

ABSTRACTS OF PAPERS

SUBMITTED FOR PRESENTATION TO THE SOCIETY

The following papers have been submitted to the Secretary and the Associate Secretaries of the Society for presentation at meetings of the Society. They are numbered serially throughout this volume. Cross-references to them in the reports of the meetings will give the number of this volume, the number of this issue, and the serial number of the abstract.

356. Mr. Garrett Birkhoff: *The composition of modular algebras.*

By a *modular algebra* is meant a group, with or without operators, ring, or hypercomplex algebra. The paper is a systematic study of the ways of building up complex modular algebras from simple constituents. The main contributions are: (1) the general formulation of some new methods of combining algebras, (2) the exact correlation of analogous methods of combination, (3) the extension and simplification of some results of Remak (Journal für Mathematik, 1930–1932) concerning canonical decompositions of finite groups. (Received October 1, 1935.)

357. Mr. Nelson Dunford: *A particular sequence of step functions.*

It is the purpose of this note to establish the existence of a sequence of positive step functions $f_n(t)$ having the property that $f_n(t) = 0$ except for a set whose measure approaches zero with $1/n$ and such that for every summable function $g(t)$, except for those in a certain set of the first category in L , the sequence $\int_0^x f_n(t)g(t)dt$ is everywhere dense in the space of measurable functions. (Received September 30, 1935.)

358. Mr. Nelson Dunford: *Linear operations on L_p to L_q .*

It is the purpose of this paper to establish the following representation theorems. The general linear operation on L to L_2 is given by the formula $Tf = \int_0^1 K(P, Q)f(P)dP$ where the kernel $K(P, Q)$ is measurable, is in L_2 for each P , and the $\text{ess. sup.} [\int_0^1 |K(P, Q)|^2 dQ]^{1/2} < \infty$. This bound gives the norm of T . For a measurable kernel satisfying the relation $[\int |K(P, Q)|^q dQ]^{1/q} \in L_{p/p-1}$ the above transformation is completely continuous on L_p to L_q and the general linear operation is always the limit of a sequence of transformations of this type. The general linear operation on L to a Hilbert space X is given by $Tf = \int x(P)f(P)dP$ where $x(P)$ is an essentially bounded and measurable function with values in X and the integration is in Bochner's sense. Every abstract measurable function $f(P)$ on $(0, 1)$ to L_q and satisfying the condition $\|f(P)\| \in L_p$ can be represented by a measurable function $K(P, Q)$ such

that $\int [f|K(P, Q)|^q dQ]^{p/q} dP < \infty$ and conversely. (Received September 30, 1935.)

359. Dr. E. J. Finan: *A theorem on matrices.*

The purpose of this article is to prove the following result: **THEOREM.** Let $A = (a_{rs})$ be a square matrix of order n with elements in a finite algebraic field $\Gamma(\theta)$ where Γ is the rational field. Let a_{ii} satisfy an equation $f(\lambda) = 0$ of lowest degree m in Γ . Now form the direct product $I \cdot x A$ which is composed of the n^2 scalar matrices $[a_{rs}, a_{rs}, \dots, a_{rs}]$. Replace the scalar matrix in the il position by the companion matrix C of $f(\lambda) = 0$. The characteristic function $\phi(\lambda)$ of the resulting matrix P contains the characteristic function of A as a factor. In fact $\phi(\lambda) = \prod_{i=0}^{m-1} f_i(\lambda)$ where $f_i(\lambda)$ is the characteristic function of A with a_{ii} replaced by its i th conjugate and $f_0(\lambda) = f(\lambda)$. (Received September 18, 1935.)

360. Professor Tomlinson Fort: *Maxima and minima of finite sums.*

In this paper the author gives a general discussion of the maxima and minima of finite sums with the development of a new sufficient condition in terms of the solutions of a recurrent relation. Several particular problems are treated including the development of a necessary form for the minimum surface of revolution obtained by revolving a broken straight line about an axis. (Received September 21, 1935.)

361. Professor Philip Franklin: *Classification of measurable functions.*

The author begins with a series of theorems, for the most part well known, on the possibility of approximating Riemann integrable, summable, and measurable functions by continuous functions. These lead to a classification of functions with some advantages over a similar one due to Carathéodory. The author concludes with some results on B. a. p. functions equivalent in the sense of Besicovitch. (Received September 30, 1935.)

362. Mr. Michael Goldberg: *A class of multi-symmetric polyhedra.*

This note shows the construction of all the multi-symmetric trihedral convex polyhedra possessing only hexagons in addition to twelve pentagons (or six quadrilaterals or four triangles) regularly and symmetrically distributed. Among these are found sets of polyhedra which have the anomalous property of being topologically different even though they possess the same number of polygons in their sea of hexagons and their twelve pentagon islands (or six quadrilaterals or four triangles) are regularly and symmetrically disposed. (Received September 13, 1935.)

363. Professor Harold Hotelling and Margaret Richards Pabst: *Rank correlation and tests of significance involving no assumptions of normality.*

The rank correlation coefficient is treated primarily as a test of the existence

of relationship between two variates, rather than as a measure of correlation. In this way it is possible to find exact probabilities without making any assumption of a bivariate normal population. A proof is given for the first time that the distribution approaches normality of form for large samples, a proposition generally assumed for practical purposes, which, however, is apparently not a corollary to any of the known general theorems on approach to normality. Other tests of significance that avoid the assumption of an underlying normal distribution are discussed. A method of combining the evidence for the association contained in a 2 by s contingency table, using both χ^2 and the order relations of the frequencies, is illustrated. (Received September 5, 1935.)

364. Professor Edward Kasner: *Relative conformal invariants.*

A fundamental problem of conformal geometry is to find the invariants of a horn angle (two curves having ordinary contact at a given point). The author, in 1910, proved the existence of a unique absolute invariant, and gave its geometric interpretation in the form $I_3 = (d\gamma_1/ds_1 - d\gamma_2/ds_2)/(\gamma_1 - \gamma_2)^2$, where γ_1 and γ_2 are the curvatures of the given curves (see Kasner, Proceedings of the International Congress of Mathematicians, Cambridge, 1912, p. 81). From the proof it is obvious that the denominator $(\gamma_1 - \gamma_2)^2$ is a relative invariant of second order, and likewise the numerator $d\gamma_1/ds_1 - d\gamma_2/ds_2$ is a relative invariant of third order. In a recent paper Ostrowski proves again the relative invariance of $\gamma_1 - \gamma_2$, which he denotes by the symbol $\beta(1, 2)$, and he gives many beautiful applications of this simple invariant (Jahresbericht der Deutschen Mathematiker-Vereinigung, 1934). In the present paper it is shown that each of these relative invariants is by itself sufficient to characterize the conformal group. The absolute invariant I_3 serves to characterize conformal among all contact transformations. If three curves are in contact, then $(\gamma_1 - \gamma_2)/(\gamma_1 - \gamma_3)$ is not only an absolute conformal invariant, but also an absolute topological (analytic restricted) invariant. An absolute topological invariant Ω_6 involving the curvatures of six curves was given by the writer in the American Journal of Mathematics, 1906, p. 210, with application to the result for the conformal case on p. 213. The equiangular group gives rise to an absolute univariant J_3 which is the dual of I_3 . In particular the dual of Ostrowski's relative invariant is $\gamma_1 - \gamma_2$, the difference of the radii of curvatures. (Received September 14, 1935.)

