THE GROUPS CONTAINING THIRTEEN OPERATORS OF ORDER TWO.

BY PROFESSOR G. A. MILLER

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It is known that a group of order 2^m cannot contain exactly thirteen operators of order 2, but that there are other groups which have this property.* Let G represent such a group. We shall first consider the case where the operators of order 2 in G form a single set of conjugates. In this case G transforms these operators according to a transitive substitution group of degree 13. The subgroup of G which corresponds to identity in this transitive substitution group will be represented by H.

If H were of even order, it would contain an operator of order 2 which would be commutative with every operator of this order contained in G. From this it would follow that each of these operators would be commutative with all of them. This is impossible since they cannot all be contained in a group of order 2^m. Hence *H is of odd order.*

The substitution group which is simply isomorphic with G/H must therefore contain substitutions of order 2. Since the group of order $2h$, h being the order of H, which corresponds to the group generated by such a substitution contains at least one operator of order 2, the substitution group of degree 13 contains exactly 13 substitutions of order 2; hence it involves only one subgroup of order 13 and its order is a divisor of 156. Moreover, *all the operators of H are commutative with every operator of order 2 contained in G.*

The product of two operators of order 2 contained in G corresponds to an operator of order 13 in the isomorphic group. Since this product is transformed into its inverse by each of its factors,† and since the operators of H are commutative with each of these factors, it follows that this product is of order 13. That is, *any two operators of order 2 contained in G generate the dihedral rotation group of order 26.* As each of the operators of this group is commutative with every operator of H, it fol-

* BULLETIN, vol. 12 (1905), p. 74.

† BULLETIN, vol. 7 (1901), p. 424.

lows that G contains the direct product of a group of odd order and the dihedral rotation group of order 26 as an invariant subgroup. The quotient group of G with respect to this invariant subgroup is contained in the cyclic group of order 6.

The order of G cannot be divisible by eight when all the operators of order 2 are conjugate. If this order is divisible by four the subgroups of order 4 are cyclic. Hence it follows that each of the Sylow subgroups of G whose order is a power of two contains only one operator of order 2 when the operators of this order in G form a single conjugate set. Conversely all the operators of order 2 must be conjugate under G whenever a Sylow subgroup of order 2^m contains only one operator of this order. Hence it follows that in the groups which remain to be considered each Sylow subgroup of order 2^m contains more than one operator of order two.

§ 1. *Groups Containing a Set of two Conjugate Operators of Order* 2.

Let s_1, s_2 be two operators of order 2 contained in G, such that s_1 is transformed into s_2 by exactly half the operators of G. As every operator which transforms s_1 into s_2 must also transform s_2 into s_1, G contains operators whose orders are powers of 2 which transform s_1 into s_2. Hence the Sylow subgroups of order 2^m contained in G are non-abelian and their orders must be divisible by 8. As the product s_1s_2 is invariant under G, it is included in every Sylow subgroup of order 2^m. The Sylow subgroups which contain s_1 must therefore also contain s_2. That is, *every Sylow subgroup of order* 2^m *includes the four group generated by* s_1, s_2.

Every Sylow subgroup of order 2^m includes other operators of order 2. Let s_3 represent such an operator. We shall first prove that s_3 cannot be commutative with s_1 and s_2. If this were the case, a Sylow subgroup S would contain at least seven commutative operators of order 2. It could not contain exactly seven operators of order 2, since 10 is not divisible by 4. It could not contain eleven operators of this order;* for there would be some other Sylow subgroup S_1 which would contain

* This is a special case of the theorem : if a Sylow subgroup of any group G contains more than one subgroup of order p without including all the operators of order p in G, then G contains at least p^2 subgroups of order p which are not in a given Sylow subgroup.

at least two operators of order 2 which are not in S. One such operator would transform S into S_2 having at least two operators of order 2 which are not in S. As S_2 could not contain this transforming operator, it follows that S_1 would contain at least two operators which are neither in S nor in S_2. Since S_2 would contain at least two other operators of order 2 which are not in S, it is clear that S could not contain more than nine operators of order 2.

