where

$$(3) \qquad b \equiv 0 \pmod 3.$$

Since a given prime is expressible in not more than one way in the form $a^2 + 3b^2$, and since

$$\Pi = (1)^2 + 3(2^\kappa)^2$$

it follows that condition (3), and therefore (1), is not satisfied.

It is easy to show that *composite* numbers of the forms $2^\kappa \cdot 3 + 1$, $2^\kappa \cdot 5 + 1$ can not be factors of Fermat's numbers.

NORTHWESTERN UNIVERSITY,
April, 1906.

THEORETICAL MECHANICS.

A Treatise on the Analytical Dynamics of Particles and Rigid Bodies; with an Introduction to the Problem of Three Bodies. BY E. T. WHITTAKER. Cambridge, The University Press; New York, The Macmillan Company, 1904. xiii+414 pp.

AT Cambridge, England, mathematics means for the most part mechanics, mathematical physics, or even physics sometimes not so very mathematical. The famous tripos — the mathematical tripos, of course, which goes back at any rate to 1747—seems, at least to an outsider, to lay its main stress on the theoretical application of mathematics rather than on pure mathematics. Very likely this is a tradition that has come down from the time of Newton, and it is certainly maintained by the eminent physicists such as Stokes, Kelvin, Maxwell, Rayleigh, J. J. Thomson, who have been high wranglers. This tripos with its great prestige gives an attractive and distinctive touch to the university and although the ever-increasing pressure of pure mathematics, with its possibilities for various kinds of unessential and mediocre work infinitely wider than those to be found in theoretical applied mathematics, will probably tell on the training at Cambridge sooner or later, may we not look forward to that date with some regrets on the general uniformizing that is taking place and lament the fact that other realms than physics are possessed of an entropy? At present, however, there seems no immediate danger. The one year of 1904 saw the publication at the Cambridge University Press of three large and highly important volumes by as many recent graduates of Cambridge typically cantabrigian in that they exhibit great mathematical power and attainments directed firmly and unerringly along the direction of physical research. It is needless to say that we refer to Walker's Analytical theory of light, Jean's Dynamical theory of gases, and Whittaker's Analytical dynamics.

Mechanics is more largely pure mathematics than the other branches of theoretical physics because the data are fewer and simpler; the transition to the systems of differential equations, less recondite. And so far is mechanics developed that probably it is the mathematician who must carry on the work rather than the physicist engrossed in advancing the experimental data of science—witness the recent workers, Poincaré, Painlevé, Hadamard, Duhem, Volterra, Levi-Cività, Gibbs, Love, Jeans, Klein, and others. For the mathematician, however, the way has unfortunately been barricaded by the kind of treatise on mechanics which has been available to the intermediate student. The famous works of Routh are not so much mathematical as distinctly mechanical; the reader learns rather how to do difficult and often artificial problems rather than how problems are done. This training is highly valuable and should precede other work: but where to find the other? It is precisely at this point that Whittaker breaks the barricade and opens the way to fruitful advance. His book, to be sure, starts at the beginning of the subject—but it is not for the beginner. A good course from the practical problem-solving aspect of Routh is almost indispensable to a ready and successful mastery of Whittaker. The book is mathematical in nature, written with a precision and developed with a logic sure to appeal to mathematicians. It is modern, thoroughly modern with its bibliographical references running quite up into the year 1904. Its subjects include the contributions of "Lie, Rayleigh, Klein, Hertz, Lorentz, Poincaré, Siacci, Bruns, Boltzmann, Larmor, Greenhill, Appell, Painlevé, Stäckel and Levi-Civita" to quote from the preface. And this is quite true. Those who are familiar with a number of treatises on mechanics may scan them in vain for a résumé of the methods introduced since 1880 just as they may look equally in vain for an advanced book arranged in what seems a logical order from the mathematician's point of view.

It is therefore a definite and strategic position which Whittaker has occupied and one may in this instance put aside all lamentations from the observation that of the making of books there is no end. To show this, needs but a detailed examination of the contents. First, however, it may be well to mention that the methods of attack which the author displays throughout the text are numerous. Be they geometric or algebraic or analytic he invariably selects that which moves most directly to the goal of solution. This diversity of method taken with the compact style makes the book hard reading for any but the somewhat advanced student. To solve the exercises requires on the part of the reader a similar versatility. Thus one may see why and how it is that the Cambridge student well versed in mechanics is also in reality thoroughly equipped in pure mathematics in so far as present practical applications are concerned. To a greater extent than his predecessors Whittaker has availed himself of the great machinery of modern mathematics, has thrown life into both the mathematics and the mechanics which he treats, and in this way has produced a book which may well become an inspiration to the next generation as Routh's treatises have been to this.

