BULLETIN

OF THE

AMERICAN MATHEMATICAL SOCIETY

THE OCTOBER MEETING IN NEW YORK

The two hundred fifty-seventh regular meeting of the American Mathematical Society was held at Columbia University, on Saturday, October 29, 1927, extending through the usual morning and afternoon sessions. Attendance included the following ninety-five members of the Society:

Alexander, Allison, R. C. Archibald, R. G. Archibald, J. W. Arnold, A. A. Bennett, Benton, Bratton, Brinkmann, R. W. Burgess, Carrié, E. T. Carroll, Coble, Cramlet, Edmondson, Eisenhart, Engstrom, Fenn, Fiske, Fite, D. A. Flanders, Forsyth, Fort, M. C. Foster, Frink, Garver, Gehman, Gill, Glenn, P. H. Graham, Gronwall, C. C. Grove, Guggenbühl, Hebbert, L. S. Hill, Hille, Hofmann, Hoyt, Huber, Dunham Jackson, Joffe, R. A. Johnson, Kasner, Kholodovsky, Kline, Koopman, Kormes, Lamson, Langman, Lefschetz, MacColl, S. P. Mead, Meder, H. H. Mitchell, Richard Morris, Mullins, Murnaghan, C. A. Nelson, Newman, K. E. O'Brien, Ore, Paradiso, Pfeiffer, Pierpont, Post, R. G. Putnam, Reddick, R. G. D. Richardson, D. E. Richmond, Ritt, Robertson, Rutledge, Schoonmaker, Seely, Siceloff, Simons, Slotnick, Smail, P. A. Smith, Virgil Snyder, Stetson, Tamarkin, J. E. Thompson, Veblen, H. E. Webb, Wedderburn, Weida, Weisner, A. P. Wheeler, Whittemore, Widder, W. A. Wilson, Margaret M. Young, Zippin, Zobel.

The Secretary announced the election of the following persons to ordinary membership in the Society:

Mr. Abraham Adrian Albert, University of Chicago;
Dr. Ralph George Archibald, Columbia University;
Mr. Armond W. Bear, Marquette University;
Mr. Richard Stevens Burington, Case School of Applied Science;
Professor Harold Hardesty Downing, University of Kentucky;
Mr. Walter Henry Gage, Victoria College, Victoria, British Columbia;
Mr. Ellis Richard Heineman, Michigan Agricultural College;
Mr. Isaac Newton Heminger, San Jose, Calif;
Professor Edward Milton Little, University of Montana;
Professor Ernest Lloyd Mackie, University of North Carolina;
Professor Marion Lee MacQueen, Southwestern College;
Professor Max Morris, Case School of Applied Science;
Mr. Kenneth Ernest Moyle, High School, Somerville, N. J.;
Professor Oystein Ore, Yale University;
Mr. Rafael Sanchez-Diaz, University of Porto Rico;

Miss Mildred Ellen Taylor, Knox College;
Mr. Carleton Russell Worth, University of Arkansas;
Professor Frank Lynwood Wren, George Peabody College for Teachers;

Nominees of the Georgia Power Company as sustaining member:

Mr. C. E. Bennett, Georgia Power Company, Atlanta;
Mr. S. C. Bleckley, Georgia Power Company, Atlanta;
Mr. F. A. Brine, Georgia Power Company, Atlanta;
Mr. W. A. Hammel, Georgia Power Company, Atlanta;
Mr. W. P. Hammond, Alabama Power Company, Birmingham.

At the meeting of the Board of Trustees, the editors of the Bulletin and of the Transactions were authorized to sign printing contracts with the Banta Publishing Company.

The council fixed the dates for regular New York meetings of the Society in New York during 1928 as follows: February 25, October 27, and, in place of the meeting usually held on the last Saturday in April or the first Saturday in May, a two-day meeting on April 6–7, the Friday and Saturday before Easter. An invitation for the summer meeting of 1931 was received from the University of Minnesota.

On nomination of the Western group, a committee consisting of Professors E. W. Chittenden (chairman), L. M. Graves, and M. H. Ingraham was appointed to choose a Symposium speaker for the Easter, 1928, meeting at Chicago.

Nominations for officers and other members of the Council was adopted, and ordered printed on the official ballot.

President Snyder presided at the morning session. In the afternoon two sectional meetings were held; Dr. Gronwall presided over the section of Analysis and Point Sets, at which papers numbered 11–26, 46–48 were read, and Professor Pierpont over that of Algebra and Geometry, at which papers numbered 27–45 were read.

