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ON A SMALL VARIATION WHICH RENDERS A LINEAR DIFFERENTIAL SYSTEM INCOMPATIBLE.

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LET us consider a homogeneous linear differential expression of the nth order*

$$L(u) \equiv l_n \frac{d^n u}{dx^n} + l_{n-1} \frac{d^{n-1}u}{dx^{n-1}} + \cdots + l_1 \frac{du}{dx} + l_0 u,$$

whose coefficients are continuous functions of the real variable x in a closed interval ab. We suppose that l_n does not vanish in this interval. We consider the $2n$ quantities

$$u(a),\ u'(a),\ \cdots,\ u^{[n-1]}(a); \qquad u(b),\ u'(b),\ \cdots,\ u^{[n-1]}(b)$$

and form n linearly independent linear forms in them, $U_1(u)$, $\cdots$, $U_n(u)$, with constant coefficients.

Consider now the homogeneous linear differential system

$$(1) \qquad L(u) = 0, \qquad U_i(u) = 0 \quad (i = 1, 2, \cdots, n).$$

This system is said to have k-fold compatibility if there are k and only k linearly independent functions which satisfy it. It is well known and immediately obvious that, if y_1, $\cdots$, y_n is any fundamental system of the equation $L(u) = 0$, a necessary and sufficient condition for k-fold compatibility is that the rank of the matrix

* No additional difficulties would be introduced if we considered the more general expressions treated in my paper, *Transactions*, vol. 14 (1913), p. 403. See in particular the latter part of § 3.

$$(2)\qquad \begin{pmatrix} U_1(y_1) & \cdots & U_1(y_n) \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ U_n(y_1) & \cdots & U_n(y_n) \end{pmatrix}$$

be $n - k$. Since all the elements, and hence all the determinants, of this matrix will be only slightly changed by a small variation of the coefficients of the system (1) (provided that, as is obviously possible, the y_i's and their first $n - 1$ derivatives are allowed to vary only slightly) we immediately infer the following important result:

THEOREM I. *If the system* (1) *has k-fold compatibility, it has no higher order of compatibility after any variation of its coefficients which is uniformly sufficiently small in ab.**

While this theorem tells us that *no* very small variation will *raise* the order of compatibility, the main result to be established in this paper refers to the possibility of *lowering* the order of compatibility, and here we shall prove not merely that there always exist arbitrarily small variations which render the system incompatible (i. e., reduce its order of compatibility to zero) but that a variation of a very simple and important type will have this effect; namely a *real* variation of the coefficient l_0 alone (so that the conditions $U_i = 0$ are not varied) and, indeed, a variation which is everywhere positive, or, what is not essentially different, everywhere negative. The proof will depend on certain preliminary lemmas.

Let us suppose that the system (1) has k-fold compatibility, and, as a matter of notation, let us suppose that the $(n - k)$-rowed determinant in the upper left-hand corner of the matrix (2) is not zero. Then every solution of the equation $L(u) = 0$ which satisfies the first $n - k$ conditions $U_i = 0$ will also satisfy the remaining conditions. Such a function is given by the determinant

$$(3)\qquad \begin{vmatrix} y_1 & \cdots & y_{n-k} & c_1 y_{n-k+1} + \cdots + c_k y_n \\ U_1(y_1) & \cdots & U_1(y_{n-k}) & c_1 U_1(y_{n-k+1}) + \cdots + c_k U_1(y_n) \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ U_{n-k}(y_1) & \cdots & U_{n-k}(y_{n-k}) & c_1 U_{n-k}(y_{n-k+1}) + \cdots + c_k U_{n-k}(y_n) \end{vmatrix}$$

* The special case $k = 0$ of this theorem tells us that if the system (1) is incompatible, it remains so after every variation of its coefficients which is uniformly sufficiently small.

Moreover this determinant vanishes identically only when $c_1, \cdots, c_k$ are all zero, since otherwise $y_1, \cdots, y_n$ would be linearly dependent. Consequently the formula (3) gives a linear family of solutions of the system (1) whose bases consist of just k functions, so that, since by hypothesis (1) has k-fold compatibility, (3) gives its general solution.