365. Professor Edward Kasner and Dr. George Comenetz: *Conformal geometry of horn angles.*

A horn angle is the figure formed by two curves which have (simple) contact at a point. Under the group of conformal transformations in the plane, a horn angle has a single absolute invariant of finite order; this is the fraction $(\delta_1 - \delta_2)/(\gamma_1 - \gamma_2)^2$, where γ_1, γ_2 are the curvatures of the two curves at the point, and δ_1, δ_2 are the rates of change of curvature with respect to arc-length (Kasner, *Conformal geometry*, Proceedings of the Cambridge Mathematical Congress (1913), vol. 2, p. 81). If n curves are drawn through a point, all in the same direction, they form $C_{n,2}$ horn angles. The $C_{n,2}$ values of the invariant are not all independent when n exceeds 3. It is found, in fact, that if four curves are drawn, forming six horn angles, the six values of the invariant must satisfy

a certain long equation of twelfth degree, which is given explicitly. (Received September 27, 1935.)

366. Professor Hans Rademacher: *On prime numbers on real quadratic fields in rectangles.*

The existence of a totally positive fundamental unit gives rise to a periodicity in certain finite sums, of which the number of prime numbers in a rectangle is a special case. The periodicity is utilized for a Fourier expansion, which leads to an identity for those finite sums. In the case of prime numbers some estimates can be obtained, one of which has been already given by the author in *Acta Arithmetica*, vol. 1 (1935). (Received September 28, 1935.)

367. Professor J. F. Ritt: *On the singular solutions of algebraic differential equations.*

This paper undertakes the study of the singular solutions of algebraic differential equations, particularly of equations of the second order, from the point of view of the theory of irreducible manifolds. (Received September 27, 1935.)

368. Dr. M. S. Robertson (National Research Fellow): *On the coefficients of an odd schlicht function.*

If $f(z) = z + \sum_{n=1}^{\infty} b_{2n+1} z^{2n+1}$ is regular and schlicht for $|z| < 1$ it is known that $|b_{2n+1}| < A$ for all n , where A is an absolute constant. In particular $|b_3| \leq 1$, $|b_5| \leq e^{-2/3} + 1/2$. When the coefficients are all real, then $|b_{2n-1}| + |b_{2n+1}| \leq 2$. In this paper the author demonstrates that the latter inequality is also true for $n=1$, $n=2$, when the coefficients are complex, numbers, and that $(|b_3| + |b_5|)/2 \leq [(|b_3|^2 + |b_5|^2)/2]^{1/2} \leq 1$. (Received September 28, 1935.)

369. Dr. M. S. Robertson (National Research Fellow): *On the univalence of the Cesàro partial sums of a univalent function.*

If $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ is regular and univalent for $|z| < 1$, the partial sums $s_n(z) = z + \sum_{k=2}^n a_k z^k$ are in general not univalent in the unit circle, but are univalent for $|z| < 1/4$ at least. Under certain conditions, as J. W. Alexander has shown, the partial sums are, however, univalent for $|z| < 1$. The author shows that whenever all the partial sums $s_n(z)$ are univalent in $|z| < 1$ so also are univalent for $|z| < 1$ all the Cesàro sums of the first order. (Received September 28, 1935.)

370. Mr. L. B. Robinson: *A system of Riquier and the tensor calculus.*

Given a complete system $X_i f \equiv A_{1i} \partial f / \partial x_1 + A_{2i} \partial f / \partial x_2 + \dots + A_{ri} \partial f / \partial x_r + 1 = 0$, ($i=1, 2, \dots, n$), where A_{ki} depends on x_1, x_2, \dots, x_k only, is rational with respect to x_1 , and is rational and integral with respect to the other variables, by quadratures one can solve such a system in finite form. Let it be called a system of class K . The tensors of a system of linear homogeneous differential equations satisfy the following system of Riquier: $\partial u_i / \partial x_1 = U_{1i} + \sum_{j=1}^r S_{1j} \partial u_j / \partial S_j$, $\partial u_i / \partial x_2 = U_{2i} + \sum_{j=1}^r S_{2j} \partial u_j / \partial S_j$, \dots , $\partial u_i / \partial x_n = U_{ni} + \sum_{j=1}^r S_{nj} \partial u_j / \partial S_j$, ($i=1, 2, \dots, n$), (see Riquier, *Les Systèmes d'Équations aux Dérivées*

Partielles, p. 502). The integration of the above system can be reduced to the integration of a system of class K . The author has obtained some similar results in the field of functional equations. (Received September 18, 1935.)

371. Professor J. A. Shohat: *On mechanical quadratures.*

First, the general mechanical quadratures formula based on the Lagrange interpolation formula is discussed, with regard to the signs of its "Cotes' numbers" and the distribution of the abscissas employed. Thus some of the results of Stekloff are obtained and extended. Second, the special mechanical quadratures formula is discussed, which uses for abscissas the zeros of the polynomial $\phi_n(x) + A\phi_{n-1}(x) + B\phi_{n-2}(x)$, where A, B are constants and $\{\phi_n\}$ are orthogonal Tchebycheff polynomials. The discussion leads to more precise statement and also an extension of recent results of Fejér (*Mathematische Zeitschrift*, vol. 37 (1933), pp. 287-309). (Received September 25, 1935.)

372. Dr. Abraham Sinkov: *On the simple group of order 5616.*

In 1931 Dr. K. E. Bisshopp obtained necessary and sufficient conditions which two operators, S and T , of periods three and two must satisfy in order to generate the simple group G_{5616} . The theorem consisted of two abstract definitions of the group, each of which required nine distinct restrictions upon the generators. Apparently no attempt was made to study the independence of these relations. The purpose of the present paper is to simplify the results given by Bisshopp and to show that five conditions are sufficient to define the group completely in either case. Four of these conditions give the periods of the generators, their product and their commutator: (3, 2, 13; 4) in one case and (3, 2, 13; 6) in the other. (Received September 10, 1935.)

373. Miss V. E. Spencer: *Note on asymptotic expressions for the zeros of Hermite and Laguerre polynomials.*

By applying the relation between Hermite and the generalized Laguerre polynomials to a result of Zernike asymptotic expressions for the largest zero x_{nn} of Laguerre polynomials with $\alpha = 1/2$ and $3/2$ are immediately obtained accurate to a term in $n^{-5/3}$. By use of Markoff's theorem and a certain argument similar to Zernike's the theory is extended to asymptotic expressions for x_{nn} for any α . (Received September 28, 1935.)

374. Dr. W. R. Thompson: *On confidence ranges for the median and other expectation distributions for populations of unknown distribution form.*

Let $A = \{x\}$ denote the real number domain; and U , called an *infinite population*, denote an unknown frequency-distribution law of draft from $\{x\}$ such that the probability of obtaining x in an arbitrary interval, $P_{(\alpha < x < \beta)} = \int_{\alpha}^{\beta} f(x) \cdot dx$, is unknown, but $f(x) \geq 0$ and bounded in A ; and, for every positive $p < 1$, there exists (α, β) such that $P_{(\alpha < x < \beta)} < p$. Let n drafts, independently thus governed, be made from A without replacements, yielding a sample denoted by $\{x_i\}$ enumerated so that $i < j$ for $x_i < x_j$, $i, j = 1, \dots, n$. Then it is shown that the well known principle of Bayes is strictly applicable to evalua-

tion of several expectations, $\bar{P}_{(R)}$, where R is a relation between sample values and population parameters; for example, if we let p_k denote $P_{(x < x_k)}$, then $\bar{P}_{(p_k < p)} = I_{p(k, n-k+1)}$, the incomplete B -function ratio; if M is the unknown population median of U , then $\bar{P}_{(x_k < M < x_{k+1})} = (1/2)^n \binom{n}{k}$; and $\bar{P}_{(x < x_k)} = k/(n+1)$. Similar relations are developed for a finite population, and best upper and lower bounds are found for some bivariate cases of expectation. (Received September 26, 1935.)

