CLASSES OF MAXIMUM NUMBERS ASSOCIATED WITH CERTAIN SYMMETRIC EQUATIONS IN n RECIPROCALS*

BY NORMA STELFORD AND H. A. SIMMONS

1. *Introduction.* In an article dealing with this subject Simmons† stated without proof two general theorems whose proofs are to be obtained by making certain modifications in the theory of his article, which will be referred to in the sequel as I (for paper I). These theorems will be stated after a few definitions from I are recalled.

Kellogg solution. If a solution $x \equiv (x_1, x_2, \cdots, x_n)$ of any given symmetric equation in n reciprocals is obtained by minimizing the variables $x_1, x_2, \cdots, x_{n-1}$ (all positive integers) in this order, one at a time, we shall denote it by w and call it the Kellogg solution of the given equation. Thus $x \equiv (2, 3, 6)$ is the Kellogg solution w of the equation $x_1^{-1}+x_2^{-1}+x_3^{-1}=1$.

E-solution. A solution $x \equiv (x_1, x_2, \cdots, x_n)$ of a given symmetric equation in n reciprocals is called an E-solution if $x_1, x_2, \cdots, x_{n-1}$ are positive integers and $x_1 \leqq x_2 \leqq \cdots \leqq x_n$.

Polynomial $P(x)$. Let $P(x_1, x_2, \cdots, x_n) \equiv P(x)$ be any polynomial which is symmetric in the n variables x_i, contains one or more positive coefficients and no negative coefficient, and is not identically a constant.

$\Sigma_{i,j}(x)$. With $i \geqq 0$ and j equal to integers, we let $\Sigma_{i,j}(x)$ stand for the jth elementary symmetric function of the i variables $x_1, x_2, \cdots, x_i$; with the customary understanding that

$$\Sigma_{i,j}(x)\begin{cases} \equiv 0 \text{ when } i < j \text{ and also when } j < 0, \\ \equiv 1 \text{ when } j = 0. \end{cases}$$

We now state the two theorems referred to above with the numbering of I.

THEOREM 4. *If in the equation*

---

* Presented to the Society, April 6, 1934.

† See H. A. Simmons, Transactions of this Society, vol. 34 (1932), pp. 876–907.

$$\Sigma_{n,r}(1/x) + \lambda_{r+1}\Sigma_{n,r+1}(1/x) + \lambda_{r+2}\Sigma_{n,r+2}(1/x) + \cdots \\ + \lambda_s\Sigma_{n,s}(1/x) = b/a, \qquad a \equiv [(c+1)b - 1], \tag{21}$$

where b and c are positive integers, either $\lambda_p = 1$, $(p = r+1, \cdots, s)$, or λ_{r+1} is an integer $\geqq 0$ and $\lambda_p = 0$, $(p = r+2, \cdots, s)$, the largest number that exists in any E-solution of the resulting equation (21) *is the w_n of the corresponding solution w defined by the recurrence relations*

$$\begin{aligned} w_p &= 1, \qquad (p = 1, \cdots, r-1), \qquad w_r = c+1, \\ w_{p+1} &= a[\Sigma_{p,p-r+1}(w) + \lambda_{r+1}\Sigma_{p,p-r}(w) + \lambda_{r+2}\Sigma_{p,p-r-1}(w) \\ &\quad + \cdots + \lambda_s\Sigma_{p,p-s+1}(w)] + 1, \quad (p = r, \cdots, n-2), \\ w_n &= a[\Sigma_{n-1,n-r}(w) + \lambda_{r+1}\Sigma_{n-1,n-r-1}(w) + \lambda_{r+2}\Sigma_{n-1,n-r-2}(w) \\ &\quad + \cdots + \lambda_s\Sigma_{n-1,n-s}(w)]. \end{aligned} \tag{23}$$

Furthermore, in each of these cases, w_n appears in but one E-solution of the equation (21) *in question.*