If S contained nine operators of order 2 and if s_3 were also commutative with s_1 and s_2, the operators of order 2 which are not in the group generated by s_1, s_2, s_3 could not be commutative with s_1. Hence s_1, s_2, s_3 would generate an invariant subgroup of S. As an operator of order 2 could not transform this subgroup into itself without being commutative with at least four of its operators, S could not contain exactly nine operators of order 2. Hence, as stated above, s_3 cannot be commutative with s_1 and s_2.

Since s_3 transforms s_1 and s_2, it, together with these two operators, generates the octic group. We shall now consider the case where S contains no operator of order 2 besides those of this octic group. As all the conjugates of S have three operators in common, there are just five distinct octic subgroups in these conjugates. These subgroups are transformed by all the operators of G according to a transitive group T of degree 5. Since T contains a complete set of five conjugate operators of order 2, it can involve only one subgroup of order 5. The operators of G which correspond to identity of T constitute an invariant subgroup H of G. The operators of H which are commutative with s_3 constitute a subgroup of half the order of H. This subgroup H_1 includes s_1, s_2.

As H_1 contains only one operator of order 2, its Sylow subgroups of order 2^m either are cyclic or they contain cyclic subgroups of half their order while the remaining operators are of order 4. They must therefore contain an odd number of cyclic subgroups of order 4 if they contain one such subgroup. If the order of H_1 were divisible by 4, H would contain an operator of order 4 which would be transformed into its inverse either by s_1 or by s_3. Hence either H would contain more than three operators of order 2, or a divisor of G corresponding to an operator of order 2 in T would contain more than two such operators. As this is impossible, *H contains only one subgroup of order 4, viz., the one generated by s_1, s_2, and the order of H is not divisible by 8.*

Two operators of order 2 in G which correspond to two distinct operators of order 2 in T have for their product an operator whose order is divisible by 5 but by no other odd prime, since s_3 is commutative with all the operators of H_1. The order of this product could not be divisible by 4 since H does not include any operator of this order. Moreover, the two given factors may be so selected as to make the order of this product either 5 or 10. In the latter case they generate the dihedral rotation group of order 20. This group contains all the operators of order 2 in G except s_1 and s_2. These two operators are commutative with the operators of its cyclic subgroup of order 10, but they transform the remaining operators into themselves multiplied by s_1s_2. Hence *the operators of order* 2 *in G generate the group of order* 40 *which contains* 13 *operators of order* 2.

As the order of H_1 is not divisible by 4 and as H_1 contains only one operator of order 2, it is the direct product of a group of odd order and s_1s_2. This group of odd order is clearly invariant under G. In fact, H is the direct product of this group and the group generated by s_1, s_2. Hence G contains the direct product of a group of odd order and the group of order 40 which contains exactly thirteen operators of order 2. If G contained any other operators, it would contain this direct product as an invariant subgroup of half its order, and the remaining operators would correspond to operators of order 4 in T. The square of such an operator would therefore transform s_1 into s_2, since s_3 has this property. This is clearly impossible as s_1 has only two conjugates under G.

We have now proved that G is the direct product of a group of odd order and the group of order 40 which contains just thirteen operators of order 2, whenever s_3 transforms s_1 into s_2 and a Sylow subgroup contains no operators of order 2 besides those contained in the octic group generated by s_1, s_2, s_3. Moreover, the direct product of *any* group of odd order and this group of order 40 possesses the properties in question. It remains to consider the case when S contains more than one octic subgroup involving s_1, s_2. Each of these octic subgroups is transformed into itself by every operator of order 2 contained in S. There could not be more than three such octic subgroups involving s_1, s_2, since S cannot involve eleven operators of order 2. If there were three, one would be invariant under S. The operators of this subgroup would transform each of the others into itself. Hence each would be transformed into itself by the other two, and the three would generate a group whose commutator subgroup would be 1, s_1s_2. As such a group would contain more than nine operators of order 2, S cannot contain more than two octic subgroups involving s_1, s_2.

