A PROPERTY OF THE LEVEL LINES OF A REGION WITH A RECTIFIABLE BOUNDARY*

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1. *Introduction.* Before stating the result of this paper let me recall that a *level line of a region*† in a plane is the locus of the equation $g(x, y; a, b)=c$, where $g(x, y; a, b)$ is the Green's function of the region which has the point (a, b) as its pole, and c is a positive constant. The set of all level lines of the region, with the point (a, b) fixed and c any positive constant, is called a *pencil of level lines* of the region, and the fixed point (a, b) is called the *pole* of that pencil. Any pencil Σ of level lines of a region which is in a plane and which has a connected boundary containing more than one point, is the image of the set of circles concentric with and interior to any circle K under any transformation Π which maps in a one-to-one and conformal way the interior of K on Σ, such that the pole of the pencil of level lines corresponds to the center of K; and, conversely, the image of the set of circles concentric with and interior to a circle K under any transformation Π which maps in a one-to-one and conformal way the interior of K on a planar region Σ is a pencil of level lines of Σ, which has the image under Π of the center of K as its pole. Thus, a pencil of level lines of Σ is a one-parameter set of simple closed curves, and the value of the parameter t of a level line G of such a pencil of level lines may be taken as the length of the radius of the circle to which G corresponds under Π. Taking K as the unit circle, then t varies between 0 and 1. The symbol $[G_t]$ denotes a pencil of level lines of the region

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* Presented to the Society, May 7, 1927, as part of a paper of the title *Certain sequences of curves which approach a rectifiable boundary from within.*

† A region in a plane is a set of points in a plane such that there exists a planar neighborhood of each point of the set which contains only points of the set.

Σ such that the level line G_t corresponds under Π to the circle which is concentric with the unit circle and which has a radius of length t $(0<t<1)$.

Now, if $\Lambda(t)$ denotes the function of t defined for $0<t<1$ and such that $\Lambda(t_1)$ is the length of the level line G_{t_1} of the pencil of level lines $[G_t]$, $0<t<1$, then it is a result of a theorem of Hardy, referred to below, that $\Lambda(t)$ is an increasing continuous function of t for $0<t<1$; that is, if $0<t_1<t_2<1$, then $\Lambda(t_1)<\Lambda(t_2)$. It is a simple consequence of the formula for the length of an analytic transform of a rectifiable curve, which is derived in §2 below, that $\lim_{t\to 0}\Lambda(t)=0$ and nothing further is said about that; but much of the following proof is devoted to showing that if the boundary of Σ is a rectifiable simple closed curve, then $\lim_{t\to 1}\Lambda(t)$ is the length of the boundary of Σ.

The result of this paper is contained in the following two theorems. The theorems are closely connected and their proofs are combined in the demonstration which follows.

THEOREM. *The function* $\Lambda(t)$ *of* t, *which is defined in the interval* $0\leqq t\leqq 1$, *and which is such that* $\Lambda(t_1)$, *if* $0<t_1<1$, *is the length of the level line* G_{t_1} *of the pencil* $[G_t]$, $0<t<1$, *of level lines of the planar region* Σ *whose boundary is a rectifiable simple closed curve and such that* $\Lambda(0)=0$ *and* $\Lambda(1)=$ *the length of the boundary of* Σ, *is an increasing* continuous function of t in the (closed) interval* $0\leqq t\leqq 1$.

DEFINITION. An *approximating sequence of regions* of the region Σ is a sequence of regions $\{\Sigma_n\}$, $n=1, 2, 3, \cdots$, such that every limit point of each Σ_n is a point of Σ and every point of Σ belongs to all but a finite number of the regions Σ_n.†

* That is, if $0\leqq t_1<t_2\leqq 1$, then $\Lambda(t_1)<\Lambda(t_2)$.

† It follows readily from this definition that if the region Σ is bounded then an approximating sequence of regions $\{\Sigma_n\}$, $n=1, 2, 3, \cdots$, of the region Σ contains a subsequence of regions $\{\Sigma_{n_i}\}$, $i=1, 2, 3, \cdots$, which is such that (a) every limit point of any region Σ_{n_i} belongs to Σ and to the succeeding region $\Sigma_{n_{i+1}}$ and (b) every point of Σ is in all but a finite number of the regions Σ_{n_i}.