THEOREM 5. *In each of the two cases of Theorem* 4, *if X is an E-solution of equation* (21) *and is different from the w of that equation, then $P(X) < P(w)$.*

Here we shall extend all theorems of I that relate to maximum numbers by considering cases in which the right members, b/a, of the equations in question are much more general than they were in I. For (a very simple) example, consider

$$\frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3} = \frac{3}{3c-2},$$

where $c \geqq 3$ is odd. The Kellogg solution of this equation is readily found to be $x = w$, where $w_1 = c$, $w_2 = 2^{-1}(3c^2 - 2c + 1)$, $w_3 = (3c-2)w_1w_2$. We show later that the theorems above hold for this solution w. This conclusion does not follow from I because $(3/(3c-2)) - (1/w_1)$ is not a unit fraction, while in I, $(b/a) - (1/(w_1 \cdots w_r))$, $(r = 1, \cdots, n-1)$, was required to be a unit fraction. Our new theorems are stated in §§6, 9, 10.

We divide the portion of this paper from §2 to the end into two parts. In Part 1, §§2 to 5, we prove a case of Theorem 4; in Part 2, we develop our extensions as far as seems desirable.

Except where the contrary is stated, we employ here the definitions and notation of I. The analogs of equation (i) and

Theorem i of I, if written here, are denoted by (ia) and Theorem ia, respectively, and proofs that are to be obtained as were analogous proofs of I are omitted. Consequently, in Part 1 we only state our modifications of certain definitions and lemmas and prove inequality (46a).

PART 1. PROOF OF ONE CASE OF THEOREM 4

2. *New Relations.* We shall prove the part of Theorem 4, §23 of I, for which $\lambda_{r+1}=\lambda_{r+2}=\cdots=\lambda_s=1$, that is, the case where equation (21) is

$$\phi_n(1/x) = b/a, \tag{21a}$$

where $\phi_n(1/x)\equiv\Sigma_{n,r}(1/x)+\Sigma_{n,r+1}(1/x)+\cdots+\Sigma_{n,s}(1/x)$, and r, s, n are positive integers such that $r<s\leqq n$.

If in the last displayed equation n is replaced by p, $r\leqq p\leqq n$, our definition of $\phi_p(1/x)$ is obtained.

The analogs here of (26), (28), (29), (30), (32) of I are as follows.

$$\begin{aligned} w_p &= 1, \qquad (p = 1,\cdots, r-1), \qquad w_r = c+1,\\ w_{p+1} &= a\big[\Sigma_{p,p-r+1}(w) + \Sigma_{p,p-r}(w) + \cdots\\ &\qquad + \Sigma_{p,p-s+1}(w)\big] + 1, \quad (p = r,\cdots, n-2),\\ w_n &= a\big[\Sigma_{n-1,n-r}(w) + \Sigma_{n-1,n-r-1}(w) + \cdots\\ &\qquad + \Sigma_{n-1,n-s}(w)\big]. \end{aligned} \tag{26a}$$

$$\begin{aligned} \phi_p(1/w) &= (bw_1\cdots w_p - 1)/(aw_1\cdots w_p),\\ &\qquad\qquad (p = r,\cdots, n-1;\ \text{see (27) of I}). \end{aligned} \tag{28a}$$

$$\begin{aligned} \phi_p(1/X) &\leqq (bX_1\cdots X_p - 1)/(aX_1\cdots X_p),\\ &\qquad\qquad (p = r,\cdots, n-1), \end{aligned} \tag{29a}$$

which is to be proved as was (29) of I.

$$\phi_{n-1}(1/X) \leqq \phi_{n-1}(1/w), \qquad (1\leqq r<s\leqq n). \tag{30a}$$

$$\begin{aligned} &\Sigma_{n-1,r-1}(1/X) + \Sigma_{n-1,r}(1/X) + \cdots + \Sigma_{n-1,s-1}(1/X)\\ &\quad < \Sigma_{n-1,r-1}(1/w) + \Sigma_{n-1,r}(1/w) + \cdots + \Sigma_{n-1,s-1}(1/w),\\ &\qquad\qquad (1\leqq r<s\leqq n). \end{aligned} \tag{32a}$$

