# SURFACES GENERATED BY CONICS CUTTING A TWISTED QUARTIC CURVE AND AN AXIS IN THE PLANE OF THE CONIC.

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THE surface generated by the family of $\infty^1$ conics which cut five director curves is the simplest generalization of a scroll. Various formulas have been derived, particularly by Stuyvaert,* and the case of a unicursal quintic as directrix was first given by Bertini,† and further developed by Nugteren.‡ It is the purpose of this note to mention an interesting configuration to which the preceding methods do not apply.

1. Given a rational non-singular twisted quartic curve $c_4$, and a straight line $l$ not cutting it. Let the planes $\pi$ through $l$ and the points $P$ on $l$ be in (1, 1) correspondence. A plane $\pi$ will cut $c_4$ in four points $Q_i$ which with $P$ uniquely determine a conic $c_2$. It is required to determine the order of the surface $F$ generated by $c_2$ when $\pi$ describes the axial pencil about $l$. Let $R_2$ be the quadric upon which $c_4$ lies; let $\kappa, \kappa'$ be the two points in which $l$ cuts $R_2$, and $s_3$, $s_3'$ the trisecants of $c_4$ passing through them respectively. The plane $(l, s_3)$ will cut $c_4$ in a fourth point $Q_4$ and $c_2$ will consist of $s_3$ and $Q_4P$, if it be assumed that $P$ is not at $\kappa$. The second generator of $R_2$ in this plane is the line $Q_4\kappa'$ and does not therefore lie on $F$ unless $P$ is at $\kappa'$. Similarly, the lines $s_3'$ and $Q_4'P'$ will make up a second conic.

Any conic $c_2$ cuts $R_2$ in four points, all lying on $c_4$, hence it can cut no trisecant apart from points on the quartic curve. When $P$ is at $\kappa$, the corresponding $c_2$ has five points on $R_2$, hence lies entirely on $R_2$, and also when $P$ is at $\kappa'$ the conic in its plane lies on $R_2$. Since every trisecant of $c_4$ lies on $R_2$ it follows that every such trisecant will cut five conics of the system, hence *$F$ is of order five.*

Since any plane through $l$ contains only $c_2$ besides $l$, it follows that *$l$ is a triple line on $F_5$.* The complete intersection of $F_5$

* "Etude de quelques surfaces algébriques engendrées par des courbes du second et du troisième ordre;" dissertation, Gand, 1902.

† "Sulle curve gobbe razionali del quinto ordine," in the Collectanea Mathematica in memoriam D. Chelini, Mediolani, 1881, pp. 313–326.

‡ "Rationale Ruimtekrommen van de fijde Orde;" dissertation, Utrecht, 1902.

and $R_2$ consists of $c_4$, $2c_2$, $s_3$ and $s_3'$, making a configuration of order 10 having fourteen actual double points and two triple points, the latter being $\kappa$, $\kappa'$.

Other straight lines lie on $F_5$ also. If in any plane $\pi$ two points of $c_4$ be collinear with $P$, the line joining them and the line joining the other points of $c_4$ in that plane will constitute the conic. To determine the number of such planes, consider the three planes $\omega$ formed by $l$ and a bisecant of $c_4$ passing through $P$. To every position of $\pi$ correspond three planes $\omega$. Conversely, in any plane $\omega$ are four points of $c_4$, making six bisecants cutting $l$. Each bisecant determines a point $P$, and thus uniquely fixes $\pi$. Between $\omega$, $\pi$ exists a (6, 3) correspondence, having therefore nine coincidences. In each plane lie two lines belonging to $F_5$ and cutting $c_4$ twice, hence: *$F_5$ contains eighteen bisecants of $c_4$.*

The bisecants of $c_4$ which cut $l$ define a scroll $R_9$ of order 9, on which $l$ and $c_4$ are each triple, and $s_3$, $s_3'$ are triple generators. The intersection of $F_5$ and $R_9$ is made up of $l$ counted nine times, $c_4$ counted three times, the two trisecants which cut $l$ each counted three times, and the eighteen bisecants mentioned above. This may be expressed thus:

$$(F_5, R_9) = l(9) + c_4(12) + 2s_3(6) + 18s_2(18).$$

2. In case $\kappa$ and the plane $(l, s_3)$ are corresponding elements, then the residual point in which the (degraded) $c_2$ cuts $l$ is indeterminate, hence this plane is a factor of $F_5$. The line $l$ is double on the other factor $F_4$. We now have

$$(F_4, R_2) = c_4(4) + 2s_3(2) + c_2(2),$$

$$(F_4, R_9) = c_4(12) + l(6) + 2s_3(6) + 12s_2(12).$$

There are now but six coincidences in the correspondence between $\omega$ and $\pi$, apart from $s_3$ which counts for three.