DEFINITION. An *approximating sequence of curves* of the region Σ is a sequence of curves $\{C_n\}$, $n=1, 2, 3, \cdots$, such that each curve C_n is the boundary of a region Σ_n of an approximating sequence of regions $\{\Sigma_n\}$, $n=1, 2, 3, \cdots$, of the region Σ.

THEOREM. *If the boundary of a planar region Σ is a simple closed curve which is rectifiable, then there exist approximating sequences of curves $\{C_n\}$, $n=1, 2, 3, \cdots$, of the region Σ such that the curves C_n are level lines of any given pencil of level lines of Σ and, if l_n denotes the length of the curve C_n, $l_n < l_{n+1}$ and $\lim_{n\to\infty} l_n$ is the length of the boundary of Σ.*

2. *The Length of an Analytic Transform of a Rectifiable Curve.* Let the function $w=f(z)$ be analytic in the interior, $i(K)$, of the unit circle, K, and map in a one-to-one way $i(K)$ on the region Σ of the theorem. Let C be a rectifiable curve in $i(K)$ and C' its image under the transformation $w=f(z)$. Then the length, l', of C' is

$$\lim_{n\to\infty} (|\Delta w_{n_1}| + |\Delta w_{n_2}| + \cdots + |\Delta w_{n_r}|),$$

where $\Delta w_{n_i}=f(z_{n_i})-f(z_{n_{i-1}})$, where $z_{n_0}, z_{n_1}, z_{n_2}, \cdots, z_{n_r}, z_{n_{r+1}}=z_0$ are points on C such that $|z_{n_i}-z_{n_{i-1}}|<\delta_n>0$, and where $\lim_{n\to\infty}\delta_n=0$.

If Δs_{n_i} is the length of the arc of C whose end points are z_{n_i} and $z_{n_{i-1}}$ and which does not contain as an inner point the point $z=z_{n_0}$, then

$$l' = \lim_{n\to\infty} \sum_{i=1}^{r} \left| \frac{\Delta w_{n_i}}{\Delta z_{n_i}} \right| \cdot \Delta s_{n_i}, \qquad \Delta z_{n_i} = z_{n_i} - z_{n_{i-1}};$$

and, because of the uniformity of the approach of $\Delta w/\Delta z$ to dw/dz along C, dw/dz being continuous on C, it follows that

$$l' = \lim_{n\to\infty} \sum_{i=1}^{r} \left(\lim_{z\to z_{n_i}} \left| \frac{\Delta w}{\Delta z} \right| \right) \Delta s_{n_i}.$$

Hence

$$l' = \int_C \left| \frac{dw}{dz} \right| ds.$$

3. *The Level Lines of a Polygonal Region.* Let $i(J)$ denote a region whose boundary is a simple polygon J, and let $w=f(z)$ be a function which is analytic in the interior $i(K)$ of the unit circle K and which maps in a one-to-one way $i(K)$ on $i(J)$. Then $w=f(z)$ is analytic at any point of the circle K whose image is not a vertex of J and if $w=f(a)$ is a vertex of J, then at any point of $i(K)$ different from $z=a$ in some neighborhood of $z=a$ the derivative of $w=f(z)$ is $(z-a)^{\mu}\lambda(z)$, where $\lambda(z)$ is analytic at $z=a$ and $-1<\mu<1$.* In fact, $\mu=\alpha/\pi-1$, where α is the measure in radians of the interior angle of the polygon J whose vertex is the point $w=f(a)$.