3. *New Definitions of Sets* σ, τ, *and of the Transformation. Sets** σ, τ. Let λ be a fixed positive integer such that $r \leqq \lambda \leqq n$, where r and n are as they are defined for (21a). We shall call $x_{1\ldots\lambda}$ a set σ (relative to the w of (26a)) if, and only if, $\phi_p(1/x) \leqq \phi_p(1/w)$ for every positive integer p such that $r \leqq p \leqq \lambda$. We shall call $x_{1\ldots(\lambda+1)}$ a set τ if, and only if, with λ a positive integer such that $r \leqq \lambda \leqq n-2$, $x_{1\ldots(\lambda+1)}$ is not, and $x_{1\ldots\lambda}$ is, a set σ.

The new transformation.† Suppose $X'_{1\ldots\nu}$ contains at least one element of each of the classes A and B defined in I. Then we define our transformation of $X_{1\ldots\nu}$ into a new set $X_1'{}_{\ldots\nu}$ by (t_{3a}) or (t_{4a}):

$$
(33\text{a}) \quad \begin{aligned}
&(t_{3a}) \qquad X_p' = X_p(p \neq q_1, q, p \leqq \nu),\ X_{q_1}' = w_{q_1}, \\
&\qquad\qquad\qquad\qquad\qquad\qquad \phi_\nu(1/X') = \phi_\nu(1/X); \\
&(t_{4a}) \qquad X_p' = X_p(p \neq q_1, q, p \leqq \nu),\ X_{1q}' = w_{1q}, \\
&\qquad\qquad\qquad\qquad\qquad\qquad \phi_\nu(1/X') = \phi_\nu(1/X),
\end{aligned}
$$

according as (t_{3a}) requires X_{iq}' to be not greater than w_{iq} or greater than w_{1q}, respectively.

4. *Analogs of Lemmas* 6, 7, 8.

LEMMA 6a. (i) *If* $X_{1\ldots k}$ *is a set* τ, $X_{1\ldots k}$ *is transformable.*‡ (ii) *If* $X_{1\ldots k}$ *is a set* τ *or a transformable set* σ *for which* $r < k \leqq n$, *and if* t *is a positive integer, application of* (33a) *with* $\nu = k$ *to* $X_{1\ldots k}$ *yields a set* $X'_{1\ldots k}$ *such that* (34a), *identical with* (34), *is true, and*

$$(35\text{a}) \qquad \phi_p(1/X') \leqq \phi_p(1/w), \textit{ for } p = r, \cdots, {}_1q - 1;$$

[(36a) is not needed];

$$(37\text{a}) \qquad \phi_p(1/X') < \phi_p(1/X), \textit{ for } p = {}_1q, \cdots, k-1;$$

$$(38\text{a}) \qquad \phi_p(1/X') = \phi_p(1/X), \textit{ for } p = k.$$

LEMMA 7a. *If* s_1, s_2, λ *are integers* $\geqq 0$ *and* $s_1 \geqq s_2, \lambda > 0$, *then*

$$(1^{s_1})(1^{s_2}) > (1^{s_1+\lambda})(1^{s_2-\lambda}).$$

---

* See pp. 889–890 of I.

† See p. 891 of I.

‡ See p. 891 of I.

In proving the next lemma by our method, Lemma 7a would be used.