375. Dr. S. E. Warschawski: *On the preservation of angles at a boundary point in conformal mapping.*

Let R be a simply connected "schlicht" region in the w -plane whose boundary contains $w=0$. Let $w=0$ be "accessible" along the Jordan curve L . Suppose that there is a circle $|w| < \rho$ such that the part of the boundary of R inside this circle lies in the angles: (*) $|\arg w - h_+| \leq k_+$, $|\arg w - h_-| \leq k_-$, ($h_- \leq h_+$). Suppose, furthermore, that L connects $w=0$ with a boundary point outside $|w| = \rho$ such that L divides R into two subregions. Let all boundary points of one subregion, which are in $|w| < \rho$ and not on L , be in one of the angles (*), and those of the other subregion, which are in $|w| < \rho$ and not on L , be in the other. Then the author proves: If $w(z)$ maps $|z-1| < 1$ conformally on R such that $z=0$ "corresponds" to the boundary point $w=0$ defined by L , then $|\arg w(z) - H(\arg z)| \leq K(\arg z) + \epsilon$, where $\epsilon \rightarrow 0$ as $z \rightarrow 0$ in any angle $|\arg z| \leq \theta < \pi/2$, and $H(\alpha) = (1/\pi) [\{(\pi/2) + \alpha\} h_+ + \{(\pi/2) - \alpha\} h_-]$, $K(\alpha) = (1/\pi) [\{(\pi/2) + \alpha\} k_+ + \{(\pi/2) - \alpha\} k_-]$. This result is an extension of a theorem of Carathéodory and Lindelöf. (Received September 27, 1935.)

376. Professor Hassler Whitney: *On a theorem of H. Hopf.*

Let K be a complex. Let the "inverse boundary" of an i -cell E^i be the sum of the (properly oriented) $(i+1)$ -cells which have E^i on their boundaries. The author defines in the obvious manner the inverse boundary of any i -chain, inverse cycles, inversely bounding cycles, and hence inverse Betti groups \bar{B}^i (compare Alexander, Proceedings of the National Academy of Sciences, August, 1935; also Whitney, *ibid.*, July, 1935). If f is a map of the n -complex K^n onto the n -sphere S^n , it may be deformed so that all $(n-1)$ -cells lie on a point of S^n ; each cell E_k^n then lies on S^n with a definite degree γ_k . Now $\sum \gamma_k E_k^n$ is an inverse cycle, determining an element of \bar{B}^n . A simple proof is given that the classes of maps of K^n onto S^n are in one-one correspondence with the elements of \bar{B}^n . It is not hard to show that this is equivalent to a theorem of Hopf (*Commentarii Mathematici Helvetici*, vol. 5 (1933)). (Received September 3, 1935.)

377. Dr. Max Zorn: *Elimination of the continuum-hypothesis from algebra.*

Given a system of elements with a definition of dependence satisfying van der Waerden's axioms (see *Moderne Algebra*, p. 204), the author shows that two bases are not only similar but may also be well-ordered such that the sections of elements of equal indices without immediate predecessors are equivalent. Weakening the axioms such that they cover rational dependence the author proves the existence of bases and, in the infinite case, the similarity of all

bases. This similarity theorem yields Steinitz's degree of transcendency, and Baer's degree of infinite algebraic extensions. The axiom of choice is used, but well-ordered is involved only by some of its cardinal consequences. The generalized continuum-hypothesis, essential in Baer's theory, is completely avoided. (Received September 27, 1935.)

378. Professor Eberhard Hopf: *Fuchsian groups and ergodic theory.*

This paper deals with geodesics on surfaces of constant negative curvature. Every geodesic is supposed to be continuable indefinitely in both directions. It is shown that the flow connected with the geodesics problem is always metrically transitive if the surface area is finite. A special case has been treated previously by Hedlund. The method developed in the present paper is novel. The principal tools are potential theory and the *NE* metric introduced by Poincaré. In the case of infinite surface area it is shown that geodesics tend, in general, towards the funnels of the surface. (Received September 10, 1935.)

379. Dr. W. T. Reid: *Sufficient conditions by expansion methods for the problem of Lagrange in the calculus of variations.*

In the present paper a sufficiency theorem for a strong relative minimum in the problem of Lagrange is established by the expansion method used by Levi (*Annali di Matematica*, vol. 21 (1913), pp. 173–218) for the simple problem of the calculus of variations in the plane. The hypotheses of the theorem do not involve any assumption of normality, and have the same generality as those used by Hestenes (*Transactions of this Society*, vol. 36 (1934), pp. 793–818) in the proof of sufficiency theorems by the usual field method for the general problem of Bolza. Use is made of the result relating the strengthened conjugate point condition and the solutions of the accessory differential equations which has been established independently by Morse (*Transactions of this Society*, vol. 37 (1935), pp. 147–160) and the author (*American Journal of Mathematics*, vol. 57 (1935), pp. 573–586). The extension of Levi's method to the problem of Lagrange is not trivial. In particular, it involves a proof of a certain inequality which is satisfied by the Weierstrass \mathcal{E} -function for large values of certain of its arguments. The expansion method may also be extended to prove the results established by Hestenes for the general problem of Bolza. (Received September 10, 1935.)

380. Mr. A. G. Swanson: *Factorial moments.*

Using the generating function of factorial moments, as given by Campbell, as a definition, a formula in the form of a derivative evaluated at a point is obtained for finding the frequency function of a variable which may take on a finite set of values and for which the factorial-moment-generating-function is known. This formula is made use of in obtaining the distribution of means in samples of N independent observations of a variable with known factorial-moment-generating-function. A generalization of a theorem of Pólya is established giving conditions under which if a sequence of factorial-moment-generating-functions has as its limit an ordinary moment generating function, the

corresponding frequency functions has as its limit the function corresponding to the limiting moment generating function. Two methods are used in finding the characteristics of the distribution of $\sigma_{(n)}$ (the n th factorial moment) in samples of N . First a system of symmetrical functions analogous to the semi-invariants of Thiele are introduced and by means of their properties a method of computation of the factorial moments of $\sigma_{(n)}$ is obtained. The second method is based on the known method of finding semi-invariants of moments about a fixed point in samples of N . The first four factorial moments of the first four factorial moments are given. (Received September 12, 1935.)

381. Mrs. G. T. Whyburn: *Rotations around a set of fixed points.*

Let $T(M) = M$ be a topological transformation of a closed set of points M into itself, and let K denote the set of fixed points under T . It is shown first that the components of $M - K$ break up into cyclic rotation groups of finite or infinite order in such a way that the boundary of any one element of a given group is identical with the boundary of any other element of that group. Furthermore, if M is a plane continuous curve, the boundary of any element of a group of order > 2 contains at most two points. If M is a topological sphere, there can exist only one rotation group of order > 1 . Furthermore this is the only rotation group and it is exactly of order 2. (Received September 12, 1935.)

382. Dr. F. G. Dressel: *A note on Young-Stieltjes integrals.*

The paper presents an expansion formula for $\int_{\mathcal{D}}^{\mathcal{A}} f(x) dg(x)h(x)$, serviceable even though $g(x)$ and $h(x)$ have common points of discontinuity. (Received October 3, 1935.)