Let the point $w=f(a_i)$ be a vertex of the polygon J and U_{a_i} a neighborhood of $z=a_i$ such that at every point of $i(K)$ which is in U_{a_i} and different from $z=a_i, f'(z)=(z-a_i)^{\mu_i}\lambda_i(z)$, where $\lambda_i(z)$ is analytic at $z=a_i$ and $-1<\mu_i<1$. Further, let Γ_i denote a circular arc concentric with K, and contained in U_{a_i}, such that its mid-point $z=b_i$ is on the radius of K through $z=a_i$. Let $z=c_i$ be an end point of this arc. Then, if $-1<\mu_i<0$ and M_i is a bound of $|\lambda_i(z)|$ in U_{a_i},

$$\int_{\Gamma}|f'(z)|\,ds \leqq M_i\int_{\Gamma_i}(|z-a_i|)^{\mu_i}ds \leqq M_i\int_{\Gamma_i}(|z-b_i|)^{\mu_i}ds.$$

If η is an arbitrary positive number, then there exists a positive number ξ such that $|z-b_i|/(\widehat{zb_i})\geqq 1-\eta$ if $\widehat{zb_i}<\xi$, where $\widehat{zb_i}$ denotes the length of the sub-arc of Γ_i whose end points are $z=z$ and $z=b_i$. Then

$$\int_{\Gamma_i}(|z-b_i|)^{\mu_i}ds \leqq (1-\eta)^{\mu_i}\int_{\Gamma_i}(\widehat{zb_i})^{\mu_i}ds$$
$$=2(1-\eta)^{\mu_i}\int_0^{l_i/2}s^{\mu_i}ds = 2(1-\eta)^{\mu_i}\frac{1}{\mu_i+1}\cdot\frac{l^{\mu_i+1}}{2^{\mu_i+1}},$$

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* Bieberbach, *Lehrbuch der Funktionentheorie*, vol. II, p. 34 and p. 37. Also Study, *Vorlesungen über ausgewählte Gegenstände der Geometrie*, Part II, p. 85.

where l_i is the length of Γ_i and if $\eta<1/2$, then

$$\int_{\Gamma_i}(|z-b_i|)^{\mu_i}ds<\frac{1}{2^{2\mu_i}(\mu_i+1)}l_i^{\mu_i+1}.$$

Again, if $0\leqq\mu_i<1$ and M_i is a bound of $|\lambda_i(z)|$ in U_{a_i}, we have

$$\int_{\Gamma_i}|f'(z)|\,ds\leqq M_i\int_{\Gamma_i}(|z-a_i|)^{\mu_i}ds\leqq M_i\int_{\Gamma_i}|c_i-a_i|\,ds$$

$$\leqq M_i\left(|a_i-b_i|+\frac{l_i}{2}\right)l_i,$$

where l_i is the length of the arc Γ_i. Hence if ϵ is any positive number there exists a positive number δ_i such that if Γ_i is any circular arc which is concentric with and either interior to or on the given circle K and contained in U_{a_i} and which has a length $l_i<\delta_i$, then

$$\int_{\Gamma_i}|f'(z)|\,ds<\epsilon.$$

Now, let σ_i denote an arc on the circle K which has $z=a_i$ as its mid-point and a length l_i which is less than δ_i and, further, such that no two arcs σ_i have a point in common and let R denote the set of all points which are interior to K and which do not belong to any sector bounded by the arc σ_i and the radii of K through its end points; also let $\overline{R}$ denote the set of points consisting of the points of R and of the boundary of R. Then $w=f'(z)$ is analytic in $\overline{R}$ and hence there exists a positive number δ such that $|f'(z_1)-f'(z_2)|<\epsilon$ if $z=z_1$ and $z=z_2$ are any two points in $\overline{R}$ such that $|z_1-z_2|<\delta$. Now, if d is a positive number less than the radius of each U_{a_i}, let K' denote a circle interior to and concentric with K and having a radius which differs from that of K by less than d and also less than δ. Then let τ_i denote any arc of K which contains no arc σ_i and whose end points are also end points of arcs σ_i and let σ_i' and τ_i' denote the arcs of K' which are composed of the points of intersection of the circle K'

and the radii of K through all the points of σ_i and all the points of τ_i respectively. It follows that

$$\int_{\tau_i'} |f'(z)|\, ds = \int_{\tau_i} \frac{r-d}{r}(|f'(z)| + \eta(z))ds,$$