LEMMA 8a. *If u, v, γ, t are integers with $u>v\geqq\gamma\geqq 1$, $0\leqq t\leqq(\gamma-1)$, and $x_i>0$ for $i=1,\cdots,u$, then*

$$\Sigma_{u,\gamma}(1/x)\Sigma_{v,t}(1/x) > \Sigma_{u,t}(1/x)\Sigma_{v,\gamma}(1/x).$$

5. *Analog of Relation* (46) *of* I. From (37a) and (38a) one may obtain (46a) below as (46) was obtained from (37) and (38) of I:

$$\text{(46a)*}\quad \begin{aligned}&[\Sigma'_{k-2,r-1}+\Sigma'_{k-2,r}+\cdots+\Sigma'_{k-2,s-1}][\Sigma'_{p-2,r-2}\\ &\quad+\Sigma'_{p-2,r-1}+\cdots+\Sigma'_{p-2,s-2}]\\ &>[\Sigma'_{k-2,r-2}+\Sigma'_{k-2,r-1}+\cdots+\Sigma'_{k-2,s-2}][\Sigma'_{p-2,r-1}\\ &\quad+\Sigma'_{p-2,r}+\cdots+\Sigma'_{p-2,s-1}].\end{aligned}$$

With

$$c_1\equiv\Sigma'_{k-2,r-1}+\Sigma'_{k-2,r}+\cdots+\Sigma'_{k-2,s-2},$$
$$c_2\equiv\Sigma'_{p-2,r-1}+\Sigma'_{p-2,r}+\cdots+\Sigma'_{p-2,s-2},$$

(46a) may be written

$$(\Sigma'_{k-2,s-1}+c_1)(\Sigma'_{p-2,r-2}+c_2)>(\Sigma'_{k-2,r-2}+c_1)(\Sigma'_{p-2,s-1}+c_2).$$

PROOF OF (46a). That this inequality is true follows from the inequalities I_1, I_2, I_3 below, which we presently prove. In arriving at (46a), one finds that it is to hold under the hypotheses $1\leqq r\leqq {}_1q\leqq p<k$, with ${}_1q\geqq 2$. These relations are understood to hold in this proof.

$$(I_1)\qquad c_1\Sigma'_{p-2,r-2}>c_2\Sigma'_{k-2,r-2},$$

$$(I_2)\qquad c_2\Sigma'_{k-2,s-1}\geqq c_1\Sigma'_{p-2,s-1},$$

$$(I_3)\qquad \Sigma'_{k-2,s-1}\Sigma'_{p-2,r-2}\geqq\Sigma'_{k-2,r-2}\Sigma'_{p-2,s-1}.$$

The terms in c_1 are $\Sigma'_{k-2,\lambda}$, $(\lambda=r-1,\cdots,s-2)$; those in c_2, $\Sigma'_{p-2,\lambda}$. Hence (I_1), (I_2) may be written

$$(I_1')\qquad \Sigma'_{k-2,\lambda}\Sigma'_{p-2,r-2}>\Sigma'_{k-2,r-2}\Sigma'_{p-2,\lambda},$$

$$(I_2')\qquad \Sigma'_{k-2,s-1}\Sigma'_{p-2,\lambda}\geqq\Sigma'_{k-2,\lambda}\Sigma'_{p-2,s-1},$$

---

* $\Sigma'_{i,j}\equiv\Sigma'_{i,j}(1/X)$.

respectively. If $p-2 \geqq \lambda$, (I_1') holds by Lemma 8a with $u=k-2$, $v=p-2$, $\gamma=\lambda$, and $t=r-2$. If $(p-2)<\lambda$, then since $p \geqq r$ the left side of (I_1') is positive and the right side is zero, so that (I_1') holds. Consider (I_2'). If $(p-2) \geqq (s-1)$, (I_2') holds with $>$ by Lemma 8a, with $u=k-2$, $v=p-2$, $\gamma=s-1$, $t=\lambda$. If $(p-2) < (s-1)$, the right side of (I_2') is zero and the left side is surely not negative. Therefore (I_2') holds with $\geqq$. The proof for (I_2) is a special case of that for (I_2').

Part 2. Extensions of Previous Theory

6. *A New Theorem on Equations of the Form* $\Sigma_{n,1}(1/x)=b/a$. The article I contains results for equations of this type only when $a=[(c+1)b-1]$ and b, c are positive integers. This case is covered by taking $\mu=1$ in the new Theorem A below.

