A GENERALIZATION OF THE SYLLOGISM*

BY B. A. BERNSTEIN

The syllogism is the proposition:

If x is y and y is z, then x is z.

In the language of boolean algebra this proposition is:

$$\textit{If } xy' = 0 \textit{ and } yz' = 0, \textit{ then } xz' = 0,$$

where the usual notations are used, with the prime indicating negation. There thus exists in boolean algebras a universal relation† R such that

$$\textit{If } xRy \textit{ and } yRz, \textit{ then } xRz. \tag{1}$$

I propose to find the most general boolean relation R which satisfies (1), and to show the connection between this relation and the syllogism relation $xy' = 0$.

I start with the fact that any universal relation between two boolean elements x, y is given by an equation of the form

$$Axy + Bxy' + Cx'y + Dx'y' = 0. \tag{2}$$

Let (2) be a relation R satisfying (1). Then the *discriminants* A, B, C, D must be such that from

$$\text{(i)} \qquad Axy + Bxy' + Cx'y + Dx'y' = 0,$$
$$\text{(ii)} \qquad Ayz + Byz' + Cy'z + Dy'z' = 0,$$

we may conclude

$$\text{(iii)} \qquad Axz + Bxz' + Cx'z + Dx'z' = 0.$$

That is, the discriminants A, B, C, D must be such that equation (iii) is the result of eliminating y from equations (i) and (ii). Now (i) and (ii) together are equivalent to the single equation

$$\text{(iv)} \quad (Ax + Cx' + Az + Bz')y + (Bx + Dx' + Cz + Dz')y' = 0.$$

The result of eliminating y from (i) and (ii) is, then, the result

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†That is, a relation given by a universal proposition.

of eliminating y from (iv). This result is

$$(Ax + Cx' + Az + Bz')\ (Bx + Dx' + Cz + Dz') = 0,$$

or

$$A(B + C)xz + (B + A)\ (B + D)xz' + (C + A)\ (C + D)x'z + (C + B)Dx'z' = 0,$$

or

$$\text{(v)} \quad A(B + C)xz + (B + AD)xz' + (C + AD)x'z + D(B + C)x'z' = 0.$$

Equating corresponding discriminants of (iii) and (v), we get, as necessary and sufficient conditions that (2) be a relation R satisfying (1),

$$A = A(B + C), \quad B = B + AD, \quad C = C + AD, \quad D = D(B + C),$$

or

$$AB'C' = 0, \quad AB'D = 0, \quad AC'D = 0, \quad B'C'D = 0,$$

or

$$(3a) \quad A + D < B + C, \quad \text{and} \quad (3b) \quad AD < BC.$$

We therefore find that the *most general boolean relation* R *which satisfies the proposition* (1) *is given by the relation*

$$(4) \quad Axy + Bxy' + Cx'y + Dx'y' = 0, \qquad A + D < B + C, \quad AD < BC.$$

We find also that *the syllogism relation* $xy' = 0$ *is the relation* (4) *in which* $A = C = D = 0$, $B = 1$.

We note that the relation of equivalence, which is $xy' + x'y = 0$, is the relation (4), with $A = D = 0$, $B = C = 1$.

The statement that the relation (4) satisfies (1) is true even if the relation (4) be non-existent. Thus, the relation $xy + xy' + x'y + x'y' = 0$, i. e., the relation $1 = 0$, is non-existent. But it satisfies (4) and it is true that *If* $1 = 0$ *and* $1 = 0$, *then* $1 = 0$. If we impose on (2) the condition that it be an *existent* relation, we must have

$$(5) \qquad ABCD = 0.$$

Since condition (3b) can be written in the form

$$AD = AD \cdot BC,$$

conditions (3b) and (5) together are equivalent to

$$(6) \qquad AD = 0,$$

and conditions (3*a*), (3*b*) become

$$(7) \qquad A+D<B+C, \qquad AD=0.$$

Hence we find that *the most general existent boolean relation R which satisfies* (1) *is given by* (2) *and* (7).

Our main results may also be stated in the following form: *The totality of transitive universal relations in a boolean algebra is given by* (4). *The totality of existent transitive universal relations is given by*

$$Axy+Bxy'+Cx'y+Dx'y'=0, \quad A+D<B+C, \quad AD=0.$$

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ON CERTAIN QUINARY QUADRATIC FORMS*

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1. *Introduction.* Except the classical theorems on the total number $N_5(n)$ of representations of the integer n as a sum of five integer squares, no explicit results on numbers of representations in quinary quadratic forms seem to have been obtained.† In general $N_5(n)$ is not expressible as a function of the real divisors alone of a single integer, but when n is a square, $N_5(n)$ is so expressible. This remarkable fact was found inductively by Stieltjes for $N_5(p^2)$, p prime, and proved for $N_5(n^2)$ by Hurwitz,‡ who showed that if $n=2^\alpha m$, m odd, $N_5(n^2)=10\zeta_3(2^\alpha)H(m)$, where

$$H(m)=[\zeta_3(p^a)-p\zeta_3(p^{a-1})][\zeta_3(q^b)-q\zeta_3(q^{b-1})]\ldots,$$

$\zeta_r(n)$ being the sum of the rth powers of all the divisors of n, and $m=p^aq^b\ldots$ the prime factor resolution of m; by convention $H(1)=1$. In the course of his proof he showed that

$$\zeta_1(m^2)+2\zeta_1(m^2-2^2)+2\zeta_1(m^2-4^2)+\cdots=H(m).$$

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† Cf. Bachmann, *Zahlentheorie*, vol. 4, pp. 565-594.

‡ COMPTES RENDUS, vol. 98 (1884), pp. 504-7; cf. Dickson's *History of the Theory of Numbers*, vol. 2, p. 311. For quadratic forms in $n>4$ variables, cf. ibid., vol. 3, chap. XI.