383. Dr. Saunders MacLane: *The Schönemann-Eisenstein irreducibility criteria in terms of prime ideals.*

By using non-archimedean absolute values in the construction of the ideal factors of a rational prime it is possible to obtain a general simplified statement of the irreducibility theorems of Eisenstein and Schönemann. This statement, which includes practically all the successive generalizations of these theorems for polynomials with rational coefficients (Bauer, Dumas, Kürschák, Ore, Rella, MacLane), runs as follows: Consider the polynomial $f(x)$ as if it were an irreducible equation defining an algebraic field, and use Newton polygons (or absolute values) to construct the prime ideal decomposition of a prime \mathfrak{p} in this purported algebraic field. If \mathfrak{p} then appears to be a power \mathfrak{p}^e of a prime ideal P of apparent degree f , where $e \cdot f$ is the degree of $f(x)$, it follows that the original polynomial $f(x)$ is actually irreducible. (Received October 5, 1935.)

384. Dr. Saunders MacLane: *A construction of prime ideals in terms of absolute values of an algebraic field.*

The problem under consideration is the construction of all the prime ideal factors of a rational prime p in an algebraic field $R(\theta)$ defined by a given equation $f(x) = 0$. Ore's convenient method for this construction in terms of Newton

polygons is known to fail in certain exceptional cases. These exceptions can be avoided by a general and abstract formulation of the procedure in terms of absolute values. This depends on the known fact that every prime ideal of $R(\theta)$ corresponds to a non-archimedean absolute value of $R(\theta)$. But each such absolute value is equivalent to an absolute value of the polynomial ring $R[x]$, in which $f(x)$ and all its multiples have the value $+\infty$. These values of $R[x]$ are in turn constructed by an extension of the method outlined in a previous paper by the author (Proceedings of the National Academy, vol. 21 (1935), p. 472). The arithmetic properties of the discriminant of $f(x)$ show that this method of construction will give all the prime ideal factors of p after a finite number of steps. The results (including the degrees and exponents of the prime ideals) can be formulated without recourse to ideal-theory. (Received October 5, 1935.)

385. Dr. Solomon Kullback: *On certain distributions derived from the multinomial distribution.*

The multinomial distribution gives the probabilities for the various combinations of events possible in N trials where in each trial one of n mutually exclusive events occur. In the N trials some of the possible events may not occur, others may occur once, twice, . . . , each. The general problem we consider is that of studying the simultaneous distribution of the number of events which occur r, s, \dots, t times each. The theory is illustrated and verified by specific examples. The theory discussed has certain practical applications. (Received November 2, 1935.)

386. Professor C. C. Latimer: *On the class number of a quaternion algebra with a negative fundamental number.*

Let A be a rational generalized quaternion algebra with a negative fundamental number, i.e., containing an element with a negative norm. It is shown that every one-sided ideal in an arbitrarily chosen set G' of integral elements in A is a principal ideal. A special set G of integral elements which contains a certain ring G_1 is considered. It is known that there is a correspondence between the classes of regular left ideals in G_1 and certain classes of Hermitian forms. From this it is shown that every left ideal in G is a principal ideal. The first result mentioned above follows from this and a result of Brandt's. Theorems on the existence of a g.c.r.d. and a g.c.l.d. of two elements and on the factorization of an element in G' are obtained. These results are similar to well known results of Hurwitz for a certain set in the classic quaternion algebra. (Received October 11, 1935.)

387. Professor L. A. Dye: *Involutorial space transformations associated with a rational ruled surface. II.*

Two involutorial transformations are studied in this paper which arise from a one-one correspondence between the cubic surfaces of a pencil and the generators of a ruled quartic surface. Each surface of a pencil of cubic surfaces having two fixed double points is tangent to a plane along the line l through the nodes. This plane cuts a line r from the surface and these lines r generate a

ruled quartic surface having l as a triple line. A point P of space selects a cubic surface and hence a line r . The plane Pr is tangent to the quartic surface at a point Q on r . The line PQ meets the quartic surface in a residual point P' , the associate of P in an involutorial transformation. A similar transformation arises when a pencil of cubic surfaces with a double line is used. In each of these transformations the generators of the quartic surface are parasitic lines of the transformation and the quartic surface is shed off from the transformation three times. (Received October 14, 1935.)

388. Professor Arnold Emch: *Two notable classes of hypersurfaces.*

Since the Cayley cubic and the Steiner quartic surfaces are dual to each other and have other interesting properties, they have always attracted considerable attention among geometricians. In this paper two dual classes of hypersurfaces in S_{2n-1} are discussed which reflect in a generalized form many of the properties of the Cayley and Steiner surfaces. The author calls these Cayley and Steiner hypersurfaces. (Received October 16, 1935.)

389. Professor R. F. Rinehart: *Some properties of the discriminant matrices of a linear associative algebra.*

MacDuffee and L. E. Bush investigated some of the properties of the discriminant matrices of a linear associative algebra by reducing them to a diagonal form by means of transformations of basis of the algebra. Thence they derived a set of cyclic relations among the constants of multiplication, by means of which it is possible to derive some interesting properties of the discriminant matrices and the characteristic equations. In the present paper a revision of the reduction of Bush (this Bulletin, vol. 38 (1932), pp. 49-51) is given and consequences of the cyclic relations are further developed, with particular attention to the case of a non-simple algebra. The implications of the equality of the discriminant matrices are considered, and sufficient conditions for their equality are derived. Applications of some of these results to the particular case of the algebra generated by an algebraic equation are noted, yielding, in particular, new viewpoints on the Newtonian identities and the Borchardt-Jacobi theorem. (Received October 21, 1935.)

390. Dr. Leo Zippin: *Note on a problem of E. Čech.*

Professor E. Čech has introduced the following definition of local connectedness: A space shall be said to be locally connected provided every finite covering by open sets contains a finite covering by connected sets. He has raised the question whether such a space (given that it is a regular topologic space) is necessarily bicomact. The author proves the following theorem: *If a compact topologic space (not necessarily regular) is locally connected in the usual sense, then it is locally connected in the sense of Čech.* This leads easily to a negative solution of the problem. However, for regular spaces, the property of Čech does imply compactness, and also local connectedness in the usual sense. (Received October 23, 1935.)

391. Dr. Leo Zippin: *On monotonic complete covering systems.*

Professor Lefschetz has raised the question of how far the projection-sequence theory of Alexandroff, which has its origins in compact metric spaces, can be carried over (admitting *uncountable* "sequences") to the homology theory of Čech, which embraces abstract spaces. The author proves a theorem which has the following consequence: in order that a Hausdorff space possess a *monotonic complete* system of finite coverings by open sets, it is necessary and sufficient that it be compact metric. In this direction, then, no generalization seems possible. (Received October 23, 1935.)

392. Dr. Solomon Kullback: *On certain factorial sums.*

The problem here considered may be stated as follows: Evaluate the sum $S_{r,s,\dots,t}(n, N, a_1, a_2, \dots, a_n) = \sum [N!/(x_1!x_2! \dots x_n!)] (a_1^{x_1}a_2^{x_2} \dots a_n^{x_n})$ where a_1, a_2, \dots, a_n are any constants and the summation is for all values of x_1, x_2, \dots, x_n , such that $x_1 + x_2 + \dots + x_n = N$ with no x equal to r, s, \dots, t . Only positive integral values (including zero) of $x_1, x_2, \dots, x_n, N, r, s, \dots, t$ are considered. (Received October 24, 1935.)