Theorem A. *Suppose* a, b *and* μ, $\mu \leqq (n-1)$, *are given positive integers, with* a *and* b, $b \leqq a$, *relatively prime, and that there exists a set of* n *numbers* $w \equiv (w_1, w_2, \cdots, w_n)$ *with the following properties*:
(1°) *it is an E-solution of the equation* $\Sigma_{n,1}(1/x)=b/a$;

$$(2°) \qquad \frac{1}{x_1}+\frac{1}{x_2}+\cdots+\frac{1}{x_n}=\frac{bw_1\cdots w_p-1^*}{aw_1\cdots w_p}$$

for every positive integral value of p *for which* $\mu \leqq p \leqq (n-1)$ *and for no smaller value of* p;
(3°) *if* $x=X$, *where* $X \neq w$, *is an E-solution of the equation in* (1°), *then for every positive integral value of* $p \leqq \mu$ *for which* $X_{1\cdots p} \neq w_{1\cdots p}$, *the inequality* $\Sigma_{p,1}(1/X) < \Sigma_{p,1}(1/w)$ *holds.*

For such a set w *the following conclusions hold*:
(i) w_n *is the largest number that exists in any E-solution of the equation in* (1°) *and* w_n *appears in no E-solution of this equation except* w;
(ii) $P(x)<P(w)$.†

Proof of (i). It follows from hypotheses $(j°)$, $(j=1, 2, 3)$, that the w described in Theorem A is the Kellogg solution of the equation in (1°). This fact will be used in the rest of the proof of (i), which we presently make by considering separately the following cases: (α) $\mu=n-1$; (β) $\mu=n-2$; (γ) $\mu \leqq (n-3)$.

* See equation (28) of I, with $r=1$.

† See p. 887 of I.

(α) In this trivial case, (3°) requires that conclusion (i) be true.

(β) If $X_{1\cdots\mu}=w_{1\cdots\mu}$, since $X\neq w$ is an E-solution of the equation in (1°) while w is its Kellogg solution, $X_{\mu+1}\equiv X_{n-1}>w_{n-1}$, and (i) is true. If $X_{1\cdots\mu}\neq w_{1\cdots\mu}$, we reach the desired conclusion as follows. From the hypotheses (j°), ($j=1$, 2, 3), and the assumption that $X_{1\cdots\mu}\neq w_{1\cdots\mu}$, with $\mu=n-2$, we have

$$\Sigma_{p,1}(1/X) \leqq \Sigma_{p,1}(1/w), \qquad (p = 1, \cdots, \mu - 1);$$

$$(1) \qquad \Sigma_{n-2,1}(1/X) < \Sigma_{n-2,1}(1/w) \equiv \frac{bw_1 \cdots w_{n-2} - 1}{aw_1 \cdots w_{n-2}};$$

$$(2) \qquad \Sigma_{n-1,1}(1/X) \leqq \frac{bX_1 \cdots X_{n-1} - 1}{aX_1 \cdots X_{n-1}}, \qquad [\text{see (29) of I}];$$

and we wish to contradict the assumption which we now make, namely that

$$(3) \qquad \Sigma_{n-1,1}(1/X) \geqq \Sigma_{n-1,1}(1/w) \equiv \frac{bw_1 \cdots w_{n-1} - 1}{aw_1 \cdots w_{n-1}}.$$

Relations (2) and (3) imply that

$$(4) \qquad X_1 \cdots X_{n-1} \geqq w_1 \cdots w_{n-1};$$

(1) and (3), that $X_{1\cdots(n-1)}$ is transformable. Since X is an E-solution, it follows now that when exhaustive applications of (33a) for $X_{1\cdots(n-1)}$ are made, the following relations hold [see (53) of I]:

$$(5) \qquad X_1 \cdots X_{n-1} < X_1' \cdots X_{n-1}' \leqq w_1 \cdots w_{n-1}.$$

Since (4) and (5) are contradictory, we conclude that (3) is false. Consequently conclusion (i) holds in case (β).