At the request of the Program Committee, Professor F. D. Murnaghan delivered an address at the end of the morning session, entitled *Modern hydrodynamical theory, with special reference to aeronautics*. This address will appear in full in an early number of this Bulletin. Titles and abstracts of the other papers read at this meeting follow below. The papers of Ayres, Bray, Copeland, Douglas, Frink (second and third papers), Garver (third paper), Hickey, Hollcroft, Michal, Roos, Seely, Stetson, Weatherburn, and Whyburn were read by title. Mr. Gourin and Mr. Wang were introduced by Professor Ritt, Miss Hickey by Professor Evans, and Professor Weatherburn by Professor Kasner.

1. Professor C. H. Forsyth: *The amount of a single sum of money at any number of rates of interest.*

In the issue of this Bulletin for June-July, 1921, the author presented certain generalizations of some of the most fundamental formulas of the mathematical theory of finance. Those generalizations were based upon the use of two and three rates of interest. The purpose of the present paper is to show the nature and form of the function which is obtained when the number of rates of interest is increased indefinitely.

2. Professor C. H. Forsyth: *The determination of monthly items from yearly items of the secular trend of a time series when that trend is assumed to be other than linear.*

In the case of a linear trend, a straight line is fitted to yearly items or totals of a time series, and the coefficients of the linear equation divided by 12 give the monthly items. Occasionally, the trend is clearly other than linear; then the yearly items are easily obtained when once the type of curve is selected, but the corresponding monthly items cannot be found by simple division. The author shows they may be found by interpolation.

3. Dr. H. P. Robertson (National Research Fellow): *An alternative to De Sitter's cosmology.*

The 4-dimensional manifold defined by the line element $ds^2 = -e^{2kct} \cdot (dx^2+dy^2+dz^2)+c^2dt^2$, $k>0$, is of constant Riemannian curvature, and is therefore mathematically equivalent to De Sitter's universe. Interpreting the variable t occurring in this form as the proper-time of an observer at a point x, y, z =constant, the "mass-horizon" and "arrest of time at the horizon" paradoxes of De Sitter are obviated. Because of the dynamical form of the line element, however, natural processes are not reversible; in particular, a shift of spectral lines in light from distant objects is predicted which is in accordance with existing data on extra-galactic nebulae and which leads to a value $k=5\cdot 10^{-28}(cm)^{-1}$. The universe may be represented as a hypersphere in a flat 5-space, hence the relations existing between freely moving observers (the analogue of the Lorentz transformations of special relativity) are equivalent to the rotations of flat 5-space.

4. Dr. H. P. Robertson: *Conditions under which a certain type of solution of Schrödinger's wave equation can be found*

The conditions under which the fundamental equation of Schrödinger's "wave-mechanics" is solvable by separation of variables are found for the case of a conservative system whose kinetic energy is expressed in orthogonal coordinates. It is shown that in addition to the conditions given by P. Stäckel for the solution of the corresponding Hamilton-Jacobi equation a certain functional relation must be satisfied.

5. Dr. L. S. Hill: *Certain matrices in finite algebraic fields, and their relation to code telegraphy.*

Let a matrix in which no determinant of any order vanishes be called, for convenience, uniform; the present paper directs attention to the bearing of such matrices, constructed in various finite algebraic fields, upon certain important problems of accuracy assurance in the transmittal of electrical communications,—telegraph, cable, radio. A special type of uniform matrix, which appears to be directly amenable to practical application, is treated with particular care. It would be easy to construct a mechanical device, similar to the ordinary adding machine but somewhat less complex, which would automatically accomplish all checking operations based upon this special type. The paper leads to a further problem which is of purely algebraic interest, and which will be studied later.

6. Professor D. E. Richmond: *A complete set of relations between type numbers of extremals joining a pair of points.*

Let A and B be a pair of points in an extremal-convex region for a calculus of variations problem. An extremal arc AB may be assigned a type number n equal to the number of points on the arc conjugate to A. Let M_n represent the number of extremals of type n joining A to B. The author has proved the completeness of a certain set of number relations between the integers M_n, when no type number n exceeds 2. When extremals of type 2 exist joining A to B, the relations are $M_0=2$, $M_0-M_1+M_2=1$. It has been shown that there exists a calculus of variations problem and a pair of points A and B such that the number of extremals of type i ($i=1, 2, 3$) is M_i, where the M_i's are arbitrary positive integers satisfying the relations stated.

