset{r}{\phi_n}(x)dx = \int_a^b q(x)\overset{r}{\phi_n}(x)dx = 0, \qquad (n \geqq r+1). \tag{4}$$

Since $\int_a^b S(x)dx = \int_a^b q(x)dx$, and $\phi_n{}^r(x)$ is of degree $n-r$, we have

$$\int_a^b x^n S(x)dx = \int_a^b x^n q(x)dx, \qquad (n \geqq 0),$$

so that $q(x) \equiv S(x)$ almost everywhere. Thus $q^r(x)$ exists almost everywhere and

$$\begin{aligned} q^r(x) &= P_r(x)p(x), \\ q(a) &= q'(a) = \cdots = q^{r-1}(a) = q(b) \\ &= q'(b) = \cdots = q^{r-1}(b) = 0. \end{aligned} \tag{5}$$

4. *Relations for the Derivatives of* $q(x)$. An integration by parts applied to the next to last equation of (2) gives

$$\begin{aligned} 0 = \int_a^b q(x)\phi_n'(x)G_{n-2r}(x)dx &= \int_a^b q'(x)G_{n-2r}(x)\phi_n(x)dx \\ &\quad + \int_a^b q(x)G_{n-2r}'(x)\phi_n(x)dx. \end{aligned}$$

Since $q(x)=P_{2r}(x)p(x)$, the second integral is zero. The other equations of (2) give us successively

$$\int_a^b q'(x)G_{n-2r}(x)\phi_n(x)dx = 0, \quad \int_a^b q''(x)G_{n-2r+1}(x)\phi_n(x)dx = 0,$$
$$\cdots, \quad \int_a^b q^{r-1}(x)G_{n-r-2}(x)\phi_n(x)dx = 0.$$

The lemma applies to these equations and we have

$$\begin{aligned} q(x) &= P_{2r}(x)p(x), & q'(x) &= P_{2r-1}(x)p(x), \cdots, \\ q^{r-1}(x) &= P_{r+1}(x)p(x), & q^r(x) &= P_r(x)p(x). \end{aligned} \tag{6}$$

5. *Determination of $p(x)$*. If we eliminate $p(x)$ from the last two equations of (6), we get

$$q^r(x) = \frac{P_r(x)}{P_{r+1}(x)} q^{r-1}(x), \qquad q^{r-1}(a) = q^{r-1}(b) = 0. \tag{7}$$

We conclude that $P_{r+1}(a)=0$. For suppose not, then $q^r(a)=0$, and (7) is a linear differential equation for $q^{r-1}(x)$, with coefficients analytic at $x=a$. The initial conditions make $q^{r-1}(x)\equiv 0$ (and therefore $p(x)\equiv 0$) in some neighborhood of $x=a$. But this contradicts the condition of a preceding footnote. Similarly, $P_{r+1}(b)=0$.

The same type of argument shows that for $n=1, 2, \cdots, r$, $P_{r+n}(x)$ has n-fold zeros at both $x=a$ and $x=b$. Hence

$$\begin{aligned} P_{2r}(x) &= k(x-a)^r(x-b)^r, \\ P_{2r-1}(x) &= c(x-a)^{r-1}(x-b)^{r-1}(lx+m). \end{aligned}$$

The first two equations of (6) now yield

$$\frac{q'(x)}{q(x)} = \frac{c(x-a)^{r-1}(x-b)^{r-1}(lx+m)}{k(x-a)^r(x-b)^r} = \frac{\alpha}{x-a} - \frac{\beta}{b-x},$$
$$q(x) = K(x-a)^\alpha(b-x)^\beta, \quad p(x) = C(x-a)^{\alpha-r}(b-x)^{\beta-r}.$$

Since this is the weight function of Jacobi polynomials, our proposition is proved.

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