393. Professor J. J. Stoker, Jr.: *The form of surfaces of positive curvature in three-dimensional space.*

The starting point of this paper is a theorem due to Hadamard (Journal de Mathématiques Pures et Appliquées, (5), vol. 3 (1897)) which is usually stated in the following form: A closed surface in three-dimensional space which is free of singularities and whose Gaussian curvature is everywhere positive is topologically equivalent to the sphere, that is, it is an ovaloid. The purpose of the present paper is to generalize this theorem for open surfaces. At the same time another proof of the theorem of Hadamard is given. An essential assumption in proving the theorems of this paper is that of the *completeness* of the surfaces. It is shown that every *open* surface which is complete possesses the following properties: (1) it is topologically equivalent to the plane, (2) it is the boundary of an unbounded convex point set, (3) the spherical representation of the surface lies on a hemisphere and the correspondence set up between the points of the surface and those of the spherical representation is one-to-one, hence the surface possesses a total Gaussian curvature $\leq 2\pi$, (4) the *entire* surface admits of a representation in the form $z=f(x, y)$ with f one-valued. (Received October 24, 1935.)

394. Professor C. N. Moore: *On necessary and sufficient conditions for convergence factors in multiple series.*

In a paper presented at the February meeting of the Society in New York (see this Bulletin, abstract 41-1-56) the author gave sufficient conditions for convergence factors in series summable by Nörlund means. The object of the present paper is to modify those conditions in such a manner as to make them both necessary and sufficient. The results thus obtained constitute a generalization to multiple series of the theorems for single series contained in a note recently published in the Proceedings of the National Academy of Sciences, vol. 21 (1935), pp. 263-266. (Received October 25, 1935.)

395. Professor C. C. MacDuffee: *A recursion formula for the polynomial solutions of differential equations with constant coefficients.*

Spampinato (Atti della Reale Accademia Nazionale dei Lincei, Rendiconti (6), vol. 21 (1935), pp. 73–76) has used the companion matrix of the auxiliary equation of a homogeneous differential equation with constant coefficients to obtain infinitely many solutions. The method is here extended to obtain a recursion formula for all homogeneous polynomial solutions of each degree. Computation by this formula is rapid. (Received October 28, 1935.)

396. Professor W. D. Baten: *Frequency distributions of means of n independent variables whose frequency distributions can be dissected into component frequency laws.*

This paper develops distributions of arithmetic averages of n independent variables under the assumption that the frequency distribution of each of the variables can be dissected into two component frequency distributions. A general situation is first presented after which special cases are considered. The last section shows that the Gaussian law is equal to the sum of an infinite number of Gaussian laws. (Received October 29, 1935.)

397. Professor L. E. Dickson: *A new method for Waring theorems with polynomial summands. II.*

Instantaneous deduction of a Waring theorem for a polynomial $f(x)$ of degree $2n$ from a known Waring theorem for a polynomial of degree n is described. In a recent number of the Transactions of this Society the case in which $f(x)$ involves only even powers of x was treated. Here the new case in which $f(x)$ contains also a term in x is discussed. (Received October 30, 1935.)

398. Dr. C. W. Vickery: *Abstract vector spaces of uncountably many dimensions.*

The author has obtained, for each ordinal number α , a set \mathfrak{G}_α of axioms in terms of the undefined notions, *vector*, $+$, \cdot , $\| \cdot \|$, χ , and Σ , such that if S is any set of objects called vectors satisfying these axioms, then S is isomorphic and isometric with the set of all vectors of space D_{ω_α} . In this treatment no topological notions are employed (overtly). The notions of an uncountable basis and the sum of an uncountable sequence of vectors are introduced. (Spaces D_{ω_α} are defined in a paper recently presented to the Society.) (Received October 30, 1935.)

399. Professor L. W. Cohen: *Cauchy convergence in topological space.* Preliminary report.

The notions of Cauchy convergence, total boundedness, and completeness are defined in topological spaces satisfying the first denumerability axiom. It is proved that if the space is complete every totally bounded set is compact. A sufficient condition for the converse is given. The problem of immersing such spaces topologically in complete spaces is discussed. (Received October 31, 1935.)

400. Mr. H. Reingold: *Generalized determinants of Vandermonde*. II.

Let G_{ij} , ($i, j=1, 2, \dots, n$), be a square matrix of g , let $G_{rs}^{(m)}$ stand for the element in the r th row and the s th column of the m th power of the matrix G_{ij} , and let $G_{ij}^{(0)}$ denote the Kronecker δ . Consider the n by n determinant $|G_{r_1 s_1}^{(m_1)} G_{r_2 s_2}^{(m_2)} \dots G_{r_n s_n}^{(m_n)}|$, ($i=1, 2, \dots, n$), in which the i th row is exhibited, where r_1, r_2, \dots, r_n and s_1, s_2, \dots, s_n are any specific permutations of the integers $1, 2, \dots, n$, and $m_1 < m_2 < \dots < m_n$. Denote this determinant by $\Delta(m_1, m_2, \dots, m_n)$. In a previous paper (see this Bulletin, abstract 41-5-230) certain properties of these determinants were considered. In the present note a more general theorem is stated of which the previous results are special instances. This last property may be stated in the following form: $\Delta(m_1, m_2, \dots, m_n) = g \cdot \Delta(m_1-1, m_2-1, \dots, m_n-1) = g^{m_1} \cdot \Delta(0, m_2-m_1, \dots, m_n-m_1)$. In case the matrix G_{ij} is symmetric, the last result of this paper may be obtained by a method supplied by Professor G. Szegő, namely, that of reducing the associated quadratic form to a sum of squares. (Received October 31, 1935.)

401. Professor F. John: *On the moments of inertia of convex regions*. I.

A convex region in the plane may be called "normal," if the ellipse of inertia corresponding to its center of gravity is a circle. If d denotes the greatest, Δ the least diameter, and A the area of a normal region R , then $d \leq 2^{1/2} \Delta$, $A \leq \Delta^2$. If R is symmetrical with respect to a point, then also $A \geq d^2/2$. Moreover, if S is a convex body, then through an interior point of S there passes a plane such that its intersection with S is normal. (Received November 1, 1935.)

402. Professor F. John: *On the moments of inertia of convex regions*. II.

Let R be a normal convex region with a center O , (i.e., R may be symmetrical with respect to a point O , and the ellipse of inertia of R corresponding to O may be a circle). If a and b are the sides of any rectangle circumscribed about R , then $a/b \leq \alpha$, where $\alpha = 3[2 + (100)^{1/3}]^{-1/2} = 1.164 \dots$ (α being the exact constant in this connection). (Received November 1, 1935.)

403. Mr. C. B. Tompkins: *A note on complete covariant differentiation*.

E. H. Cutler (Transactions of this Society, vol. 33 (1929), p. 832) has defined by example complete covariant differentiation. It is a method of adding quantities to the derivatives of a complete tensor to make a complete tensor. A set of quantities, $A_{\alpha\beta \dots \phi}^{ab \dots \sigma}$ whose indices may have several ranges forms a complete tensor if it transforms as a tensor under transformations of coordinates to which each of its index ranges appertains. In this paper complete covariant differentiation is considered from the standpoint of the coefficients of connection. To do this a Douglas space of paths is set up; it leads to unsurprising conclusions. This space has a number of dimensions equal to the sum of the numbers of dimensions appertaining to the various types of indices encountered. It

may be possible to reduce the number of dimensions of this introduced space. This problem is equivalent to finding conditions for the tensor analysis of a space to be isomorphic with the tensor analysis of a subspace and a variety in that subspace. These conditions are similar to generalized Gauss and Codazzi equations of the variety. (Received November 1, 1935.)