7. Dr. A. H. Copeland: *Admissible numbers in the theory of probability.*

The problem of establishing the consistency of the fundamental assumptions of the theory of probability is proved in this paper to be equivalent to that of showing the existence of certain sets of numbers, $A(p)$. These assumptions are consistent in the case of a given probability, p, if and only if the corresponding set, $A(p)$, is non-empty. Admissible numbers are the numbers that belong to the sets $A(p)$. Every admissible number must satisfy an infinite set of independent equations and hence the consistency of the fundamental assumptions seems doubtful. Not only is the consistency of the assumptions open to question, but also the customary definition of independence is so loosely stated that, if one takes the most obvious interpretation of this definition, the theorem of compound probability is false. However, it is shown that this definition of independence can be so stated that this theorem is true. It is also proved that the sets $A(p)$ are non-empty and hence that the fundamental assumptions are consistent. Moreover, it is proved that each point x of the interval $0<x<1$ is a point of condensation of each of the sets $A(p)$ where $0<p<1$.

8. Dr. A. H. Copeland: *Stability of the motion of a gyroscope in contact with a smooth horizontal plane.*

The problem considered in this paper is the stability of the motion of a gyroscope which is a smooth body of revolution and which is acted upon only by gravity and a constraint which keeps it in contact with a smooth horizontal table. The motion is described in terms of a curve Γ traced by the axis of the gyroscope on the unit sphere whose center is at the center of mass of the gyroscope and whose orientation is fixed in space. The north pole is taken at the top of the sphere. The curve Γ is bounded by two parallels of latitude. It is proved that for slight changes in the initial conditions large changes can occur in the positions of these parallels, that is, the motion can be unstable; but these unstable motions can be cut out by giving the gyroscope a sufficiently large spin. The method used is an extension of the method employed in the author's paper entitled *Types of motion of the gyroscope* (see abstract No. 10, below).

9. Dr. A. H. Copeland: *The motion of a gyroscope with variable spin.*

This paper is a discussion of the motion of a gyroscope which is acted upon by gravity, by a constraint which keeps one point on its axis fixed in space, by a torque, S, which tends to increase or decrease the spin of the gyroscope about its axis, and by no other forces. The case in which S is identically zero gives a classical problem. In this paper it is assumed that S is a function of the time, the position of the gyroscope, and the velocities of the coordinates determining the position. The effect of S is traced throughout the entire motion. The major portion of this paper is devoted to the case in which the spin is monotone and nowhere stationary. All the possible types of motions of gyroscopes with always increasing spins, or always decreasing spins, and all the possible changes in these types are determined. It is also shown that some types of motion are unstable and break down into other types when small changes are made in the spin.

10. Dr. A. H. Copeland: *Types of motion of the gyroscope.*

A method of graphical representation of gyroscopic motion is developed in this paper. A simple representation is given of the complete history of all motions of all gyroscopes subject to the following restrictions: each gyroscope is acted upon by gravity and by a constraint which keeps one point on its axis fixed in space, and by no other forces. With the aid of Osgood's intrinsic equations, this history is extended to include new intrinsic properties of the space cone traced by the axis of the gyroscope. By means of these methods it is possible to list the properties of the motion more fully than has hitherto been done, exhibit their dependence upon the initial conditions, and show how one type of motion changes into another as the initial conditions are varied continuously.

11. Professor Edward Kasner: *Non-monogenic (or polygenic) functions.*

The author proposes the term *polygenic* for arbitrary functions $w=\phi(x, y)+i\psi(x, y)$ of the complex variable $z=x+iy$ where the derivative depends on the slope m as well as the point z. To each point z we have then ∞^1 values of $dw/dz=\alpha+i\beta$, which when plotted as points in a new plane (α, β) form a circle called the *derivative circle*. Thus to the entire polygenic function corresponds a *congruence* of circles. This in special cases degenerates; e.g., if the given function is monogenic the circles all become points; for a function of $x-iy$, we obtain in the (α, β) plane ∞^1 circles with centers at the origin; if the centers all lie on the axis of α, the function must obey merely one of the Cauchy-Riemann equations, etc. The *mean derivative* of any polygenic function is defined as $H+iK$, where H, K are the coordinates of the center of the derivative circle. The mean derivative is monogenic when and only when ϕ and ψ obey the Laplace equation. The transformation is then termed general harmonic by the author; this includes conformal as a very special case. The point (α, β) on any derivative circle always moves, as m changes, with angular velocity twice that of m and in the opposite direction. For each circle a certain initial radius must be fixed; the complete graph is thus a congruence not of circles but of *clocks*, each clock with four coordinates. A general theory of such congruences is given.