As one octic group would transform one other octic group into itself, it would transform two of the five contained in G into themselves. Hence G would have to transform the five octic subgroups according to a transitive group of degree 5 containing a transposition. This is clearly impossible. Hence *G contains a set of two conjugate operators of order 2 only when it is the direct product of the group of order 40 which contains thirteen operators of order 2 and some group of odd order.*

§2. *Proof that there is no Group in which the Number of Operators of Order 2 in a Complete Set of Conjugates is Three, Four or Five.*

If there were a set of three operators of order 2 which formed a complete set of conjugates under G, they would be transformed according to a group of degree 3 by all the operators of G. The subgroup H which would correspond to the identity in this group of degree 3 could not involve these three conjugates; for, if they were all contained in a four group, H could not contain the thirteen operators of order 2 since $13 \not\equiv 3 \mod 4$. Neither could H contain just seven operators of this order, since they would be included in an invariant subgroup of order 8 and hence G would have to involve more than thirteen operators of order 2. Since the number of operators of order 2 which are in G without being also in H is divisible by 6, no other numbers require consideration. If the three given operators generated a group of order 8, this would be invariant, and hence this case is impossible. If the three conjugate operators in question were not in H, there would be just one more operator of order 2 in a divisor corresponding to a substitution of order 2 than in H. This is again impossible because $13 \not\equiv 3 \mod 4$. Hence there cannot be a complete set of three conjugate operators of order 2 under G.

If G contained a complete set of four conjugate operators of order 2, at least two of these would be in a four group. If three of them were in this group, they would all be commutative and hence would generate an invariant subgroup of order 8. As this is impossible, the four group which contained two of these four conjugates would not contain any other. It is easy to see that this four group would be transformed into itself by the other conjugates and that the only case which requires further consideration is when these four conjugates generate an invariant octic group, and when S contains nine operators of order 2. No two Sylow subgroups could have only five common operators of order 2, since one of the remaining operators of order 2 in one of these Sylow subgroups could not transform the other into itself and hence would transform it into one having additional operators of order 2. As this is impossible, any two Sylow subgroups would have at least seven common operators of order 2.

The eight operators of order 2 which are not in the octic subgroup generated by the four conjugate operators in question would therefore be divisible into four pairs such that each pair with this octic group would generate a subgroup involving just seven operators of order 2. As the number of subgroups involving nine operators of order 2 would be odd, and as all would be conjugate under G, this is impossible. The impossibility of constructing such a group G can also be proved by observing that the nine operators of order 2 in S would generate a group of order 32 which would have a commutator subgroup of order 2 and contain the given octic group as invariant subgroup. As such a group cannot be constructed, it is proved that G could not contain a complete set of four conjugate operators of order 2.

If G contained a complete set of five conjugate operators of order 2, it would transform them according to a transitive substitution group of degree 5. Let H represent the subgroup of G corresponding to identity in this transitive substitution group. As H cannot involve the five conjugate operators of order 2, it contains just three operators of this order. This is impossible, since each of these three operators would be commutative with each one of the five conjugate operators of order 2, and hence each divisor corresponding to a substitution of order 2 would contain just four operators of order 2.

§ 3. *Groups in which there is a Complete Set of Six Conjugates of Order* 2.

If G contains a complete set of six conjugate operators of order 2 it must transform them according to a transitive substitution group T of degree 6. The conjugate operators of order 2 cannot be contained in the subgroup H which corresponds to identity of T, since G cannot contain an invariant subgroup of order 8 which involves no operator of order 4. If these six conjugates correspond to the same substitution of T, H would contain five operators of order 2. The remaining two operators could not correspond to this substitution of T. Hence they would correspond to another invariant substitution. This is impossible since T cannot contain two invariant substitutions of order 2. As every non-invariant substitution of order 2 has at least three conjugates under T, the six conjugate operators in question correspond either to three or to six substitutions of T.