404. Professor W. J. Trjitzinsky: *Theory of linear differential equations containing a parameter.*

An equation of n th order is considered with coefficients indefinitely differentiable, in x , on a certain interval. These coefficients are given asymptotic to series in negative integral powers of a complex parameter λ , with a finite number of positive powers of λ possibly present. Important results, under various restrictions, were previously obtained by G. D. Birkhoff, J. D. Tamarin, and P. Noaillon. In the present work *the fundamental existence theorem establishes the asymptotic character of the solutions under no restrictions on the formal series solutions.* This theorem is applied to several problems. The paper will appear in the Acta Mathematica. (Received November 1, 1935.)

405. Professor Tobias Dantzig: *Note on the history of elliptic functions.*

One of the earliest contributions to the history of elliptic integrals was made by di Fagnano in 1716. The idea was later generalized by Euler, and eventually, at the hands of Legendre and Jacobi, led to the addition theorems. Let $F(x) = \int_0^x [(Ax^2+B)/(Cx^2+D)]^{1/2} dx$: Fagnano considers the relation $F(x) + F(\bar{x}) = k \cdot \bar{x}$, and shows that for certain values of the constant k this generally transcendental expression reduces to a biquadratic in x and \bar{x} . If $F(x)$ be regarded as the length of an arc in a central conic, the relation establishes an algebraic correspondence between two points of the conic. In the present paper, there is given a geometrical interpretation of the Fagnano correspondences. It turns out that of the two cases studied by Fagnano, one reduces to $R\bar{R} = ab$, and the other to $y\bar{y} = b^3/a$, where a and b denote the semi-axes of the conic, and y, \bar{y} , and R, \bar{R} are the ordinates and the radii of curvature at corresponding points. (Received November 1, 1935.)

406. Professor L. E. Dickson: *On Waring's problem and its generalization.*

By a modification of the latest method of Vinogradov in the current Annals of Mathematics, his small number of n th powers whose sum yields every large integer N is reduced by ten or more. The same conclusion is obtained when the n th powers are multiplied by any given positive integer. A C is found for which the above results hold for every $N > C$. Universal Waring theorems are then deduced. If q denotes the greatest integer $< (3/2)^n$, then $q^{2^n} - 1$ is the sum of $I = q + 2^n - 2$, but no fewer, n th powers. No integer has been found which requires more than I powers. This ideal I is 4223, 8384, 16673 when $n = 12, 13, 14$. When $n = 12$ to 17 it is proved that every positive integer is a sum of I n th powers. Moreover, for $n = 12$ or 13, every integer $\geq 2 \cdot 3^n$ is a sum of 2757 or 4342 n th powers; every integer $\geq 4^{14}$ is a sum of 5238 fourteenth powers. These results fail for slightly smaller integers. (Received October 15, 1935.)

407. Professor Hassler Whitney: *A function not constant on a connected set of critical points.*

A function $f(x, y)$ with continuous first derivatives is defined such that $\partial f/\partial x = \partial f/\partial y = 0$ at every point of a nonrectifiable arc, while the function is not constant on the arc. The paper will appear in the Duke Mathematical Journal. (Received October 25, 1935.)

408. Dr. E. L. Post: *Polyadic groups.* Preliminary report.

In 1928 W. Dörnte generalized the concept of abstract group to allow for an operation involving an arbitrary number of elements m . In the present investigation the concepts of substitution and transformation are also generalized. An identity is seen to be an $(m-1)$ -ad of elements, and corresponding definitions of inverse and transform result. The author shows that every abstract m -adic group (Dörnte's m -group) can be represented as an m -adic substitution group. There results the following theorem. Every abstract polyadic group can be represented as a coset of an invariant subgroup of an ordinary group (and conversely for finite groups). Further work is largely restricted to finite groups and includes, for abstract groups, a complete generalization of the theory of cyclic groups and of the first and third parts of Sylow's theorem under the condition g/p^B prime to $(m-1)$. For substitution groups results on odd and even substitutions, sets of intransitivity, and Jordan's theorem for regular groups have been generalized, while for linear groups the elementary theorems, other than for similarity transformations, go over. Still in progress is the determination of the m -adic linear groups in two variables for the case $m=3$. (Received October 26, 1935.)

409. Professor Louis Weisner: *Criteria for the compositeness of finite groups.*

This paper includes a number of compositeness criteria which are closely related to those of Frobenius and Burnside. Let G be a group of order $p^\alpha m$, where p is a prime that does not divide m , let H be a subgroup of order p^α of G , and let $N(P)$ denote the normalizer in G of a subgroup P of G . The following are among the criteria given. 1. A necessary and sufficient condition that G have an invariant subgroup of index p that does not include a particular element S of H is that H have a maximal subgroup I that does not include S , such that every conjugate of S under G that is contained in H have the form SV , where V is an element of I , while every conjugate of S_p^e ($e=1, \dots$) under G that is contained in H is contained in I . 2. If H is an abelian group, a necessary and sufficient condition that G have an invariant subgroup of index p is that $N(H)$ have an invariant subgroup of index p . 3. If a subgroup L of the central of H is invariant under $N(H)$ and under every subgroup of order p^α of G that contains L , and if $N(L)$ has an invariant subgroup of index p , then G has an invariant subgroup of index p . (Received October 26, 1935.)

410. Mr. Alvin Sugar: *A cubic analogue of the Cauchy-Fermat theorem.*

Fermat stated and Cauchy proved the following theorem for polygonal

numbers: every positive integer is a sum of $m+2$ values of $m(x^2-x)/2+x$. Recently, James conjectured the existence of similar theorems for cubic polynomials (American Journal of Mathematics, vol. 56 (1934), p. 305). This paper establishes such a theorem for the cubic polynomials $P(x) = m(x^3-x)/6+x$. The writer evaluated Dickson's constant (Transactions of the American Mathematical Society, vol. 36 (1934), pp. 1-12) and by developing a powerful ascension theorem and ascension methods was able to prove that every positive integer is a sum of $m+3$ values of $P(x)$ for $m \geq 16$. Since for $m \geq 3$ there will always exist a positive integer $I(m)$ which is not a sum of fewer than $m+3$ values of $P(x)$, it follows that the above theorems are ideal universal Waring Theorems. (Received October 12, 1935.)

411. Professor H. P. Robertson: *Kinematics and world-structure*.

The idealized cosmological problem, in which nebulae are considered as particles in homogeneous flow, is analyzed from the standpoint of the operational methodology, allowing the fundamental observers the use only of clocks, theodolites, and light-signals. It is found, as an extension of the Helmholtz-Lie solution of the problem of physical space, that such a space-time necessarily admits the introduction of an invariant Riemannian metric of precisely the form and generality of that on which the general relativistic theory of cosmology is based, and in terms of which all given elements can be interpreted in the same way as in the relativistic theory. (Received October 14, 1935.)

412. Professor Morgan Ward: *A calculus of sequences*.

An extensive generalization of portions of the formal parts of algebraic analysis and the calculus of finite differences is obtained by systematically replacing the ordinary binomial coefficient $n(n-1) \cdots (n-r+1)/1 \cdot 2 \cdots r$ in formulas by the expression $[n, r] = u_n u_{n-1} \cdots u_{n-r+1}/u_1 u_2, \cdots, u_r$ where $u_0=0, u_1=1, u_2, \cdots, u_n, \cdots$ is a fixed sequence of complex numbers subject to the single restriction $u_n \neq 0, n > 1$. As an example of the type of results obtained, if the difference $\Delta\phi(x)$ of a polynomial $\phi(x) = \sum_{r=0}^n a_{n-r} x^r$ is defined to be $\sum_{r=0}^n \sum_{s=1}^r a_{n-r} [r, s] x^{r-s}$ and the derivative $D\phi(x)$ is defined to be $\sum_{r=1}^n c_{n-r} [r, 1] x^{r-1}$, then symbolically $\Delta = \mathcal{E}(D) - 1$ where $\mathcal{E}(t) = 1 + \sum_{i=1}^{\infty} \alpha^i t^i / u_1 u_2 \cdots u_n$ is the series replacing the exponential $e^t = 1 + \sum_{i=1}^{\infty} t^i / 1 \cdot 2 \cdots i$. (Received October 21, 1935.)