3. If $\kappa$, $(l, s_3)$ and $\kappa'$, $(l, s_3')$ are both pairs of corresponding elements, the surface is a cubic, on which $l$ is a simple line. The equations are

$$(F_3, R_2) = c_4(4) + 2s_3(2),$$

$$(F_3, R_9) = c_4(12) + l(3) + 2s_3(6) + 6s_2(6).$$

Of the ten lines on $F_3$ which cut $l$, eight lie on $R_9$; the other two are the residuals in the planes $(l, s_3)$, $(l, s_3')$ and have but one point on $c_4$. $F_3$ contains sixteen other lines which do not lie on $R_2$ nor on $R_9$.

4. Now suppose no restriction be put upon the relation between $P$ and $\pi$, but that $c_4$ passes through $\kappa$. The conic in every plane $\pi$ will pass through $\kappa$. The surface is now of order four, and $l$ is a double line upon it. The only conic common to $F_4$ and $R_2$ is in the plane $\pi$ when $P$ is at $\kappa'$; that in the plane corresponding to $\kappa$ touches $l$ at $\kappa$, but does not lie on $R_2$. In the plane of $(l, s_3')$, $l$ is itself part of $c_2$. The point $\kappa$ is a triple point on the surface. The $R_9$ of bisecants breaks up into a cubic cone $K_3$ having its vertex at $\kappa$ and having $s_3$ for a double edge, and an $R_6$ having $l$ for triple line, $c_4$ for double curve, $s_3'$ for a triple generator, and $s_3$ for a simple generator.*

We now have

$$(F_4, R_6) = c_4(8) + s_3(1) + s_3'(3) + l(6) + 6s_2(6).$$

The correspondence $(\omega, \pi)$ is now $(3, 3)$ with six coincidences, which account for twelve lines on $F_4$, but only six belong to $R_6$, the other six lying on $K_3$,

$$(F_4, K_3) = c_4(4) + s_3(2) + 6s_1(6).$$

The surfaces $F_4$, $K_3$ furnish a monoidal representation of $c_4$ when the common vertex is a point on the curve.

If in the correspondence $(P, \pi)$, $\kappa$, $(l, s_3)$ are corresponding elements, nothing new will result, since the line joining $Q_4$ of $c_4$ to $\kappa$ is not a generator of $R_2$. The surface is not changed except that one of the six bisecants of $c_4$ mentioned above now passes through the triple point $\kappa$.

5. If $\kappa'$, $(l, s_3')$ are corresponding elements however, the surface reduces to a cubic on which $l$ is a simple line and $\kappa$ is a double point;

$$(F_3, R_2) = c_4(4) + 2s_3(2); \quad (F_3, R_6) = l(3) + c_4(8) + s_3(1)$$
$$+ s_3'(3) + 3s_2(3), \quad (F_3, K_3) = c_4(4) + s_3(2) + 3s_1(3).$$

$F_3$ and $K_3$ furnish a monoidal representation of $c_4$.

6. If $l$ is a bisecant of $c_4$, every $c_2$ must pass through two fixed points. Since it must also pass through $P$ on $l$, the latter is a factor of every $c_2$, and the residual is a straight line joining

* This surface is type 80 in my enumeration of sextic scrolls, *Amer. Jour. of Math.*, vol. 27, p. 101.

the other two points of $c_4$ in $\pi$. The surface is now of order three and $l$ is a double line above it. Since from every point of $l$ two bisecants to $c_4$ can be drawn, apart from $l$ itself, the surface is a cubic scroll of the first kind, $R_3$. The $R_6$ breaks up into a cubic cone, with vertex at $\kappa'$, and this same $R_3$.

Finally, if $l$ has three points upon $c_4$, $F$ becomes the $R_2$ upon which $c_4$ lies.

