THE BITANGENTS AT A POINT OF A DESMIC SURFACE

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A plane tangent to a surface of the fourth order cuts the surface in a curve of the fourth order with a singular point at the point of contact. Through this point there can then be drawn six lines tangent to the curve elsewhere. So through a point on a surface of the fourth order there are in general six bitangents.

The second points of contact of these bitangents lie on a conic which, for a desmic surface, degenerates into two lines. Thus we have two sets of three bitangents; numerous geometrical properties of one triple have been given by G. Humbert and others.* Concerning the second triple only some general properties that apply to both sets are known; no characterizing geometric properties have been published. This paper presents one.

For brevity in the equations we shall use the symbol $\overline{ij}$ to denote $(x_i^2 - x_j^2)$.

The pencil of desmic surfaces in variables (y) and parameter (x),

$$(1)\qquad \begin{vmatrix} (y_0^2 - y_1^2)(y_2^2 - y_3^2), & (y_0^2 - y_2^2)(y_1^2 - y_3^2) \\ \overline{01}\cdot\overline{23} \quad , & \overline{02}\cdot\overline{13} \end{vmatrix} = 0,$$

determines a cubic complex, and the cubic cone of this complex of which an arbitrary point $P(x)$ is the vertex, cuts the plane $y_3=0$ in the cubic curve

$$(2)\quad \overline{03}y_1y_2(x_2y_1 - x_1y_2) + \overline{13}y_2y_0(x_0y_2 - x_2y_0) + \overline{23}y_0y_1(x_1y_0 - x_0y_1) = 0.$$

* Humbert, G., *Sur la surface desmique du quatrième ordre*, Journal de Mathématiques, (4), vol. 7 (1891), pp. 353–398.

Jessop, C. M., *Quartic Surfaces with Singular Points*, Cambridge, 1916.

Mathews, R. M., *Cubic curves and desmic surfaces*, Transactions of this Society, vol.28 (1926), pp. 502–522, and ibid., vol. 30 (1928), pp. 19–23.

The tangent plane at $P(x)$ to the desmic surface through P is

$$(3)\quad \overline{13}\cdot\overline{23}\cdot\overline{12}x_0y_0 + \overline{23}\cdot\overline{03}\cdot\overline{20}x_1y_1 + \overline{03}\cdot\overline{13}\cdot\overline{01}x_2y_2 + \overline{01}\cdot\overline{12}\cdot\overline{20}x_3y_3 = 0,$$

and cuts $y_3=0$ in a line ABC with

$$(4)\quad \begin{cases} A: \overline{03}x_1x_2 : \overline{13}x_2x_0 : \overline{23}x_0x_1, \\ B: \overline{03}(x_1x_2 + x_0x_3) : \overline{13}(x_0x_2 + x_1x_3) : \overline{23}(x_0x_1 + x_2x_3), \\ C: \overline{03}(x_1x_2 - x_0x_3) : \overline{13}(x_0x_2 - x_1x_3) : \overline{23}(x_0x_1 - x_2x_3). \end{cases}$$

The points A, B, C are on the cubic curve (2) and the lines PA, PB, PC are the known bitangents.*

The cubic curve (2) has for Hessian curve the cubic

$$(5)\quad \begin{vmatrix} \overline{23}x_1y_1 - \overline{13}x_2y_2, & \overline{23}(x_1y_0 - x_0y_1), & \overline{13}(x_0y_2 - x_2y_0) \\ \overline{23}(x_1y_0 - x_0y_1), & \overline{03}x_2y_2 - \overline{23}x_0y_0, & \overline{03}(x_2y_1 - x_1y_2) \\ \overline{13}(x_0y_2 - x_2y_0), & \overline{03}(x_2y_1 - x_1y_2), & \overline{13}x_0y_0 - \overline{03}x_1y_1 \end{vmatrix} = 0.$$

To write the coordinates of an arbitrary point Q on the line ABC, it is convenient to find $\overline{A}$ as the harmonic conjugate of A with respect to B and C, namely

$$(6)\qquad \overline{A} : \overline{03}x_0x_3 : \overline{13}x_1x_3 : \overline{23}x_2x_3$$

and then to write the coordinates for

$$Q \equiv \rho A + \sigma\overline{A}.$$

On solving $A\overline{A}$ simultaneously with the Hessian (5), we find that the values of the ρ/σ ratio for the points of intersection are roots of the cubic

$$(7)\quad 2\rho^3x_0x_1x_2x_3 + \rho^2\sigma\sum x_0^2x_1^2 + 6\rho\sigma^2x_0x_1x_2x_3 + \sigma^3\left(\sum x_0^2x_1^2 - \sum x_0^4\right) = 0.$$

Next, the line PQ is an arbitrary line through P on the tangent plane at P. We write the coordinates of an arbitrary point R on PQ in the form

$$R \equiv \lambda Q + \mu P$$

* For these equations and results see Mathews, loc. cit.

and substitute in the equation of the desmic surface. The result is a fourth degree equation in the ratio λ/μ whose four roots give the intersections R of PQ with the surface. Since P is a double point on the intersection of the plane and surface this quartic equation reduces to the quadratic

$$(8)\quad \left\{\begin{aligned} &\lambda^2\{(z_0^2 - z_1^2)z_2^2 n - (z_0^2 - z_2^2)z_1^2 m)\} \\ &\quad + 2\lambda\mu\{[(z_0^2 - z_1^2)z_2x_2 + z_2^2(x_0z_0 - x_1z_1)]n \\ &\quad - [(z_0^2 - z_2^2)z_1x_1 + z_1^2(x_0z_0 - x_2z_2)m]\} \\ &\quad + \mu^2\{[(z_0^2 - z_1^2)\overline{23} + z_2^2\overline{01} + 4(x_0z_0 - x_1z_1)x_2z_2]n \\ &\quad - [(z_0^2 - z_2^2)\overline{13} + z_1^2\overline{02} \\ &\quad + 4(x_0z_0 - x_2z_2)x_1z_1]m\} = 0, \end{aligned}\right.$$

where $m=k\ \overline{02}\cdot\overline{13}$, $n=k\ \overline{01}\cdot\overline{23}$, ($k$ a factor of proportionality), and where the z's are the coordinates of Q.

Now, when PQ is a bitangent to the surface the two roots of this quadratic are equal. We set the discriminant equal to zero, substitute for the z's the coordinates of Q as

$$Q = \rho A + \sigma\overline{A}$$

and obtain an equation of the sixth degree in the ratio ρ/σ. Three of the roots are $\sigma=0$ and $\rho/\sigma=\pm 1$ giving A, B, C, the points for the known bitangents, and leaving for the determination of the other three exactly equation (7) which is the condition that Q lie at the intersections of the line ABC with the Hessian cubic.

The cubic cone at P cuts an arbitrary plane, not through P, in a cubic curve which has a definite Hessian; when the points of the latter are joined to P we get a cubic cone which may be called the Hessian cone corresponding to the first. Thus we have the following theorem.

THEOREM. *The bitangents to a desmic surface at a point thereon lie in two triples; the lines of one triple lie on the cubic cone of the system proper to P, and of the other on the corresponding Hessian cone.*

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