413. Professor W. M. Whyburn: *Complexes and manifolds represented by functions of real variables*.

In this paper it is shown that the equation $f(x_1, x_2, \cdots, x_n) = 0$, where the real function f is continuous in the real variables (x_1, \cdots, x_n) throughout the interior and boundary of an n -dimensional rectangle H , defines an $(n-1)$ -complex within and on the boundary of H provided that at least one point of H satisfies this equation and, furthermore, one of the difference quotients $Q_i = [f(x_1, \cdots, x_{i-1}, x'_i, x_{i+1}, \cdots, x_n) - f(x_1, x_2, \cdots, x_n)] / [x'_i - x_i]$ is definite in sign throughout a neighborhood of each point of the locus of this equation

that belongs to H and its boundary. A maximal connected subset of the above described locus which is interior to H is shown to be an $(n-1)$ closed manifold in the analysis situs sense (see Veblen, Colloquium Publications, Volume V, Part II, page 92). The locus defined by k equations of the foregoing type is considered and this locus is shown to be an $(n-k)$ complex when hypotheses of the above type are met by certain k th order difference Jacobians. The paper makes use of a general implicit function theorem that is due to Hedrick and Westfall (Bulletin de la Société Mathématique de France, Vol. 44, pp. 1-14). (Received October 28, 1935.)

414. Professor Clifford Bell: *On a determinant function involving the parameter of a plane curve.*

The function $H_{ij}(t)$ is defined as the determinant $|f_i(t) g_j(t)|$ where the subscripts in the determinant indicate the order of the derivatives. The properties of this function are developed, especially those of geometric significance, where $f(t)$, $g(t)$ are taken as the coordinates of any point on an analytic curve. It is shown that $H_{ij}(t)$ may be used in finding inflection points and those multiple points at which two or more tangent lines are coincident. (Received October 29, 1935.)

415. Professor E. T. Bell: *Distributivity of polynomial compositions.*

All polynomial solutions $P(u, v)$, $S(u, v)$ of the functional equations for the associativity and distributivity (right, or left, or both) of P , S are determined, where u, v are elements of a domain of integrity (commutative ring with no nilfactors). This problem arises in the theory of numerical functions. There are precisely twelve solutions. Among the solutions, the fields constitute a three-parameter family. Commutativity of P , S is not assumed. (Received October 31, 1935.)

416. Mr. D. L. Webb: *Definition of Post's generalized negative and maximum in terms of one binary operation.*

In volume 43, pages 163-185 of the American Journal of Mathematics, E. L. Post demonstrates that it is possible to construct a function for any order table in a system of m truth-values by the use of two primitive functions which are generalizations of the negative and maximum in two-valued logic. In this paper Post's generalized negative and maximum are defined in terms of one binary operation. (Received Oct. 31, 1935.)

417. Professor Harry Bateman: *Two systems of polynomials.*

A study is made of the polynomials whose generating function is a power of $1-t^2$ multiplied by a sine or cosine of $2xt(1-t^2)^{-1/2}$. Alternative generating functions are found with the aid of generalized hypergeometric functions and by using the set of generating functions thus extended numerous recurrence relations are obtained. (Received November 1, 1935.)

418. Mr. I. E. Highberg: *A note on abstract polynomials in complex spaces.*

The definition of polynomials in an "espace algébrophile" given by Fréchet is compared with the definition considered by Martin and Michal. It is to be noted that Fréchet's definition is essentially that a function is a polynomial if it is continuous, and if, for some integer n , the $n+1$ th difference of the function vanishes identically. On the other hand Martin defines $p(x)$ to be a polynomial if it is continuous and if $p(x+\mu y)$ is a polynomial in μ with coefficients depending on x and y . The main result of this note is to show that in complex "espaces algébrophiles" the two definitions are equivalent if we add the requirement of Gateaux differentiability *everywhere* to Fréchet's conditions. (Received November 1, 1935.)

419. Dr. R. D. James: *The number of representations of an integer as a sum of six, ten, and fourteen squares.*

Let $N(2^\alpha m, s)$ denote the number of representations of an integer $2^\alpha m$, $\alpha \geq 0$, m an odd integer > 0 , as a sum of s squares. Let $\xi_r(m)$ denote the excess of the sum of the r th powers of the divisors of m which are $\equiv 1 \pmod{4}$ over the sum of the r th powers of the divisors of m which are $\equiv 3 \pmod{4}$. The following results are proved in this paper: (1) $N(2^\alpha m, 6) = 4 \{ (-1)^{(m-1)/2} 2^{\alpha+2} - 1 \} \xi_2(m)$, $m \equiv 1$ or $3 \pmod{4}$; (2) $5N(2^\alpha m, 10) = -4(2^{4\alpha+4} - 1)\xi_r(m)$, $m \equiv 3 \pmod{4}$; (3) no relations of the form $N(2^\alpha m, 10) = c(\alpha)\xi_i(m)$, $m \equiv 1 \pmod{4}$ or $N(2^\alpha m, 14) = d(\alpha)\xi_i(m)$, $m \equiv 1$ or $3 \pmod{4}$ are possible with $c(\alpha)$ and $d(\alpha)$ independent of m . The results for representations by six squares and ten squares have been stated before in different forms (for reference see L. E. Dickson, *History of the Theory of Numbers*, vol. II, chapter IX) but the method of proof here presented is believed to be new. The result for fourteen squares is new. (Received November 1, 1935.)

420. Mr. Wm. A. Mersman: *Abstract integration.*

This is a continuation of a previous paper (see this Bulletin, abstract 40-11-382). The definition of integrability is generalized in such a way that no special treatment is required for unbounded functions or for sets of infinite measure (see also Garrett Birkhoff, *Transactions of this Society*, vol. 38 (1935), pp. 357-378). It is shown that this integral is included in that of Birkhoff, while, conversely, if his "unconditional convergence" is equivalent to absolute convergence, then his integral is included in this. The present approach to the subject seems to be just as powerful and admits of much simpler proofs. By means of a known method (*Bulletin of this Society*, vol. 37 (1931), p. 561) the validity of Egoroff's theorem is established for such spaces and functions and is even extended to the case of functions of two continuous variables instead of one continuous and one integral variable. By using this generalization of Egoroff's theorem, the extension of Lebesgue's general convergence theorem is obtained. (Received November 1, 1935.)

421. Professor A. D. Michal and Mr. V. Elconin: *Differential properties of abstract transformation groups with abstract parameters.*

For any x, α in domains D, Δ of two Banach spaces let $f=f(x, \alpha)$ be in D . It is shown in this paper that if the transformations $y=f(x, \alpha)$ form a differentiable group, then for any x, α in D, Δ we have $d_{\xi}^{\alpha} f=U(f, V(f, \Omega(\alpha, \xi)))$, where the left member is the Fréchet differential of f with the increment indicated, and the functions $U(x, y), V(x, \alpha), \Omega(\alpha, \beta)$ are linear in their second arguments. It is further shown that under certain hypotheses on U, V, Ω , the solutions $f(x, \alpha)$ of this differential equation which satisfy $f(x, 0)=x$ generate a differentiable transformation group. Hence two fundamental theorems in the classical theory of Lie are generalized to abstract transformation groups with abstract parameters. (Received November 1, 1935.)