Let us now suppose that the coefficients of $L(u)$ are continuous functions of (x, λ) and that the coefficients of the U_i's are continuous functions of λ; and that when $\lambda = \lambda_0$ and for a certain neighborhood of this value the system (1) has just k-fold compatibility. If we arrange the notation so that when $\lambda = \lambda_0$ the $(n - k)$-rowed determinant in the upper left-hand corner of (2) is not zero, we can take the neighborhood of λ_0 so small that this same determinant does not vanish in this neighborhood, provided that, as is surely possible, $y_1, \cdots, y_n$ are so chosen that they and their first $n - 1$ derivatives are continuous functions of (x, λ). Then (3) gives the general solution of the system (1) for all values of λ in a certain neighborhood of λ_0, and it is clear that for any special determination of the c_i's, either as constants or as continuous functions of λ, the function (3) is continuous in (x, λ). Hence

LEMMA I. *If throughout a certain range of values of λ the coefficients of L are continuous functions of (x, λ) and the coefficients of $U_1, \cdots, U_n$ are continuous functions of λ, and if for all values of λ in this range the system* (1) *has exactly k-fold compatibility; then if $u_0(x)$ denotes any particular solution of the system* (1) *when $\lambda = \lambda_0$, there exists a function $u(x, \lambda)$ continuous in (x, λ) which, throughout a certain neighborhood* of λ_0, satisfies* (1), *and is such that its limit for $\lambda = \lambda_0$ is $u_0(x)$, this limit being approached uniformly in ab.*

We turn next to

LEMMA II. *If v is any solution of the system*

$$M(v) = 0, \qquad V_i(v) = 0 \quad (i = 1, 2, \cdots, n) \tag{4}$$

adjoint† *to* (1), *and u_g is any solution of the system*

* This will be a one-sided neighborhood if λ_0 is an extremity of the range in question.

† For a definition of the adjoint system cf. for instance the paper already cited, where a more detailed statement of Green's theorem will also be found.

$$L(u) = gu, \qquad U_i(u) = 0 \quad (i = 1, 2, \cdots, n), \tag{5}$$

then

$$\int_a^b g u_g v \, dx = 0. \tag{6}$$

The proof consists in applying Green's theorem

$$\int_a^b [vL(u) - uM(v)]dx = \sum_{i=1}^{2n} U_i(u) V_{2n+1-i}(v)$$

to the two functions u_g and v, when it reduces at once to (6).

LEMMA III. *If the system* (1) *has k-fold compatibility* ($k \geqq 1$), *and* ϵ *is an arbitrarily given positive constant, a continuous, real function* $g(x)$ *exists such that* $0 \leqq g(x) < \epsilon$, *and that the system* (5) *has less than k-fold compatibility.*

To prove this, let u be a non-identically vanishing solution of (1) and v a similar solution of (4), which surely exists since (1) and (4) always have the same order of compatibility.* Since, by a fundamental (though seldom explicitly stated) theorem concerning homogeneous linear differential equations, neither u nor v has more than a finite number of zeros in ab, we can select a point p at which the product uv does not vanish. Either the real or the pure imaginary part of uv does not vanish at p; and without loss of generality we may assume that the former is the case as otherwise we might have multiplied v by a pure imaginary constant before beginning. Since uv, and therefore its real part, is a continuous function of x, we can surround p by an interval $a'b'$ so short that the real part of uv does not vanish there. Now define φ as a real continuous function of x which vanishes everywhere outside of $a'b'$ and is positive but less than ϵ everywhere within. We see then that

$$\int_a^b \varphi u v \, dx \neq 0. \tag{7}$$

We now define the function g, whose existence is asserted in our lemma, by the equation

$$g = \lambda \varphi,$$

where λ is an, as yet undetermined, positive constant less than 1; and we see from (6) that

* Loc. cit., Theorem I.

$$\int_a^b \varphi u_g v\,dx = 0, \tag{8}$$

where u_g is any solution of (5).