12. Professor J. F. Ritt: *Certain points in the theory of Dirichlet series.*

The results of this paper, all of which deal with Dirichlet series, bear upon the following topics: Picard's theorem, integral functions, asymptotic convergence. The first part contains generalizations of the theorems of Landau and Schottky. The second part deals with integral functions defined by everywhere absolutely convergent Dirichlet series, and develops the relations between the growth of the modulus of the function and the coefficients of the series. The third part carries over to Dirichlet series the theory of asymptotic convergence of power series.

13. Dr. B. O. Koopman: *Some theorems on inversion.*

Consider the substitution (1). $y_i=f_i(x_1, \cdots, x_n)$, $i=1, 2, \cdots, n$, where the f's are analytic functions of $(x_1, \cdots, x_n)=(x)$ for $(x)=(0)$. This substitution has only been studied completely when the Jacobian J does not vanish for $(x)=(0)$, when the transformation has a unique analytic inverse. The object of our paper is to investigate the substitution in the case where $(x)=(0)$ is a non-specialized point of the complex locus $J=0$. By means of non-singular substitutions of the variables (x) and (y) we reduce the substitution (1) to a standard form, from which it appears that it may be inverted by means of algebroid functions. The results are then applied to the extension of the theorem of functional dependence. (See Osgood, *Lehrbuch der Funktionentheorie,* II, chap. 2, §§19, 20, 23.)

14. Mr. Eli Gourin: *On irreducible polynomials.*

Given an irreducible polynomial $Q(y_1, y_2, \cdots, y_p)$ of p variables $y_1, y_2, \cdots, y_p$, the object is to determine all positive integers $t_1, t_2, \cdots, t_p$ for which the polynomial $Q(y_1^{t_1}, y_2^{t_2}, \cdots, y_p^{t_p})$ is reducible.

15. Mr. Eli Gourin: *Periodic Weierstrass zeta functions.*

This paper presents a study of conditions for which Weierstrass's zeta function becomes a singly periodic function.

16. Mr. S. C. Wang: *On a linear addition theorem in three arguments.*

It is shown in this paper that all continuous solutions of the functional equation of real variables $f(x+y+z)=f(x+y)+f(x+z)+f(y+z)-f(x)-f(y)-f(z)$ are of the form $f(t)=At^2+Bt$, where A and B are two independent arbitrary constants.

17. Dr. T. C. Benton: *On continuous curves homogeneous except for certain points.*

This paper proves the following theorems: (1) A point set M is homogeneous except for n points $c_1, c_2, \cdots, c_n$ if for any pair of points a_1, a_2 distinct from all c_i a continuous (1-1) correspondence ψ exists such that $\psi(M)=M$ and $\psi(a_1)=a_2$; and for every i there exists a point x_i such that no correspondence π exists such that $\pi(M)=M$ and $\pi(x_i)=c_i$. (2) A bounded continuous curve homogeneous except for one point c_1 is a countable set ($\geqq 2$) of simple closed curves having only c_1 in common and such that only a finite number of them are of diameter greater than any given positive number. (3) A bounded continuous curve homogeneous except for c_1 and c_2 is a finite number ($\neq 2$) of arcs joining c_1 to c_2. (4) A bounded continuous curve homogeneous except for c_1, c_2, c_3 is either two sets homogeneous except for two points which have only one c_i in common and have the same number of arcs, or three such sets joined cyclically with the same number of arcs in each set. (5) An unbounded continuous curve homogeneous except for c_1 is a finite number of rays from c_1. (6) An unbounded continuous curve homogeneous except for c_1, c_2 does not exist.

18. Professor H. M. Gehman: *On extending a continuous (1-1) correspondence.* Second paper.

In continuation of results recently published (Transactions of this Society, vol. 28 (1926), pp. 252–265), it is proved that if any two plane point sets which are closed and bounded and which have the properties that (1) every maximal connected subset is a continuous curve, and (2) not more than a finite number of maximal connected subsets are of diameter greater than any positive number, are in continuous (1-1) correspondence in such a way that the correspondence preserves sides *in the* *same sense*, then a continuous (1-1) correspondence of their planes can be defined which is identical with the given correspondence for points of the two sets.

19. Professor W. A. Wilson: *On bounded regular frontiers in the plane.*

This paper appears in full in the present number of this Bulletin.

20. Professor R. G. Putnam: *Note on countable sets of points of non-separable metric spaces.*

It is shown in this note that every countable set of points of a non-separable metric space is homeomorphic with a linear set.