The first chapter of the text is on kinematic problems. Here, condensed into twenty-five pages, may be found Euler's, Hamilton's and Chasles's theorems on displacements, with their applications to instantaneous axes and centers; also the various methods of representing a displacement analytically, whether by the angle of version, the coördinates of Rodrigues, the angles of Euler, or the parameters of Klein, with the formulas for transformation from one to another and the expressions for the velocity and accelerations in the different systems of coördinates. It should be stated that the chapter further contains about twenty-five problems of all sorts and grades for the reader to work. This is a feature of the book. The problems cannot all be equally well selected and equally instructive. A large number of them are branded "(Coll. Exam.)" and frequently they would have shown it only too obviously without the mark. On the other hand many are taken from published memoirs of prominent writers and are excellent, though often difficult. The solution of a considerable number of problems chosen throughout the work indicates that the answers are usually correct. The second chapter is on the equations of motion and is of equal length—or rather brevity. It does for dynamics what the first did for kinematics. From the simple newtonian equations for the motion of a particle the author proceeds by rapid strides to the ideas of work and of generalized coördinates, to the distinction between holonomic and non-holonomic systems, and to the lagrangian equations for holonomic systems, to conservative systems, kinetic potential, quasi coördinates, and finally to impulsive motion. It is needless to point out that for one not already somewhat familiar with the subject these twenty-five pages must be taken in small doses mixed with similar portions taken from the fourth and later chapters. This second chapter is purely on the theory and not on the practice of the equations of dynamics.

In the discussion of the principles available for integration, which makes up Chapter III, the author continues his mathematical investigations in much the same style. The reader will again do well to intersperse some parts of the later chapters with his reading. With this and the ten or a dozen problems given in immediate connection with the investigations there is no very great difficulty to impede one's progress. After a word about what is meant by the solution of a system of differential equations, the question of ignorable coördinates is taken up in detail, and the theorem is proved that a dynamical system with n degrees of freedom which has k ignorable coördinates can be reduced to a dynamical system which has only $n - k$ degrees of freedom. And this possibility of ignoring some coördinates turns out to be the most general reason for the actual solvability of some problems and non-solvability of others. Too much emphasis cannot be laid upon this principle of the ignoration of coördinates, for it gives a definite and regular method of simplifying the problem, and frequently it alone may be made to yield the desired information about a system the actual complete solution of which we are either unable to obtain or able to obtain only by the expenditure of too much effort. Simple examples of the principle are found in systems possessing one or more integrals of momentum or of angular momentum. In this connection it should be noted that the kinetic energy of the ignored coördinates appears as potential energy in the modified system. This has given rise to a question which we are not yet in a position to answer satisfactorily, namely whether or not all potential energy may be explained as kinetic energy of some ignored coördinates, and how? It is obvious to all, that potential energy is to a considerable extent merely a convenient name for the manifestations of something the mechanism of which we do not understand and frequently "explain" by action at a distance. This instructive chapter closes with a discussion of the integral of energy in a conservative system and its use in reducing by one the number of degrees of freedom of the dynamical system.

The forty-five pages of Chapter IV on the solvable problems of particle dynamics form, when taken with the general theory of the preceding chapters, a veritable treatise on dynamics of a particle. From the simplest case of the pendulum the author goes through the various forms of central motion to motion on different kinds of surfaces. So many of the problems are soluble in finite form only in terms of the elliptic functions that some knowledge of these functions alike in the legendrian, the jacobian, and the weierstrassian analysis is indispensable. Fortunately the author has his own excellent work on modern analysis on which to fall back,* and references to it are frequent In this chapter there are about seventy problems, many of which should be interspersed with Chapters II and III, purely for pedagogic purposes. In fact, although the arrangement in the text is admirable for theoretical and logical purposes, instruction requires the interlarding of one part with another. There follows a short chapter of fifteen pages, with about as many problems, on the dynamical specification of bodies. It contains the usual theorems on moments, products, and ellipsoids of inertia, on kinetic energy, and on the independence of the motion of the center of gravity and of the motion relative to it. This leads directly to Chapter VI on the soluble problems of rigid dynamics — a complete analogue to Chapter IV as regards length, number of problems, and so forth. Naturally the top comes in for a lion's share of the space, and the elliptic functions are as indispensable as before.

Up to this point, about two fifths of the whole work has been covered with very little of a really advanced mathematical character except perhaps the technique of the elliptic functions. From here on, however, the amount of mathematics which the reader must know, or assimilate from the meager indications in the text, is by no means inconsiderable — calculus of variations, elementary divisors, integral invariants, and contact transforma-

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* E. T. Whittaker, A Course in Modern Analysis, reviewed by Bôcher in the BULLETIN, vol. 10, pp. 351–354.

tions may be mentioned as examples. It is, then, just at this point that modern mechanics begins.

The seventh chapter of some thirty-five pages deals with the theory of vibrations whether about a state of equilibrium or of motion. The present method of treatment goes back to Weierstrass and requires a tolerable knowledge of the theory of quadratic forms, linear transformations, and elementary divisors. At times the analysis itself, apart from what it presupposes, is somewhat complicated. Nevertheless a good student even poorly equipped ought to get all the chief points with the aid of the forty or so problems which he may solve in connection with the text. The more difficult of these are a task for the best; but the easier are very simple and illustrate the principles about as well. It is remarkable how much mathematics that one does not fully understand will cheerfully be accepted on faith if only a number of problems come out successfully when the indicated method is applied—and this is often the best way to get to understand the mathematics.

