CURVATURES IN RIEMANNIAN SPACE*

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Let C be a curve in Riemannian n-space with curvatures $1/\rho_\alpha$, $(\alpha=1, 2, \cdots, n-1)$. We shall designate by $\lambda_{1|}{}^i$, $(i=1, 2, \cdots, n)$ the components of the unit tangent vector and by $\lambda_{k|}{}^i$, $(i=1, 2, \cdots, n)$, those of the unit $(k-1)$st normal. Let P and P' be two points of C, and λ^i and λ'^i the components of two directions, one at P and the other at P'. We shall speak of the *angle* between these directions, meaning the angle between λ and the direction obtained at P by parallel displacement along the curve of the direction λ'.† We denote by θ_k the angle, so defined, between the $(k-1)$st normals at P and P' (or, when $k=1$, between the tangents).

In euclidean three-space it is well known that

$$\lim \frac{\theta_1^2}{s^2} = \frac{1}{\rho_1^2}, \qquad \lim \frac{\theta_3^2}{s^2} = \frac{1}{\rho_2^2}, \qquad \lim \frac{\theta_2^2}{s^2} = \frac{1}{\rho_1^2} + \frac{1}{\rho_2^2},$$

where s is the arc of C from P to P'. In fact, the curvatures are frequently defined by means of the first two of these equations. In extending the concept of curvature to a general Riemannian space this simple geometric interpretation was lost sight of, primarily because the earlier writers did not have available Levi-Civita's infinitesimal parallelism. As a consequence the curvatures, except the first, are usually defined by formal analytic methods. In this note we shall show that the above geometric interpretations of curvatures are valid in a general Riemannian space with a positive definite fundamental form. In the case of an indefinite form there is a very slight modification.

Let the curve C be given parametrically by $x^i=x^i(s)$, $(i=1, 2, \cdots, n)$, where s is the arc measured from an arbitrary point P_0. Let g_{ij} be the components of the fundamental tensor and Γ^i_{jk} the Christoffel symbols of the second kind, so that

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† Since the angle between two directions at a point remains constant when the two directions are displaced parallel to themselves the above interpretation of the angle between two skew directions does not depend upon the order in which the two directions are taken.

$$\frac{\partial g_{ij}}{\partial x^k} = g_{jh}\Gamma^h_{ki} + g_{ih}\Gamma^h_{kj}.* \tag{1}$$

We take for the x's a system of normal coordinates with origin at P_0, so that

$$\left(\frac{\partial g_{ij}}{\partial x^k}\right)_0 = 0, \qquad (\Gamma^h_{ij})_0 = 0,\dagger \tag{2}$$

where the parenthesis with the subscript nought means that the expression within is evaluated at the origin P_0.

Following Eisenhart's notation‡ we can write the generalized Frenet equations

$$\lambda_{k|}{}^i{}_{,j}\frac{dx^j}{ds} = -\frac{e_{k-1}}{\rho_{k-1}}\lambda_{k-1|}{}^i + \frac{e_{k+1}}{\rho_k}\lambda_{k+1|}{}^i, \tag{3}$$

where the e's are ± 1. Recalling the definition of covariant differentiation and making use of equations (2), we find that

$$\left(\frac{d\lambda_{k|}{}^i}{ds}\right)_0 = \left(-\frac{e_{k-1}}{\rho_{k-1}}\lambda_{k-1|}{}^i + \frac{e_{k+1}}{\rho_k}\lambda_{k+1|}{}^i\right)_0. \tag{4}$$

Differentiating equations (3) and eliminating the first derivatives of the λ's by virtue of (3), we obtain the values of the second derivatives of the λ's; at the origin they become, on account of (2),

$$\begin{aligned}\left(\frac{d^2\lambda_{k|}{}^i}{ds^2}\right)_0 = \Bigg(&-\frac{\partial\Gamma^i_{hj}}{\partial x^l}\lambda_{k|}{}^h\frac{dx^j}{ds}\frac{dx^l}{ds} + \frac{e_{k-1}e_{k-2}}{\rho_{k-1}\rho_{k-2}}\lambda_{k-2|}{}^i \\ &-\left(\frac{e_{k-1}}{\rho_{k-1}}\right)'\lambda_{k-1|}{}^i - e_k\left(\frac{e_{k-1}}{\rho^2_{k-1}} + \frac{e_{k-1}}{\rho^2_k}\right)\lambda_{k|}{}^i \\ &+\left(\frac{e_{k+1}}{\rho_{k+1}}\right)'\lambda_{k|}{}^i + \frac{e_{k+1}e_{k+2}}{\rho_k\rho_{k+1}}\lambda_{k+2|}{}^i\Bigg)_0,\end{aligned} \tag{5}$$

the primes in these equations denoting derivatives with respect to s.

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* As usual an index appearing both as a subscript and a superscript is to be summed from 1 to n.

† See, for example, Eisenhart, *Riemannian Geometry*, p. 55.

‡ Loc. cit., p. 101.