21. Dr. W. L. Ayres: *An elementary property of bounded domains.*

This paper will appear in full in an early number of this Bulletin.

22. Dr. W. L. Ayres: *Concerning subsets of a continuous curve which can be connected through the complement of the continuous curve.*

Let H denote the sum of the boundaries of the complementary domains of a continuous curve M in a plane S. These theorems are proved: (1) if an M-domain is simply connected with respect to M, there exists a simply connected domain R such that $R \cdot M = D$; (2) if every two points x and y of H may be connected through $x+y$ and $S-M$, then H is a continuous curve and either (a) H is the boundary of one of the complementary domains of M, or (b) M consists of three arcs with common end points and no two having any other point in common, or (c) M consists of three rays with the same vertex and no two having any other only its common, or (d) M consists of an open curve plus an arc having point in end points in common with the open curve; (3) if every two points x and y of M may be connected through $x+y$ and $S-M$, then $M=H$; (4) if every three points of H lie together on some complementary domain of M, then H is the boundary of one of the complementary domains of M.

23. Dr. G. T. Whyburn and Dr. W. L. Ayres (National Research Fellow): *On continuous curves in n dimensions.*

This paper will appear in full in an early number of this Bulletin.

24. Professor A. D. Michal: *Invariant functionals of functional forms.*

In the theory of algebraic invariants of algebraic forms in n variables, the underlying group is the general group of linear homogeneous transformations with non-vanishing determinants. This paper develops the first essentials of a corresponding theory in the space of real continuous functions of a real variable x. The "algebraic" *functional* forms are restricted to normal types and the fundamental group is taken to be the group of all linear functional Fredholm transformations for which unity is not a characteristic value. "Tensor" invariant functionals of the coefficients of the functional forms are considered as well as scalar ones. The laws of transformation of the coefficients of the functional forms incidentally give rise to a theory of certain interesting integral equations which are not of the classic types.

25. Professor A. D. Michal: *Affinely connected function space manifolds.*

The first part of this paper develops a theory of functional tensors in the space of real continuous functions of a real variable x. The analogues of affine connection, infinitesimal parallelism, covariant differentiation, geodesics, normal coordinates, and tensor extension of n-dimensional geometry are then considered. The functional invariants of the functional affine connection arise as the integrability conditions of functional equations with funcional derivatives. The "geodesics" satisfy non-linear integro-differential equations of static type. The theory of the Fredholm integral equation underlies the theory of functional normal coordinates. The paper closes with a discussion of functional quadratic differential forms followed by a development of a theory of "Riemannian" function space manifolds. This paper combined with a part of the preceding paper will appear in the American Journal of Mathematics.

26. Miss Deborah M. Hickey: *The equilibrium point of Green's function for an annular region.*

This paper derives the necessary and sufficient condition for an equilibrium point in an annular region in terms of Fourier series and in terms of elliptic functions. Experimental investigations by W. S. Vaughn seemed to indicate that the equilibrium point remained fixed when the pole in the region was moved. A necessary and sufficient condition that this point be fixed is $2\zeta(\omega')-\omega'\wp(\omega')=0$. By means of Fourier series it is shown that this equation is satisfied for no ratio of the half periods ω, ω'. Nevertheless the experimental results are justified by obtaining theoretically the upper bound of the shift, which is very small.

27. Professor Oystein Ore: *Newton polygons in the theory of algebraic fields.*

The determination of the development of an algebraic function in power series corresponds to a certain extent to the determination of the prime-ideal decomposition of a prime p in an algebraic field. By generalizing the method of Newton polygons and by using the author's investigations on algebraic numbers (Mathematische Annalen, vols. 96 and 97) the complete decomposition of all primes p is obtained for every field given by an arbitrary equation.

28. Professor Oystein Ore: *Some relations between groups and ideals in Galois fields.*

As a generalization of the "Zerlegungsgruppe" of Hilbert, a group $G\{\mathfrak{A}\}$ is introduced corresponding to an arbitrary ideal $\mathfrak{A}$. The group $G\{\mathfrak{A}\}$ embraces all the substitutions for which the system of numbers in $\mathfrak{A}$ as ideal is invariant. The construction of $G\{\mathfrak{A}\}$ is determined, and some properties of the corresponding subfield studied.