(γ) The demonstration that is required here is essentially the same as that given under (β) for the case $X_{1\cdots\mu}\neq w_{1\cdots\mu}$; induction integers k, $(k+1)$, where $\mu\leqq k\leqq(n-2)$, here take the place of $(n-2)$, $(n-1)$, respectively, there. Hence conclusion (i) holds in case (γ).

Proof of (ii). This result can be reached by the method that was employed in proving Theorem 3 of I, since inequalities analogous to those in (53) of I hold here. Hence Theorem A is true.

7. *An Example of Takenouchi.* The Takenouchi equation*

$$\frac{1}{x_1}+\frac{1}{x_2}+\frac{1}{x_3}=\frac{5}{11} \tag{6}$$

has, as we presently prove, no solution w of the type hypothesized in Theorem A. For the Kellogg solution of (6) is $X=w=(3, 9, 99)$, and it is easily seen that for this X there does not exist an integer μ of the type specified in (2°). One may also negate the existence here of the w of Theorem A by observing with Takenouchi that the set (4, 5, 220) satisfies (6), while $(4^{-1}+5^{-1})>(3^{-1}+9^{-1})$, so that $(3, 9)=(w_1, w_2)$ does not accord with hypothesis (3°).

8. *Corollaries of Theorem* A. We have seen in I that when $a=[(c+1)b-1]$, the w of Theorem A always exists (case $\mu=1$). When $a\neq[(c+1)b-1]$ there are, as we point out in the corollaries below, numerous cases for which the w of Theorem A exists with $\mu=2$. In cases where $\mu>2$, we have no better method than that of trial and error for testing a given E-solution X to see whether it satisfies hypothesis (3°). However, when $\mu=2$ the following Auxiliary Theorem shows that hypothesis (3°) is always satisfied when (1°) and (2°) are. For Theorem A only the case $r=1$, $\lambda_{r+1}=\lambda_2=0$ of this Auxiliary Theorem is needed; other cases of it are similarly useful in connection with Theorem ia, $(i=2, 3, 4, 5)$, of I.

AUXILIARY THEOREM. *Let X be an E-solution of the equation*

$$\Sigma_{n,r}(1/x) + \lambda_{r+1}\Sigma_{n,r+1}(1/x) + \cdots + \lambda_s\Sigma_{n,s}(1/x) = b/a,$$

where $X\neq w$, w being the Kellogg solution of this equation; the λ_i, $(i=r+1, \cdots, s, s>r)$, are integers $\geqq 0$; $n\geqq r+2$;† and a, b are as in Theorem A *except that we now suppose that*

$$\frac{1}{w_1\cdots w_r}\neq\frac{bw_1\cdots w_r-1}{aw_1\cdots w_r};$$

and that

* See p. 92 of loc. cit. in footnote 3, p. 876, of I.

† This restriction insures the presence of at least two terms free of x_n in the left member of the equation just displayed. When $n=r+1$, Theorem A has no content beyond the case $\mu=1$, which was handled in I.

$$\frac{1}{w_1 \cdots w_r} + \frac{1}{w_{r+1}}\left[\Sigma_{r,r-1}(1/w) + \frac{\lambda_{r+1}}{w_1 \cdots w_r}\right]$$
$$= \frac{bw_1 \cdots w_{r+1} - 1}{aw_1 \cdots w_{r+1}}.$$

Then we conclude that* $(X_{1\ldots(r+1)}$ *is a set* σ *in the sense that)*

$$\frac{1}{X_1 \cdots X_r} \leqq \frac{1}{w_1 \cdots w_r};$$

$$\frac{1}{X_1 \cdots X_r} + \frac{1}{X_{r+1}}\left[\Sigma_{r,r-1}\frac{1}{X} + \frac{\lambda_{r+1}}{X_1 \cdots X_r}\right] < \frac{bw_1 \cdots w_{r+1} - 1}{aw_1 \cdots w_{r+1}}.$$

The proof of this theorem can be made by reasoning of the type that was used on p. 897 of I. Each of the three corollaries below is a case $\mu=2$ of Theorem A.