where r is the radius of K and $|\eta(z)| < \epsilon$. If h denotes the length of the polygon J and h' the length of the transform of K' under the transformation $w = f(z)$ and n the number of vertices of J, then

$$h' = \sum_{i=1}^{n} \int_{\sigma_i'} |f'(z)|\, ds + \sum_{i=1}^{n} \int_{\tau_i'} |f'(z)|\, ds$$

$$< n\epsilon + \sum_{i=1}^{n} \int_{\tau_i} |f'(z)|\, ds + \frac{r-d}{r}\epsilon h,$$

and

$$h' > \sum_{i=1}^{n} \int_{\tau_i} |f'(z)|\, ds - \frac{d}{r}h - \frac{r-d}{r}\epsilon h.$$

Since

$$h - n\epsilon < \sum_{i=1}^{n} \int_{\tau_i} |f'(z)|\, ds < h,$$

it follows that

$$h - n\epsilon - \frac{d}{r}h - \frac{r-d}{r}\epsilon h < h' < n\epsilon + h + \frac{r-d}{r}\epsilon h.$$

Now if $d < \epsilon$, then

$$h - \left(n + \frac{h}{r} + h\right)\epsilon < h' < h + (n + h)\epsilon,$$

or

$$| h - |h' < \left(n + \frac{h}{r} + h\right)\epsilon.$$

Hence if ρ is any positive number and d is sufficiently small then $|h - h'| < \rho$.

From this result follows immediately, as far as it concerns the $\lim_{n\to\infty} l_n$, the special case of the second theorem of the paper in which the region Σ is the interior of a simple polygon in a plane.

4. *The Region Σ in General.* Let Σ be a region in the w-plane and let the boundary of Σ be a rectifiable simple closed curve, C. Then no level line of Σ has a length greater than the length of the boundary of Σ. For let there exist a level line of Σ, say G, which has a length, g, which is greater than the length, l, of C. It is assumed that $w=0$ is an interior point of G; no loss of generality follows from this assumption. Then there exists a function $w=f(z)$ which is analytic in the unit circle and which maps in a one-to-one and conformal way the interior of the unit circle on Σ such that $f(0)=0$ and $f'(0)=1$ and such that G is the image under the transformation $w=f(z)$ of a circle, H, concentric with the unit circle, K. Further, let $\{P_n\}$, $n=1, 2, 3, \cdots$, be a sequence of simple polygons inscribed in C which are such that $\lim_{n\to\infty} d_n=0$, where d_n is the length of a side of P_n which is not shorter than any other side of P_n, and such that $w=0$ is an interior point of each P_n. Then there exists a function $w=f_n(z)$ which maps the interior of the circle $|z|=1$ on the interior of P_n in a one-to-one and conformal way such that the point $w=0$ corresponds to the point $z=0$ and the derivative of the function $w=f_n(z)$ at $z=0$ is unity. By a theorem* of Carathéodory, and the fact that there is only one function which maps the interior of the unit circle on Σ in a one-to-one and conformal way such that $w=0$ corresponds to $z=0$ and the derivative of the mapping function at $z=0$ is unity, it follows that the sequence of functions $\{w=f_n(z)\}$, $n=1, 2, 3, \cdots$, approaches the function $w=f(z)$, $|z|<1$, uniformly on any closed set of points which is contained in the interior of the unit circle. Consequently the sequence of derivatives of the functions $w=f_n(z)$, $\{w=f_n'(z)\}$, $n=1, 2, 3, \cdots$, converges uniformly to the derivative of $w=f(z)$ on any closed set of points which is in the interior of the unit circle.

* See Mathematische Annalen, vol. 72 (1912), pp. 120–126. Also Bieberbach, *Lehrbuch der Funktionentheorie*, vol. II, pp. 12–15.