29. Professor O. E. Glenn: *The invariancy of infinite series.*

The terms of a convergent infinite series $a, b, c, \cdots, \infty$ may be considered to be the coefficients of a polynomial f of order n in an infinite number of variables. Let f, with $n=1$ for example, be transformed by the infinite cyclic transformation $x=\lambda x'+\mu y'+\nu z'+\cdots\infty$, $y=\lambda y'+\mu z'+\nu t'+\cdots\infty$, $z=\lambda z'+\mu t'+\nu u'+\cdots\infty, \cdots$, where the series $\lambda, \mu, \nu, \cdots, \infty$ is convergent. Then certain series in the form of infinite polynomials in the terms $a, b, c, \cdots$ are relative invariants. The sum of the series is an invariant, the invariantive relation being $a'+b'+c'+\cdots=(\lambda+\mu+\nu+\cdots)\cdot(a+b+c+\cdots)$. The property of convergence is an invariant property. A theorem on the product of two series is a theorem on the transformation of one series by the infinite cyclic transformation determined by the other.

30. Dr. Louis Weisner: *Quadratic fields in which cyclotomic polynomials are reducible.*

For any particular value of m, the quadratic fields in which the polynomial whose zeros are the primitive mth roots of unity is reducible can be determined by methods described in the second volume of Weber's *Algebra*; but it does not seem practicable to carry out the computations for a general m. The present writer avoids these computations by proving two lemmas from which it is inferred that the cyclotomic polynomial is reducible in certain quadratic fields. The number of these fields is found to be precisely the number of intransitive subgroups of index 2 of the group of the cyclotomic equation relative to $R(1)$. The quadratic fields determined are the only ones in which the cyclotomic polynomial is reducible.

31. Dr. R. G. Archibald: *Diophantine equations in division algebras.*

This paper obtains necessary and sufficient conditions for the solvability in integers of diophantine equations, of the type $U^2-aR^2=(A^2-abB^2)\cdot(F^2-bC^2)$, which arise in an attempt to satisfy the associativity conditions for an algebra Γ based on a quartic equation $x^4+px^2+n^2=0$, irreducible in the field of rational numbers. The irreducibility of this equation imposes the condition that each of $a=2n-p$, $b=-2n-p$, ab be different from a perfect square. The associativity conditions for such algebras Γ were obtained by Dickson (*New division algebras*, Transactions of this Society, April, 1926). In the present paper numerical examples are given which satisfy all the conditions obtained.

32. Dr. Orrin Frink (National Research Fellow): *An algebraic method of differentiation.*

The theory of analytic functions of a hypercomplex variable in the case of certain familiar commutative linear algebras containing nilpotent units is shown to lead to a method of obtaining the standard formulas of the differential calculus by purely algebraic means, without the use of limiting processes. The method can be used to find higher derivatives directly. It is shown that differentials may be treated as absolute infinitesimals when no higher derivatives are involved. In some cases the method can be used to find functions with prescribed addition theorems.

33. Dr. Orrin Frink: *A general type of algebras.*

The algebras studied here, which may be considered generalizations of Boolean algebras, are commutative and associative and obey the special laws $a+a=a$ and $a^2=a$ for all a's. Many examples are given, which are discussed with relation to supplementary postulates which some of them satisfy, such as $a+ab=a$, the dual of the distributive law, and the existence of identities of addition and multiplication. An important case is one in which the addition and multiplication operations are interpreted to mean least common multiple and greatest common divisor.

34. Dr. Orrin Frink: *Concerning the fundamental regions of elliptic functions.*

It is first shown that the only convex fundamental regions of elliptic functions are parallelograms and hexagons. Then the problem is solved of finding all possible sets of points in the plane such that a period parallelogram of fixed size, shape, and orientation will always contain exactly n of them no matter what its position in the plane.

35. Dr. Hazel E. Schoonmaker: *Non-monoidal involutions having a congruence of invariant conics.*

The purpose of this paper is to derive all birational involutorial point transformations of space which have the following properties: (a) each transforms every conic of a linear congruence into itself; (b) the transformations cannot be reduced birationally to the monoidal type.

36. Dr. Raymond Garver: *The binomial quartic as a normal form.*

This paper has appeared in the November-December number of this Bulletin.

37. Dr. Raymond Garver: *A rational normal form for certain quartics.*

This paper appears in full in the present number of this Bulletin.

38. Dr. Raymond Garver: *Tschirnhausen transformations on certain rational cubics.*

Normal forms for particular cubics are obtained, as well as a necessary and sufficient condition that a rational cubic may be transformed rationally into a binomial cubic.

