that every relative integral invariant is the sum of an absolute integral invariant and the integral of a symbolic total differential, etc.

The last three chapters are devoted to recent results in the study of systems of Pfaff, a great part of which is due to Cartan. The first question considered in Chapter 6 is as to the maximum number of dimensions ρ of an integral manifold of a system of r forms of Pfaff.

There is a rich amount of material, too varied to lend itself to brief discussion, on systems of linear differential expressions, contact transformations, derived systems, the problem of Monge, second order partial differential equations, etc. The final chapter is devoted to the classification of the integral elements of a system of Pfaff and to the existence theorem.

The book affords an excellent introduction to the study of a problem which occupies a very central position in the theory of differential equations. It gives a detailed survey of the classical theory, of its connections with other domains, of its most modern delelopments, and of the directions in which further advances may be made.

A. DRESDEN

WATSON ON BESSEL FUNCTIONS

A Treatise on the Theory of Bessel Functions. By G. N. Watson. Cambridge, University Press, 1922. viii + 804 pages.

The purpose of this book is twofold: to develop certain applications of the fundamental processes of the theory of functions of complex variables for which Bessel functions are admirably adapted; and secondly, to compile a collection of results which shall be of value to the increasing number of mathematicians and physicists who encounter Bessel functions in the course of their researches. The author believes that the existence of such a collection is demanded by the greater abstruseness of properties of Bessel functions (especially of functions of large order) which have been required in recent years in various problems of mathematical physics.

In his exposition the author has endeavoured to accomplish two specific results: to give an account of the theory of Bessel functions which a pure mathematician would regard as fairly complete; and to include all formulas, whether general or special, which, although without theoretical interest, are likely to be required in practical applications. An attempt is made to give the latter results, as far as possible, in a form appropriate for use in the applications. These exalted aims the author seems to have achieved with a remarkable success. The great amount of material thus to be included and the necessity for keeping the size of the book within bounds have made necessary the employment of a concise and compact exposition: but this has been attained without undue sacrifice of clarity. Nearly all parts of the book are as easily read as one has a right to expect in the case of material of this sort. The general theory on which the special results in this volume are based is to be found in the *Course of Modern Analysis* by Whittaker and Watson. It is greatly to the reader's convenience to have a single volume for reference for the basic theory on which the exposition is founded.

Concerning the choice of canonical functions the author speaks as follows in his preface: "It is desirable to draw attention here to the function which I have regarded as the canonical function of the second kind, namely the function which was defined by Weber and used subsequently by Schläfli, by Graf and Gubler, and by Nielsen. For historical and sentimental reasons it would have been pleasing to have felt justified in using Hankel's function of the second kind; but three considerations prevented this. The first is the necessity for standardizing the function of the second kind; and, in my opinion, the authority of the group of mathematicians who use Weber's function has greater weight than the authority of the mathematicians who use any other one function of the second kind. The second is the parallelism which the use of Weber's function exhibits between the two kinds of Bessel functions and the two kinds (cosine and sine) of trigonometric functions. The third is the existence of the device by which interpolation is made possible in Tables I and III at the end of Chapter XX, which seems to me to make the use of Weber's function inevitable in numerical work."

In connection with each section references are given to the memoirs in which the results were first announced but the methods of proof employed are often different from those of the original investigators. A very complete bibliography of thirty-six pages is given at the end of the book.

Space does not allow us to give an analysis of the contents of so large a volume. But we may give a bare outline of the contents by means of chapter headings. These are as follows: Bessel functions before 1826 (pages 1–13), the Bessel coefficients (14–37), Bessel functions (38–84), differential equations (85–131), miscellaneous properties of Bessel functions (132–159), integral representations of Bessel functions (160–193), asymptotic expansions of Bessel functions (194–224), Bessel functions of large order (225–270), polynomials associated with Bessel functions (271–307), functions associated with Bessel functions (308–357), addition theorems (358–372), definite integrals (373–382), infinite integrals (383–449), multiple integrals (450–476), the zeros of

Bessel functions (477–521), Neumann series and Lommel's functions of two variables (522–550), Kapteyn series (551–575), series of Fourier-Bessel and Dini (576–617), Schlömilch series (618–653), the tabulation of Bessel functions (654–664), tables of Bessel functions (665–752), bibliography (753–788), index of symbols (789–790), list of authors quoted (791–795), general index (796–804).

The author's work is well done throughout. The reviewer did not detect any omissions or serious blemishes of any kind. In some respects the exposition is probably more disjointed than is necessary. Some parts would profit by being brought into closer connection with related general theories relative to differential equations; but the author is right in avoiding the use of any large part of the general theory of differential equations. It appears to the reviewer that the exposition of the theory of the zeros of Bessel functions would profit by a closer connection with the general oscillation theory of the solutions of linear homogeneous differential equations of the second order, especially since that theory has taken such simple form in the hands of Bôcher. In a similar manner the general notions involved in the Birkhoff theory of expansions in terms of orthogonal and biorthogonal functions satisfying linear differential equations would serve to give greater unity to the four long chapters on expansions in terms of Bessel functions. But, even so, there is something to be said in favor of the direct treatment employed by the author.

Many of the results recorded appear to be of the nature of special instances of general propositions yet to be discovered. In the case of these a disjointed exposition is inevitable until the general theory has been brought into existence. Perhaps the book will serve a third important purpose in addition to the two which the author had in mind in preparing it. It is suited to suggest further researches in two ways: it will raise interesting questions concerning Bessel functions and the differential equation which they satisfy; it will also suggest general theories concerning linear differential equations, theories special instances of which are afforded by the Bessel equation. In this respect the expansion theory seems to hold out the greatest promise. We may well close the review by emphasizing the importance in this respect of the four excellent chapters devoted to expansions of functions in series of Bessel functions.

R. D. CARMICHAEL