NOTE ON CONJUGATE POTENTIALS.

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IF $u(r, \vartheta)$ is a potential function of the unit circle and $v(r, \vartheta)$ its conjugate, and if $f(\vartheta)$ and $g(\vartheta)$ are the values approached by these functions as $r \doteq 1$, the following relations given by Hilbert * hold:

$$f(\vartheta) = \frac{1}{2\pi}\int_0^{2\pi} g(\phi) \cot \tfrac{1}{2}(\vartheta - \phi)d\phi + \frac{1}{2\pi}\int_0^{2\pi} f(\phi)d\phi,$$

$$g(\vartheta) = -\frac{1}{2\pi}\int_0^{2\pi} f(\phi) \cot \tfrac{1}{2}(\vartheta - \phi)d\phi + \frac{1}{2\pi}\int_0^{2\pi} g(\phi)d\phi,$$

where by the integration symbols in the first terms of the right hand sides the Cauchy principal value is meant. They give, to within an additive constant, the boundary values of a potential in terms of its conjugate.

They have been established for the case that $f(\vartheta)$ and $g(\vartheta)$ are integrable throughout and are continuous at all but a finite number of points and possess derivatives subject to the same conditions.† Because of the interest attaching to them from their connection with potential theory, and as examples of integral equations of the first kind, ‡ it seems worth while to point out that they hold if only $f(\vartheta)$ and $g(\vartheta)$ are integrable from 0 to 2π and are such that

$$\int \frac{f(\vartheta_0 + \delta) - f(\vartheta_0)}{\delta} d\delta \quad \text{and} \quad \int \frac{g(\vartheta_0 + \delta) - g(\vartheta_0)}{\delta} d\delta$$

are convergent when extended over an interval including $\delta = 0$ in its interior and this for all but a finite number of values of

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* Vorlesungen über Potentialtheorie, Göttingen, winter semester, 1901–02.

† See *Math. Annalen*, vol. 58, p. 442; also my dissertation for the doctorate: "Zur Theorie der Integralgleichungen und des Dirichlet'schen Prinzips," Göttingen, 1902, p. 17.

‡ Hilbert: "Grundzüge einer allgemeinen Theorie der linearen Integralgleichungen," *Gött. Nachrichten*, 1904, p. 49. Encykl. d. Math. Wiss. II, A, 11, pp. 803, 816.

ϑ_0 in the interval from 0 to 2π.* To justify this assertion it will only be necessary to point out how a minor change in the article in the *Mathematische Annalen* or in the dissertation just referred to permits the generalization in question.

A slightly different notation is there used: instead of the arguments ϑ and ϕ, $s = \vartheta/2\pi$ and $t = \phi/2\pi$ are employed. The reasoning may remain unchanged except for the matter of showing that the integral

$$I = \int_{-\epsilon}^{+\epsilon} g(t) \frac{\partial}{\partial t} \log \rho \left({}_{x,}^{t}{}_{y} \right) dt$$

is a continuous function of x and y in the neighborhood of $x = 1$, $y = 0$ (*Mathematische Annalen*, page 445; dissertation, page 21). It will be sufficient to show that this integral vanishes with ϵ in a sufficiently small neighborhood of (1, 0).

To do this, let us use instead of x and y the coördinates r and s, where $x = r \cos 2\pi s$, $y = r \sin 2\pi s$. Then $\rho^2({}_{r,}^{t}{}_{s}) = 1 + r^2 - 2r \cos 2\pi(s-t)$, and we may write the integral $I = I_1 + I_2$, where

$$I_1 = g(s) \int_{-\epsilon}^{+\epsilon} \frac{\partial}{\partial t} \log \rho({}_{r,}^{t}{}_{s}) dt,$$

and

$$I_2 = \int_{-\epsilon}^{+\epsilon} [g(t) - g(s)] \frac{\partial}{\partial t} \log \rho({}_{r,}^{t}{}_{s}) dt.$$

The first of these gives

$$I_1 = g(s) \log \frac{\rho({}_{r,}^{\epsilon}{}_{s})}{\rho({}_{r,}^{-\epsilon}{}_{s})}, \dagger$$

and, as we are at a point where $g(s)$ is continuous, this factor is finite, and if we restrict the point (r, s) to that neighborhood of (1, 0) given by $0 < 1 - r < \epsilon^2$, $|s| < \epsilon^2$, the logarithm will be found to vanish at least as fast as a constant times ϵ. Hence I_1 vanishes with ϵ.

I_2 may be written

$$\int_{-\epsilon}^{+\epsilon} \left[\frac{g(t) - g(s)}{t - s} \right] \frac{2\pi r \sin 2\pi(s-t)}{1 + r^2 - 2r \cos 2\pi(s-t)} \cdot (t-s) dt.$$

* These conditions are even less restrictive than those announced in my paper as read before the Society.

† This on the understanding that if $r = 1$ the Cauchy principal value is to be taken.

Now

$$\frac{2\pi r \sin 2\pi(s-t)}{1+r^2-2r\cos 2\pi(s-t)}\cdot(t-s)$$

is a continuous function of t for any fixed values of r and s in the above defined neighborhood of (1,0). This is at once evident when $r < 1$, and becomes evident for $r = 1$ when we give it the form

$$-\frac{\pi(s-t)}{\sin \pi(s-t)}\cos \pi(s-t).$$

Hence we may infer that I_2 vanishes with ϵ, for

$$\frac{g(s)-g(t)}{s-t},$$

being continuous for $t \neq s$, and having a convergent integral even over an interval including $t = 0$ by hypothesis, the convergence of I_2 may be made apparent by integrating by parts, and, being convergent, it vanishes with its limits $+\epsilon$ and $-\epsilon$.

To recapitulate, the integral

$$\int_0^1 g(t)\frac{\partial}{\partial t}\log \rho\left(\begin{smallmatrix} t \\ r, s\end{smallmatrix}\right)dt,$$

which we wished to show continuous in r and s in the neighborhood of (1, 0), breaks up into two parts

$$\int_\epsilon^{1-\epsilon} \quad\text{and}\quad \int_{1-\epsilon}^{+\epsilon},$$

or J and I as we may denote them. As I vanishes with ϵ, as we have just shown, we may so restrict ϵ and r and s that $I < \delta/4$. Then the variation of I in the neighborhood of (1, 0) thus determined will be less than $\delta/2$. But ϵ being fixed, J is continuous, and hence r and s may be so restricted that its variation is less than $\delta/2$. Thus the variation of the whole integral is less than δ, and it is therefore continuous at the point (1, 0), and that for any manner of approach whatever within or upon the periphery of the circle.

The formulas are thus established under the given conditions.

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