# THE THIRD EDITION OF PICARD'S TRAITÉ

*Traité d'Analyse*. By Émile Picard. Troisième édition. Revue et augmentée avec la collaboration de Gaston Julia. Volume I. Paris, Gauthier-Villars, 1922. xvii + 593 pp.

Professor Picard's *Traité d'Analyse* has for many years past been regarded as one of the classics of modern mathematical literature. It is therefore to be presumed that most of the readers of the BULLETIN are familiar with the second edition of the present volume, which appeared more than twenty years ago and was reviewed* at that time by the late Professor Bôcher. Hence the present review will be concerned only with the changes and additions that have been made in the third edition.

The additional material comprised in the present volume amounts to about one hundred and ten pages, nearly one-fourth the content of the first volume of the second edition. With the exception of a few pages on non-euclidean geometry added to part three, the additions are scattered through parts one and two, the great bulk of them being found in part two, and particularly in that portion of it which deals with trigonometric series.

In part one the additional material is found mainly in more complete discussions of certain standard topics already treated in earlier editions, such as the first and second laws of the mean, functions of bounded variation, and change of variables in double integrals. Aside from the introduction of this supplementary material, a number of changes in the exposition have been introduced in certain portions of the text.

Part two is devoted mainly to a discussion of potential theory and trigonometric series. As mentioned before, the additions are much more extensive than in part one, covering in all some eighty-two pages. Moreover, the changes in that portion of the text carried over from the second edition are much greater than in the case of part one. In those sections dealing with potential theory, the order of presentation has been considerably changed and much rewriting has been done. But it is the chapter dealing with trigonometric series (now Chapter X) that has undergone the most extensive alteration. This was to be expected, of course, since this is the particular topic of volume one whose theory has been most enriched by the researches of the past twenty years. The major portion of the new material introduced is related to certain of these researches, and as only a few could be utilized in this fashion, it is interesting to note what choice Professor Picard

* Cf. this BULLETIN (2), vol. 8 (1901–02), p. 124.

has made. We discover that the inspiration for most of the additions is to be found in the writings of three men, Lebesgue, Bôcher and Fejér.

In discussing the Jordan criterion for the convergence and uniform convergence of Fourier's series and related topics connected with the representation of arbitrary functions, the author has adopted the point of view introduced by Lebesgue in his fundamental memoir of 1910, *Sur les intégrales singulières*. The essence of this method consists in proving first a number of general theorems regarding the behavior of integrals of the type $\int_a^b f(\alpha)\varphi(\alpha, n)\,d\alpha$ when the parameter $n$ of the so-called kernel, $\varphi(\alpha, n)$, becomes infinite. It is then found that many highly important results, such as the classical results regarding the convergence of Fourier's series, Fejér's theorem regarding their summability, and the behavior of Poisson's integral, appear as special cases of one or the other of these general theorems. The advantages of this form of treatment are considerable. The reader is spared needless repetition of proofs that are essentially similar, and he is shown the underlying unity of a certain group of fundamental theorems.

One of the interesting properties of Fourier's series that for a long time remained unnoticed, is the fact that in the neighborhood of a finite jump of the function developed, the oscillation of the approximation curves (the curves $y = S_n(x)$ where $S_n(x)$ is the sum of the first $n$ terms of the series) does not approach as a limit the value of the finite jump, but a quantity exceeding it by about eighteen per cent of its magnitude. This property was first pointed out by Gibbs in 1899, in a letter to NATURE. Gibbs's statement was extremely brief and apparently attracted no general attention. In 1906 Bôcher included in his well known monograph, *Introduction to the theory of Fourier's series,* a discussion of the property in question (called by him Gibbs's phenomenon), which exhibited in a very illuminating manner its most important features. In the volume under review, section 16 of Chapter X is devoted to an exposition of Gibbs's phenomenon which follows quite closely the treatment in Bôcher's monograph. In the title heading of the section, however, the property in question is referred to as "Phénomène de Du Bois-Reymond et Gibbs", and at the end of the section reference is made to an article by Du Bois-Reymond in volume 7 of the MATHEMATISCHE ANNALEN. An examination of that article has failed to convince the reviewer that there is any justification for associating the name of Du Bois-Reymond with the phenomenon under discussion. It is true that there are certain formulas in the article from which Gibbs's phenomenon might have been deduced, but the deduction was not made. It is a little difficult to interpret the meaning of such discussion of the formulas as is given,* inasmuch as this discussion contains several

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* Loc. cit., p. 254.

erroneous statements. The most natural interpretation, however, would lead one to conclude that Du Bois-Reymond was under the impression that in the neighborhood of a finite jump the approximation curve approaches a limiting curve in which the vertical portion is the line joining the loose ends of the curve representing the function, and that he was entirely unaware of the fact that this vertical part extends beyond the loose ends in question by an amount that bears a definite ratio to the magnitude of the finite jump. Therein lies the whole point of Gibbs's noteworthy discovery.

In the case of so keen an analyst as Du Bois-Reymond, who had, moreover, penetrated deeply into a number of questions concerning the convergence and divergence of Fourier's series, one is at liberty to suppose that he may actually have noted Gibbs's phenomenon and that this fact is concealed by the errors above mentioned. But at best this is mere conjecture, and is hardly a sufficient reason for attempting to give him priority over Gibbs in the matter.

One of the most important of the well established contributions of Du Bois-Reymond to the theory of Fourier's series is his discovery of continuous functions whose developments in Fourier's series diverge at certain points. The examples he gave of such functions are quite complicated, and as the property is such a fundamental one it is highly desirable to have simpler examples, particularly for use in instruction. This need has been admirably met by Fejér, who, starting from certain principles of construction introduced by Lebesgue and Haar, succeeded in forming very simple examples of continuous functions that exhibit the so-called singularity of Du Bois-Reymond. One of these examples is discussed in detail by Professor Picard in section 17 of Chapter X. This example leads naturally to the consideration of methods of summing Fourier's series in the cases where it is divergent. In sections 19-22, Cesàro summability of order one is discussed, and Fejér's theorem regarding its application to Fourier's series is established.

Most of the other new material in Chapter X does not deal with recent researches, but is based on results obtained before the publication of the second edition. Among the topics discussed may be mentioned Dini's condition for the convergence of Fourier's series, and de la Vallée Poussin's theorem regarding the series formed from the squares of the Fourier coefficients of a given function.

With the exception of the introduction of a brief discussion of non-euclidean geometry, mentioned above, the remainder of the book exhibits no material change from the second edition.

C. N. Moore