Now, let ν be a positive number less than $g-l$ and p a positive integer such that

$$|f'(z) - f_p'(z)| < \frac{\nu}{2\pi},$$

for z on H. Then

$$|f_p'(z)| = |f'(z)| + \eta(z),$$

where $|\eta(z)| < \nu/2\pi$ for z on H, and

$$\int_H |f_p'(z)|\, ds = \int_H |f'(z)|\, ds + \int_H \eta(z) ds.$$

But

$$\int_H |f'(z)| ds = g \text{ and } \int_H |f_p'(z)|\, ds$$

is the length of the image of H under the transformation $w=f_p(z)$, $|z|<1$. The latter image is a level line of the pencil of level lines of the polygonal region bounded by P_p, which has the point $w=0$ as its pole. If the length of this level line is denoted by g_p, then $g_p > g-\nu$ and hence $g_p > l$.

According to the result for polygonal regions which was obtained above, there exists a circle, H', with center $z=0$ and a radius of length less than unity but greater than the length of a radius of H such that the length, g_p', of the image of H' under the transformation $w=f_p(z)$ differs from the length, l_p, of P_p by an amount less than $g_p - l$. Since $l_p < l$ it follows that $g_p' < g_p$. But this result contradicts the fact that by a theorem* of Hardy in connection with the formula for the length of an analytic transform of a rectifiable curve, which is given above. It follows that $g_p' > g_p$.

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* See Proceedings of the London Mathematical Society, (2), vol. 14 (1915), pp. 269–277. Also Landau, *Ergebnisse der Funktionentheorie*, p. 85. The theorem is that if $w=f(z)$ is analytic and not constant for $|z|<R$, then $\int_0^{2\pi} |f(re^{i\theta})|\, d\theta$, $z=re^{i\theta}$, $0 \leqq \theta \leqq 2\pi$ and $0<r<R$, is a continuous increasing function of r for $0<r<R$.

Only a special case of this theorem is used above. The functions concerned are only those which map in a one-to-one and conformal way the interior of the unit circle on the interior of a simple polygon.

Thus the length of any level line of Σ is not greater than the length of the boundary of Σ. The theorem of Hardy then implies that the length of any level line of Σ is less than the length of the boundary of Σ.

That any sequence of level lines $\{G_{t_n}\}$, $t_n < t_{n+1}$, $n = 1, 2, 3, \cdots$, and $\lim_{n\to\infty} t_n = 1$, which belong to any given pencil of level lines $[G_t]$, $0 < t < 1$, of level lines of any planar region Σ whose boundary is connected and contains more than one point is an approximating sequence of curves of Σ follows easily from the one-to-one conformal mapping of the interior of the unit circle on Σ, which determines the sequence of level lines $\{G_{t_n}\}$, $t_n < t_{n+1}$, $n = 1, 2, 3, \cdots$, as the image of a sequence of circles, $\{H_n\}$, $n = 1, 2, 3, \cdots$, which are concentric with and interior to the unit circle and such that $\lim_{n\to\infty} r_n = 1$, where r_n is the length of the radius of the circle H_n. Evidently, the level line G_{t_n} is in the interior of the level line $G_{t_{n+1}}$.

Now, if the boundary of Σ is a rectifiable simple closed curve of length l, it follows readily from certain known results* that if l_n is the length of the rectifiable curve C_n of the approximating sequence of curves $\{C_n\}$, $n = 1, 2, 3, \cdots$, of the region Σ and if $\lim_{n\to\infty} l_n$ exists, then we have $\lim_{n\to\infty} l_n \geqq l$. Hence, with what has preceded, if l_{t_n} is the length of the level line G_{t_n} of the sequence of level lines $\{G_{t_n}\}$, $t_n < t_{n+1}$, $n = 1, 2, 3, \cdots$, $\lim_{n\to\infty} t_n = 1$, then $l_{t_n} < l$ and, since by the theorem of Hardy $l_{t_n} < l_{t_{n+1}}$, $n = 1, 2, 3, \cdots$, it follows that $\lim_{n\to\infty} l_{t_n} = l$. Thus $\lim_{t\to 1} \Lambda(t) = l$ and the sequence of level lines $\{G_{t_n}\}$, $t_n < t_{n+1}$, $n = 1, 2, 3, \cdots$, $\lim_{n\to\infty} t_n = 1$, is an approximating sequence of curves of Σ as specified in the second theorem.

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* In particular, Theorem V, p. 519 of Hahn, *Theorie der reellen Funktionen*, vol. I.