COROLLARY 1. *For the equation* $\Sigma_{n,1}(1/x)=5/17$, *with* $n>2$, *the* w *of Theorem* A *is* $w_1=4$, $w_2=23$, $w_{i+1}=17w_1 \cdots w_i+1$, $(i=2, \cdots, n-2)$, $w_n=17w_1 \cdots w_{n-1}$.

COROLLARY 2. *For the equation* $\Sigma_{n,1}(1/x)=3/(3c-2)$, *with* $n>2$, *if* $c \geqq 3$ *is odd, the* w *of Theorem* A *is defined by the equations* $w_1 = c$, $w_2 = 2^{-1}(3c^2 - 2c + 1)$, $w_{i+1} = (3c - 2)w_1 \cdots w_i + 1$, $(i = 2, \cdots, n - 2)$, $w_n = (3c - 2)w_1 \cdots w_{n-1}$.

COROLLARY 3. *For the equation* $\Sigma_{n,1}(1/x)=b/a$, *with* $n>2$ *and* $a\equiv[(c+1)b-2]$, *if* $b>1$ *is odd and* c *is even, the set* w *of Theorem* A *is defined by* $w_1=c+1$, $w_2=2^{-1}[a(c+1)+1]$, $w_{i+1}=aw_1 \cdots w_i+1$, $(i=2, \cdots, n-2)$, $w_n=aw_1 \cdots w_{n-1}$.

9. *Generalizations of Theorem* 2, 3 *of* I.

THEOREM 2a. *Suppose* a, b *and* μ, *where* $r \leqq \mu \leqq (n-1)$, *are given positive integers, with* a *and* b, $b \leqq a$, *relatively prime, and that there exists a set of* n *numbers* $w \equiv (w_1, \cdots, w_n)$ *with the following properties*:
(1°) *it is an E-solution of the equation*

$$\Sigma_{n,r}(1/x) = b/a, \qquad (1 < r < (n-1));\dagger$$

* If the sign $<$ in the second relation below were replaced by the sign $\leqq$, $X_{1\ldots(r+1)}$ would still be a set σ in a sense perfectly analogous to that in which the term *set* σ was used in I.

† The case $r=1$ has been treated in Theorem A; the case $r=(n-1)$, in I. Hence only the cases $1<r<(n-1)$ remain for consideration.

$$(2°) \qquad \Sigma_{p,r}(1/w) = \frac{bw_1 \cdots w_p - 1}{aw_1 \cdots w_p}$$

for every positive integral value of p for which $\mu \leqq p \leqq (n-1)$ *and for no positive integral value of p less than* μ;
(3°) *if* $x = X$, *where* $X \neq w$, *is an E-solution of the equation in* (1°), *then for every positive integral value of p such that* $r \leqq p \leqq \mu$ *for which* $X_{1\cdots p} \neq w_{1\cdots p}$, *the relation* $\Sigma_{p,r}(1/X) \leqq \Sigma_{p,r}(1/w)$ *holds.*

Then w_n *is the largest number that exists in any E-solution of the equation in* (1°) *and* w_n *appears in no E-solution of this equation except w.*

THEOREM 3a. *If* $x \neq w$ *is any E-solution of the equation considered in Theorem* 2a, $P(x) < P(w)$. [See Theorem 3 of I.]

Since the methods which we have used in proving Theorem A together with those of I suffice to prove the last two theorems we omit their proofs.

The following corollary shows that Theorem *i*a, ($i = 2, 3$), has content for a case in which $\mu > r$.

COROLLARY 4. *The Kellogg solution of the equation* $\Sigma_{n,3}(1/x) = 5/11$, $n \geqq 5$, *is* $x = w$, *where* $w_1 = w_2 = 1$, $w_3 = 3$, $w_4 = 14$, $w_{i+1} = 11\Sigma_{i,i-r+1}(w) + 1$, $(i = 4, \cdots, n-2)$, $w_n = 11\Sigma_{n-1,n-r}(w)$.