39. Dr. Jesse Douglas (National Research Fellow): *The geometry of systems of K-spreads.*

This paper constructs for systems of K-spreads in N-space a theory analogous to the general geometry of paths ($K=1$) developed in a previous paper presented to the Society, May 7, 1927. If G is any group of coordinate transformations in the N-space, and H any group of parameter transformations on the K-spreads, we can have a G, H, geometry of K-spreads. This paper considers the affine, equivoluminar, and descriptive geometries of K-spreads, i.e., where G is the total group $\bar{x}^i=\bar{x}^i(x)$, and H is respectively (1) $u^\alpha=A^\alpha_\beta v^\beta$, (2) $u^\alpha=\phi^\alpha(v)$, where $|\partial u/\partial v|=\text{constant}$, (3) $u^\alpha=\phi^\alpha(v)$, ϕ^α arbitrary. For the affine geometry of K-spreads the fundamental partial differential equations are (4) $\partial^2x^i/\partial u^\alpha\partial u^\beta=H^i_{\alpha\beta}\ (x,\ \partial x/\partial u)$, where $H^i_{\alpha\beta}$ must satisfy the following condition: if $q^j_\alpha=A^\beta_\alpha p^j_\beta$, then $H^i_{\alpha\beta}(x,q)=A^\rho_\alpha A^\sigma_\beta H^i_{\rho\sigma}(x,p)$. A double tensor analysis arises in which Greek indices run from 1 to K, Latin indices from 1 to N. The formula for the affine connection is $\Gamma^i_{jk}=(K(K+1))^{-1}\partial^2H^i_{\alpha\beta}/\partial p^j_\alpha\partial p^k_\beta$. The integrability conditions for (4) are $B^i_{jkl}p^j_\alpha p^k_\beta p^l_\gamma=0$, where B^i_{jkl} is the curvature tensor based on the Γ's. A parameter transformation on the K-spreads converts (4) into $\partial^2x^i/\partial v^\alpha\partial v^\beta$ $=H^i_{\alpha\beta}(x,\ \partial x/\partial v)+(\partial x^i/\partial v^\lambda)G^\lambda_{\alpha\beta}(x,\ \partial x/\partial v)$. The equivoluminar geometry of K-spreads is characterized by the vanishing of the contracted G's: $G^\lambda_{\lambda\beta}=0$.

40. Dr. Jesse Douglas: *One-to-∞^1 surface transformations of space.*

The equations (1) $X=X(x,\ y,\ z,\ p,\ q,\ a)$, $Y=Y(x,\ y,\ z,\ p,\ q,\ a),\cdots,$ $Q=Q(x,\ y,\ z,\ p,\ q,\ a)$ associate with each surface element $x,\ y,\ z,\ p,\ q$ a series of ∞^1 surface elements $X,\ Y,\ Z,\ P,\ Q$. The ∞^2 elements of any surface Σ (or union) are thereby converted, in general, into ∞^3 elements so distributed that there is one and only one through each point of a certain region of the $X,\ Y,\ Z$ space, thus defining a Pfaff equation in the $X,\ Y,\ Z$ space. Under what circumstances will the Pfaff equation corresponding to an arbitrary Σ be exact, so that the transformation (1) may be regarded as converting any surface Σ into ∞^1 surfaces? The present paper proves that this occurs when and only when there exists a transformation of the parameter a, (2) $a=a(x,\ y,\ z,\ p,\ q,\ b)$, such that the equations which result by substituting (2) in (1): (1′) $X=\xi(x,\ y,\ z,\ p,\ q,\ b)$, $Y=\eta(x,\ y,\ z,\ p,\ q,\ b),\cdots,$ $Q=\kappa(x,\ y,\ z,\ p,\ q,\ b)$, define a *contact* transformation for each fixed value of b. As far as the present author knows, this result is not contained in Goursat's monograph *Le Problème de Bäcklund*, to the order of ideas of which the present paper belongs.

41. Dr. M. M. Slotnick (National Research Fellow): *A contribution to the theory of fundamental transformations of surfaces.*

Graustein has introduced a projective invariant of transformations F which we call the invariant C. The dual of this invariant, that is, the cross ratio in which a pair of corresponding tangent planes is divided by the focal planes of the line of the harmonic congruence, is called the invariant H. These two invariants are in the relation $CH=e_1g/(g_1e)$, where e, g, e_1, g_1 are the coefficients of the second fundamental quadratic forms of the surfaces. A study of transformations F based on these two invariants yields dual fundamental theorems, and renders many facts easier to prove than by the classical methods. The perspective transformation for which $H=1$ enters as the dual of the radial transformation for which $C=1$; and in the same way the transformation $K(C=-1)$ and the transformation $\Omega(H=-1)$ are duals of one another. The theorems obtained are applied to transformations R in general, and also to those transformations R which are also K or Ω or both. Some attention has also been paid to transformations F of various special types when the nets are isothermal-conjugate.