Chapter VIII on non-holonomic systems and dissipative systems is admirable and admirably placed for theoretical purposes. In elementary and practical treatments of mechanics it is necessary to introduce such systems earlier because so many of the simplest systems are of this type; but the difficulties in the proper theoretical treatment of these systems render it advisable to postpone the discussion of them to a relatively later time. The clean-cut division of problems into holonomic and non-holonomic, which has become possible since the work of Hertz, is very desirable and has been adopted by more than one author. The neatness of treatment of dissipative systems is much enhanced by the timely introduction of Rayleigh's dissipation function. To be sure this function does not always exist; but in the simple and probably the most important case, that in which the resistances are proportional to the velocity, it does; and gives a noteworthy simplicity to the equations of motion. The collection of thirty-five or forty problems appears to be particularly well selected and certainly presents all grades of difficulty.

The principles of Hamilton and Gauss receive careful attention in the brief space of twenty pages. The reader will perhaps find some difficulty in appreciating the methods owing to the practical impossibility of finding more than a very few problems on which to try the text. This lack may be supplied in part by having recourse to problems in earlier chapters. The author goes in some detail into the methods of the calculus of variations. He distinguishes between the applications of the principle of Hamilton to holonomic conservative systems, and its extension to dissipative and non-holonomic systems. It is in fact one of the admirable traits of this book to state with the maximum clearness just what are the conditions under which the different theorems and methods are obtained. Some treatises on dynamics merely give the equations in their simplest form and others, which presumably are more complete, fail to state the limitations under which the equations are derived. For reference purposes this is fatal. Whittaker's treatise is typographically and otherwise arranged so as to facilitate reference. To continue, however, with the present chapter we would state that the author takes up the questions of kinetic foci, of whether the integrals are actual minima (connected with the subject of fields and conjugate points in the parlance of the calculus of variations), and the way of representing dynamical systems by means of geodesics. The refinements of the modern rigorous treatment of the calculus of variations are, of course, out of the question in a work of this kind. A more natural use of space is the treatment of Bertrand's and Thomson's theorems with which the chapter closes.

The remaining one hundred and fifty pages of the book become successively more and more full of the most recent researches. Chapter X on hamiltonian systems and their integral invariants, Chapter XI on the transformation theory of dynamics with its important theorem that the whole course of a dynamical system can be regarded as the gradual self-unfolding of a contact transformation, and Chapter XII on the properties of the integrals of dynamical systems are all of tolerably recent development, although the germs of some of the methods are to be found in the works of Jacobi, Hamilton, Poisson, and even so far back as Lagrange. There are a few problems indicated for solution which must be warmly appreciated by all who know how hard it is to find specific problems in so new and advanced a domain. The last four chapters in the reduction of the problem of three bodies, the theorems of Bruns and Poincaré, the general theorem of orbits, and integration by trigonometric series are naturally essential parts of the field of mathematical astronomy, with which the most valuable part of the author's own researches have been connected. How much the student of mechanics proper may be interested in this matter is a question. The advisability of including such a treatment here cannot, however, be well open to doubt. Mathematical astronomy as one conclusion of mechanics is quite in the spirit of the present time.

Personally we should have been happy to see introduced, before these pages crammed with necessarily complicated analysis, a treatment of two problems which are of prime importance in physics and without which no student of mechanics for its applications in physics rather than in astronomy can feel himself well equipped. The first is the study of the application of Lagrange's equations and the kinetic potential to problems in physics and chemistry. This was early introduced by Maxwell into electricity and magnetism, and is treated at length in J. J. Thomson's little book on the subject. It goes a long way toward giving the lagrangian function real physical interest. Such a chapter might properly find its place just before the tenth. The second problem is that of statistical mechanics which both in theory and its applications to the kinetic theory of gases and to thermodynamics is well available, though in too great detail for the débutant, in the works of Gibbs and Jeans. A chapter on this subject, placed perhaps right after the tenth, would go far on the way toward completing a treatise on dynamics in a direction likely soon to be of greater general interest than the special methods of astronomy. May we not hope that the author's interest in astronomy (he was recently appointed astronomer royal of Ireland, a position illustrious with great names) will not prevent him from giving adequate treatment of these two physical problems in later editions of his book?

EDWIN BIDWELL WILSON.

YALE UNIVERSITY,
April, 1906.

SOME RECENT FOREIGN TEXT BOOKS.

A Course in Practical Mathematics. By F. M. SAXELBY. London, Longmans, Green, and Co., 1905. 8vo. viii + 2 unnumb. + 430 pp. Price, $2.25.

Die Planimetrie für das Gymnasium. By GUSTAV HOLZMÜLLER. Leipzig and Berlin, B. G. Teubner, 1905. 8vo. viii + 240 pp. Price, M. 2.40.