Now assume Lemma III to be false. Then for all positive values of λ less than 1 the system (5) would have at least k-fold compatibility, while by Theorem I it cannot have more than k-fold compatibility for sufficiently small values of λ. Let us then restrict λ to values so small that (5) has always exactly k-fold compatibility. Then, by Lemma I, we can take for u_g a continuous function of (x, λ) which approaches $u(x)$ uniformly as λ approaches zero through positive values. Consequently

$$\lim_{\lambda=+0} \int_a^b \varphi u_g v\,dx = \int_a^b \varphi u v\,dx.$$

This, however, is in contradiction with formulas (7) and (8). Thus our lemma is proved.

We have indeed proved more than is stated in the lemma, for we have shown that g may be taken as identically zero except in the interval $a'b'$, which interval could be taken as short as we please and in any position we please provided it avoids a finite number of points. If now the order of compatibility of (5) is not zero, we can start afresh with this system, in place of (1), and, applying Lemma III to it, form a new system

$$L(u) = gu + g_1 u, \qquad U_i(u) = 0 \quad (i = 1, 2, \cdots, n)$$

which has a still lower order of compatibility and where $0 \leqq g_1 < \epsilon$. Moreover g_1 can be made to vanish everywhere except in an interval $a''b''$ as short as we please and not overlapping the interval $a'b'$. Hence the function $g + g_1$ satisfies the same inequality as g and g_1. Proceeding in this way step by step, we finally come to a system which is incompatible. Since all the intervals $a'b'$, $a''b''$, etc., which we use may be taken, if we wish, within an arbitrarily chosen subinterval of ab, we may state our final result as follows:

THEOREM II. *If ϵ is an arbitrarily given positive constant, a continuous, real function $g(x)$ exists such that $0 \leqq g(x) < \epsilon$ and such that the system* (5) *is incompatible. This function g may be taken to be identically zero except in an arbitrarily chosen subinterval of ab.*

We have proved this theorem, it is true, only when $k > 0$. If $k = 0$ it is, however, merely an obvious consequence of Theorem I.

We come now at last to our most important result, though one which is, at bottom, less far reaching than Theorem II, namely

THEOREM III. *If ϵ is an arbitrarily given positive constant, a continuous, real function $g(x)$ exists such that $0 < g(x) < \epsilon$ and such that the system* (5) *is incompatible.*

The proof consists simply in noticing that if we add to the function $g(x)$ determined in Theorem II a sufficiently small function everywhere positive (not zero), the system (5) will, by Theorem I, remain incompatible.*

This theorem is useful in making connection, by the method originally given in special cases by Hilbert, between the system (1) and an integral equation of the second kind.

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THE SMALLEST CHARACTERISTIC NUMBERS IN A CERTAIN EXCEPTIONAL CASE.

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THE characteristic numbers of the system

$$\frac{d}{dx}(ku') + (\lambda g - l)u = 0, \quad (k > 0,\ l \geqq 0), \tag{1}$$

$$\alpha u'(a) - \alpha' u(a) = 0, \quad (\alpha\alpha' \geqq 0,\ |\alpha| + |\alpha'| > 0), \tag{2}$$

$$\beta u'(b) + \beta' u(b) = 0, \quad (\beta\beta' \geqq 0,\ |\beta| + |\beta'| > 0) \tag{3}$$

are those values of λ for which (1) has a solution not identically zero which satisfies (2) and (3). We assume that k, g, l are continuous real functions of x in the interval $a \leqq x \leqq b$,

* A similar method enables us to deduce from Theorem II a great variety of other results, for instance:

If ϵ is an arbitrarily given positive constant, and $x_1, \cdots, x_p$ are arbitrarily given points in ab, there exists a continuous, real function $g(x)$ which vanishes and changes sign at each of the points x_i but vanishes nowhere else in ab, which satisfies the condition $|g(x)| < \epsilon$, and for which (5) *is incompatible.*