42. Dr. M. M. Slotnick: *A method of applying tensor analysis to the study of rectilinear congruences.*

Some interesting results are obtained when the methods of tensor analysis are applied to deal with congruences of straight lines in a euclidean 3-space, the lines being represented by Study's "dual coordinates." The fundamental tensor is that of the linear element of the spherical representation of the congruence. We write $m_i=p_ia+q_i^l a_l$, where m represents the coordinates of the middle point of the line of the congruence, a its direction cosines; and the subscripts on m and a represent differentiation with respect to the corresponding parameters. The coefficients of linear combination p_i and q_i^l are tensors which form the basis of the study. These are evaluated in terms of the fundamental tensor, and a second tensor whose components are the coefficients of the second fundamental quadratic form of the congruence.

43. Professor T. R. Hollcroft: *On nets of manifolds in i dimensions.*

The characteristics of a net in two dimensions have been fully determined, but little has been done in three dimensions and the author has been able to find nothing for dimensions greater than three. In this paper, the characteristics of a net of manifolds in any number of dimensions are obtained. This is done by establishing a (1, 1) correspondence between the lines of a plane and the manifolds of a net, and obtaining the characteristics of the branch-point curve. The number of manifolds of the net that have two hypernodes, one hyperbinode, two simple contacts or a stationary contact with another member of the net, are obtained. A multiple basis point of a net reduces all the characteristic numbers except the order of the jacobian. When the basis point is simple, this reduction occurs only for a net of curves in two dimensions. The amount of this reduction is found for any number of basis points of any given multiplicity.

44. Dr. J. M. Stetson: *Periodic conjugate nets.*

A net which can be transformed into itself by a sequence of Levy transformations, p of them using derivatives with respect to u, and q of them using derivatives with respect to v, we call periodic (p, q). Such nets exist in spaces of $p+q-1$ or fewer dimensions. We find many types of transformations of nets periodic (p, q), all having theorems of permutability. Reciprocally derived nets and nets periodic under Laplace transformations seem to be the only cases hitherto considered. All the transformations known for these are simple special cases of the results of this paper.

45. Professor C. E. Weatherburn: *On curvilinear congruences.*

In an earlier paper (*On congruences of curves*, Tôhoku Mathematical Journal, 1927) the author has shown how the theory of curvilinear congruences in ordinary space of three dimensions may be extended along the lines followed for rectilinear congruences, making use of oblique curvilinear coordinates. The present paper contains a further extension of the theory.

46. Dr. C. F. Roos (National Research Fellow): *A dynamical theory of economics.*

In his recent papers in mathematical economics, the author has been primarily concerned with the mathematical aspects of the problems of the new dynamical economics, developed by G. C. Evans and himself, and references to existing economics papers have been avoided. It seems advisable, therefore, to give a history of these problems from the view point of the mathematical economist. In an attempt to do this, the present paper gives ample references to all existing related works known to the author, and shows how the work of Evans and the author is a generalization of existing theories of static equilibrium.

47. Professor H. E. Bray: *On uniform absolute continuity.*

Given a summable function $f(x)$ and a family of continuous functions $f(\mu, x)$ such that $\lim_{\mu\to 0} f(\mu, x) = f(x)$ almost everywhere; if $f(\mu, x)$ is non-negative, a necessary and sufficient condition that $\lim_{\mu\to 0}\int_a^b f(\mu, x)dx = \int_a^b f(x)dx$ is (a): that the absolute continuity of the indefinite integral $\int f(\mu, x)dx$ be uniform for all μ(de la Vallée Poussin). Another well known condition which is sufficient, being more restrictive than (a), is (b): that a summable function $g(x)$ exist such that $f(\mu, x) < g(x)$ for all μ. In this paper we find a necessary condition, bearing upon $f(x)$ itself, in order that a function $f(\mu, x)$ belonging to a certain class and satisfying (a) may also satisfy (b). The class of approximations considered includes the Poisson integral and also the Landau and Weierstrass integrals as special cases.

48. Dr. Caroline E. Seely: *Kernels of positive type.*

The author shows that if a kernel $K(s, t)$ is such that an infinite number of its iterated kernels are of positive type with respect to all linear combinations of its principal functions, then the resolvent kernel has all its poles real and simple.

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