422. Professor A. D. Michal and Mr. D. H. Hyers: *The solution of a second order differential system in general analysis as a function of the boundary values.*

This paper is a sequel to a previous one (see abstract 41-9-310) wherein the existence of the solution of $d^2x/dt^2=F(t, x, dx/dt)$ subject to a two-point condition was proved, x being a variable in a Banach space. Here the solution is shown to be continuous and under certain conditions differentiable in the boundary values. As an instance, a simple differential equation in a normed ring is considered. Applications are made to systems of integro-differential equations and to the differential geometry of certain function spaces. (Received November 1, 1935.)

423. Professor A. D. Michal and Mr. E. W. Paxson: *Maps of abstract topological spaces in Banach spaces.*

Homeomorphic maps of topological neighborhoods are made on regions in Banach spaces and in subsets of these spaces. The topological space is metrized through the norm map in the Banach space B . With suitable restrictions placed on the mappings, for example, that the maps in B of the neighborhoods be open sets, bounded or not, and that the maps be of given class, some of the analysis of the topological space may be discussed. (Received Oct. 30, 1935.)

424. Mr. A. E. Taylor: *Analytic functions in general analysis.* Preliminary report.

As has been remarked by Wiener, it is not difficult to show that the ordinary theory of analytic functions of one or more complex variables may be extended to the case that the values of the functions lie in a complete vector space. With this in mind the author has defined analytic functions on one (complete) vector space to another, in such a way that the above theory plays a fundamental role. The resulting theory has been developed from the Cauchy point of view. Results thus far obtained include the following: An analytic function has

Gateaux differentials of all orders, which are analytic, and linear in the increments; all of these functions have an integral representation as in the classical theory. Liouville's theorem is valid, and so also is the Cauchy-Taylor expansion theorem, when suitably expressed in terms of differentials. (Received November 1, 1935.)

425. Mr. Newman A. Hall: *A new class of functions of two variables involving Bessel functions of half an odd integer.*

Functions $C_n(a, b)$ with the generating function $\cos [a(1+x^2)^{1/2} - bx]$ arise in the evaluation of certain integrals connected with the theory of the conduction of heat. These are discussed together with three other sets of functions generated by associated expansions. All these functions satisfy the telegraphic equation in addition to several equations of mixed differences. Finite expressions in terms of Bessel functions of half an odd integer, and several integral representations are given. Certain infinite integrals involving these functions are evaluated. (Received November 2, 1935.)

426. Mr. Alvin Sugar: *On Dickson's method of ascension.*

In a previous paper by the author (to appear in this Bulletin) there is displayed a fundamental set of equations connecting $a=2^8$, $b=3^8$, \dots , $e=6^8$. From the existence of this set one is able to conclude that every positive integer is a sum of 566 eighth powers. It is of interest to examine whether there would always exist a finite fundamental set of equations for the case of n th powers and to investigate how such a set could be obtained. Dickson has indicated the solution to these problems in his papers on seventh (American Mathematical Monthly, vol. 41 (1934), pp. 547-555) and ninth powers (Bulletin of this Society, vol. 40 (1934), pp. 487-493). This note offers a detailed and elementary solution to these problems with particulars for eighth powers. (Received October 28, 1935.)

427. Professor Raymond Garver: *Postulates for special types of groups.*

In vol. 40, pp. 698-701, of this Bulletin, I presented a three-postulate definition of group. I now consider simplifications which can be effected in this definition when it is applied to finite or commutative groups, an appropriate postulate being added, of course, in either case. Several results are obtained and compared with results of other writers. (Received October 30, 1935.)

428. Professor E. T. Bell: *A functional equation in arithmetic.*

It is proved that the only polynomial solutions of the functional equation for associativity are two non-symmetric solutions and a certain symmetric bilinear solution. Hence, the only *polynomial* compositions of numerical functions satisfying the five postulates of D. H. Lehmer (Transactions of this Society, vol. 33, 1931, pp. 945-957) are Cauchy multiplication and Dirichlet multiplication. (Received October 31, 1935.)

429. Professor E. T. Bell: *The reciprocal of a numerical function.*

An immediate proof is given of the author's theorem (Tôhoku Mathematical Journal, vol. 17, 1920, pp. 221-231) that $fg=h$, where f, g, h are numerical functions, and fg is the Dirichlet product of f, g , has a unique solution g when f, h are given and $f(1) \neq 0$. The new proof gives a simpler expression for g as a function of f, h . (Received October 31, 1935.)

430. Professor B. C. Wong: *On certain varieties whose curve sections are hyperelliptic curves.*

This note is concerned with two algebraic varieties whose curve sections are shown to be hyperelliptic curves. One of these varieties is the V_n^{2n+1} in S_{2n+1} which has already been studied by Babbage and by Wong (Proceedings of the Cambridge Philosophical Society, vol. 27 (1931), pp. 399-403, and American Journal of Mathematics, vol. 56 (1934), pp. 219-224, respectively) and the other is the $V_k^{2n-2k+1}$ in S_n which is the residual intersection of $n-k$ cubic hypersurfaces having in common the intersection M_{n-2}^4 of two quadric hypersurfaces of S_n . A few detailed properties of these varieties are described, especially those of their surface sections some of which are already known. For an account of these latter properties the reader is referred to an article by Castelnuovo (see Rendiconti del Circolo Matematico di Palermo, vol. 4 (1890), pp. 73-88). (Received October 31, 1935.)

431. Mr. I. E. Highberg and Mr. A. E. Taylor: *On postulate systems for normed vector spaces.*

In part I of this paper we present a set of eleven independent postulates for real normed vector spaces, and a set of twelve independent postulates for complex normed vector spaces, the latter set being obtained from the former by addition of the postulate $ab(x+y) = a(bx) + (ab)y$. In this part of the paper equality is dealt with as a primitive relation. In part II we show that the equality of two elements x, y is equivalent to $\|x + -1 \cdot y\| = 0$. We adopt this as a *definition* of equality and by using this concept we give sets of nine and ten independent postulates for real and complex spaces respectively. (Received November 1, 1935.)

432. Mr. I. E. Highberg and Mr. A. E. Taylor: *An independent set of postulates for abstract linear spaces.*

In this paper we complete the work previously reported to the Society in a previous paper on postulate systems for abstract linear spaces (see abstract 41-3-159). A set of twelve independent postulates has been obtained for linear spaces. Equality is treated postulationally. This set differs essentially from other sets in that the two postulates $a(bx) = (ab)x$, $a(x+y) = ax + ay$ are combined into the single postulate $ab(x+y) = a(bx) + a(by)$. (Received November 1, 1935.)

433. Professor A. D. Michal and Mr. V. Elconin: *Hypercommutative rings*.

If xy is the ring multiplication, then $x \odot y \equiv xy - yx = 0$ is the usual commutativity relation (hypercommutativity of order zero). In the first part of this paper we study rings whose elements satisfy the hypercommutativity relation of order one $x \odot y \odot z = 0$ but not $x \odot y = 0$. By exhibiting examples we show that there exist linear algebras and matrix algebras that are hypercommutative of order one but not of order zero and are *not* nilpotent. Other triadic relations satisfied by associative or non-associative rings are also studied. In the second part of the paper we consider normed vector rings and study analytic functions defined by differential equations in such abstract rings. Among other theorems we prove: *If $A(x, f)$ is a power series in each variable for each value of the other, then the differential equation $\delta f(x) = A(x, f)\delta x + \delta x A(x, f)$ with Fréchet differentials is completely integrable in a normed vector ring with a hypercommutativity of order one.* (Received November 1, 1935.)