It is easy to show that in this example the μ of Theorem 2a exists and equals 4.

10. *Generalizations of Theorems* 4, 5.

THEOREM 4a. *Suppose a, b,* μ *are as in Theorem* 2a *and that there exists a set of n numbers* $w > (w_1, \cdots, w_n)$ *with the following properties*:
(1°) *it is an E-solution of the equation* $\Sigma_{n,r}(1/x) + \lambda_{r+1}(1/x) = b/a$, *in which* λ_{r+1} *is an integer* $\geqq 0$;

$$(2°) \qquad \Sigma_{p,r}(1/w) + \lambda_{r+1}\Sigma_{p,r+1}(1/w) = \frac{bw_1 \cdots w_p - 1}{aw_1 \cdots w_p}$$

for every positive integral value of p for which $\mu \leqq p < n$ (*and for no positive integral value of p less than* μ *when* $\mu > r$);
(3°) *if* $x = X$, *where* $X \neq w$, *is an E-solution of the equation in* (1°), *then for every positive integral value of p such that* $r \leqq p \leqq \mu$ *for which* $X_{1\cdots p} \neq w_{1\cdots p}$, *the relation*

$$\Sigma_{n,r}(1/X) + \lambda_{r+1}\Sigma_{p,r+1}(1/X) \leqq \Sigma_{p,r}(1/w) + \lambda_{r+1}\Sigma_{p,r+1}(1/w)$$

holds.

In such a set w, w_n is the largest number that exists in any E-solution of the equation in (1°) *and w_n appears in no E-solution of this equation except w. Furthermore a similar statement holds when the left member of the equation in* (1°) *is replaced by*

$$\Sigma_{n,r}(1/x) + \Sigma_{n,r+1}(1/x) + \cdots + \Sigma_{n,s}(1/x),$$

where, as heretofore, s is a positive integer and $r < s \leqq n$.

THEOREM 5a. *In each of the two cases of Theorem* 4a, *if X is an E-solution of the given equation and $\neq w$, the Kellogg solution of that equation, then $P(X) < P(w)$.*

The following corollaries show that the theorems of this section have content in cases where $\mu = 2$ when $r = 1$.

COROLLARY 5. *For the equation $\Sigma_{n,1}(1/x) + 3\Sigma_{n,2}(1/x) = 5/17$, wiih $n > 2$, the set w of Theorem* 4a *is given by $w_1 = 4$, $w_2 = 40$, $w_{i+1} = 17[\Sigma_{i,i}(w) + 3\Sigma_{i,i-1}(w)] + 1$, $(i = 2, \cdots, n-2)$, and $w_n = 17[\Sigma_{n-1,n-1}(w) + 3\Sigma_{n-1,n-2}(w)]$.*

COROLLARY 6. *For the equation $\Sigma_{n,1}(1/x) + \Sigma_{n,2}(1/x) = 4/13$, with $n > 2$, the w of Theorem* 4a *(see last sentence of that theorem) is given by $w_1 = 4$, $w_2 = 22$, $w_{i+1} = 13[\Sigma_{i,i}(w) + \Sigma_{i,i-1}(w)] + 1$, $(i = 2, \cdots, n-2)$, and $w_n = 13[\Sigma_{n-1,n-1}(w) + \Sigma_{n-1,n-2}(w)]$.*

NORTHWESTERN UNIVERSITY

ERRATA

The following changes should be made in the present volume (Vol. 40) of this Bulletin:

Page 93, last line of Theorem 2, insert before the words "*is that*" the words "*and that $f_m(x)$ be continuous.*"

Pages 413–416, change f to f_0 in the following places: in the statement of Theorem 2 on p. 413; in the statement of Theorem 6 on p. 415; and in five places occurring in the first six lines of p. 416.