7. If $c_4$ is of genus one it has no trisecants. As no line can cut more than two conics apart from the two lying on $R_2$, the surface is of order four and $l$ is a double line upon it. The scroll of bisecants is now $R_8$ on which $l$ is a double directrix and $c_4$ a triple curve. The correspondence $(\pi, \omega)$ is now $(2, 6)$ with eight coincidences and 16 lines, hence

$$(F_4, R_8) = c_4(12) + l(4) + 16s_2(16).$$

If one or both points $\kappa$, $\kappa'$ correspond to the plane containing a bisecant passing through them, the conics in these planes break up, but no important changes in the form of the surfaces occur.

8. If $l$ intersects $c_4$ at $\kappa$, $R_8$ breaks up into an elliptic $K_3$ and an $R_5$ having the symbol $l_2^2 + c_4^2$.* $F$ is a cubic on which $l$ is a simple line and $\kappa$ is a node.

$$(F_3, R_5) = l(2) + c_4(8) + 5s_2(5), \quad (F_3, K_3) = c_4(4) + 5s_1(5).$$

The correspondence $(\pi, \omega)$ is now $(2, 3)$ and the residual lines in the planes of the coincidences belong to $K_3$. If $l$ intersects $c_4$ twice the surface reduces to $R_2$ containing $c_4$ and $l$.

9. If $c_4$ has a node, and no restrictions as to $(\pi, P)$, the surface is $F_4$, having $l$ for double line. Two lines through the node cut $l$ and cut $c_4$ again;

$$(R_8, F_4) = l(4) + c_4(12) + 16s_2(16).$$

If $\kappa$ ($l$ node) are corresponding elements, the surface is $F_3$,

$$(F_3, R_8) = l(2) + c_4(12) + 10s_2(10),$$

since the line joining the node to $\kappa$ and cutting $c_4$ again counts for three coincidences. Both points $\kappa$, $\kappa'$ in which $l$ cuts $K_2$ on which $c_4$ lies cannot give rise to coincidence, because they lie in the same plane $\kappa$.

* This scroll is type B, iv of Schwarz's classification in *Crelle's Journal*, vol. 67, p. 37.

10. If $l$ cuts $c_4$ in $\kappa$, the surface is a nodal cubic, having one node at $\kappa$, and another at the node of $c_4$. $R_8$ breaks up into $K_3$ and $R_5$, the latter having the symbol $l_2 + c_4^2 + 3^2$ (Schwarz's A, vii), the double generation being the line joining the node to the fourth point in the plane containing $l$. Thus,

$$(F_3, R_5) = l(2) + c_4(8) + g_2(2) + 3s_2(3).$$

If $c_4$ has a cusp, the second nodal point becomes uniplanar. Further specializations result in quadrics and quadric cones.

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# OPERATION GROUPS OF ORDER $p_1^{m_1\mu_1}p_2^{m_2\mu_2}$.

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IT is desired to make certain generalizations concerning the groups of order the product of powers of two primes $p_1$, $p_2$, such that $p_1 \equiv 1 (\text{mod } p_2)$, these groups possessing abelian subgroups $H_i$ of type $[\mu_i, \mu_i, \cdots, \mu_i]$ $(i = 1, 2)$. It is possible to specify for these groups those subgroups (here called basic subgroups) from which it is necessary and sufficient that generating operations be selected in order that they may generate the whole group $G$. This general problem connected with groups of composite order seems to merit more attention than it has thus far received. If

$$H_1 = \{P_1, P_2, \cdots, P_{m_1}\}, \quad H_2 = \{Q_1, Q_2, \cdots, Q_{m_2}\},$$

then the number of operations of order $p_i^{\mu_i}$ in $H_i$ is

$$\sum_{j=0}^{m_i-1} {}_{m_i}C_j[p_i^{\mu_i} - \Phi(p_i^{\mu_i})]^j [\Phi(p_i^{\mu_i})]^{m_i-j}$$
$$= [p_i^{\mu_i-1} + \Phi(p_i^{\mu_i})]^{m_i} - p^{m_i(\mu_i-1)},$$

so that the number of cyclical subgroups of order $p_i^{\mu_i}$ in $H_i$ is

$$N_{p_i}^{\mu_i} = \frac{p_i^{m_i(\mu_i-1)}(p_i^{m_i}-1)}{\Phi(p_i^{\mu})}$$
$$= p_i^{(m_i-1)(\mu_i-1)}(p_i^{m_i-1} + p_i^{m_i-2} + \cdots + p_i + 1).$$