In the latter case H would be of odd order; for, if it were of even order the operators of order 2 which would correspond to two commutative conjugate substitutions of degree < 6 could not be commutative since their product could not be of order 2. That is, there would not be a complete set of six conjugates of order 2 in G. If H were of odd order, only one operator of G would correspond to each of the six conjugates in question, and at least one operator of order 2 would correspond to every substitution of order 2 in T. It follows directly from the properties of the T's which have six conjugates of order 2 that this is impossible, hence the six conjugate operators of order 2 in question correspond to three conjugate substitutions whose degree is < 6, and H involves only one operator of order 2. If these three substitutions of T were commutative, the corresponding operators of order 2 would also be commutative, or else they would generate a group whose commutator subgroup is of order 2.

The former of these two alternatives is clearly impossible. In the latter, the six conjugates in question would generate a subgroup of order 16, and a Sylow subgroup of order 2^m would involve at least 11 operators of order 2. As this is impossible, it follows that the three conjugate substitutions of T which correspond to the six conjugates in question must generate a group of order 6. These conjugates must therefore generate the dihedral rotation group of order 12, and hence this is an invariant subgroup of G, and T contains an invariant subgroup of order 3. From this it follows that T is one of the two groups of order 12.

We shall now prove that the order of H cannot be divisible by 4. If it were divisible by 4, a Sylow subgroup of H would contain an odd number of cyclic subgroups of order 4. Hence one of them would be invariant under S. If an operator of order 2 not included among the six conjugates in question transformed the generator of this invariant subgroup into its third power, it would correspond to the invariant substitution of T and together with the given generator would generate the octic group. The two operators of order 2 which would be in the same divisor but not in this octic group would transform this octic group into itself. Hence this octic group would be invariant under the group generated by all the operators of order 2 in G. As the operators of G which would correspond to the invariant substitution of T without being in this octic group could not transform the operator of order 4 in this octic group into its inverse they would be commutative with it, and hence this divisor would contain operators of order 4. Such operators would be transformed into their inverse by some operator belonging to the six conjugates in question. As this would lead to operators of order 2 in a divisor corresponding to another substitution of T, it is impossible.

If an operator of order 2 corresponding to the invariant substitution of order 2 in T were commutative with the given operator of order 4, it would transform an operator of order 4 corresponding to one of the six conjugates in question into its inverse. As this is clearly impossible, it is proved that H could not involve an operator of order 4 in case an operator of order 2 in G corresponds to an invariant substitution in T. We shall now prove that the order of H is not divisible by 4 when each of the operators of order 2 in T corresponds to a non-invariant substitution in T. If H contained operators of order 4, such operators would correspond to each non-invariant substitution of order 2 in T. From this it follows directly that there would be operators of order 2 in the divisor corresponding to the invariant substitution of T. As this is impossible, it is proved that *the order of H is not divisible by* 4.

If the operators of order 2 which are not in the given set of six conjugates correspond to non-invariant substitutions of T, two operators of order 2 can be found such that their product is of order 12, since no operator of order 2 corresponds to the invariant substitution in T. Hence the operators of order 2 in G generate the dihedral rotation group of order 24, and G is the direct product of this group and the largest subgroup of odd order contained in H. Moreover, every direct product of this dihedral rotation group and an arbitrary group of odd order contains just thirteen operators of order 2 which have the property in question.

If six operators of order 2 correspond to the invariant substitution of order 2 in T, they generate the dihedral rotation group of order 12, and H contains an invariant subgroup of order 3. As the largest subgroup of odd order in H transforms these six operators according to a group of odd order which has two transitive constituents of degree 3, it must contain an invariant subgroup of one-third its order which does not include the operator of order 3 in the said dihedral rotation group of order 12. Hence H is the direct product of a subgroup of order 2, a subgroup of order 3, and some group of odd order. Moreover, G contains the direct product of the dihedral rotation group of order 12, generated by the six operators of order 2 which are transformed according to T, and the invariant subgroup of order 3 in H.

The six operators of order 2 which correspond to the invariant substitution of T transform this direct product of order 36 into itself and generate with it the group of order 72 which contains exactly 13 operators of order 2 and is generated by these operators. This subgroup of order 72 is invariant in G and has only identity in common with the given group of odd order in H. Hence G is the direct product of some group of odd order and this group of order 72 whenever some of its operators of order 2 correspond to the invariant substitution of T, and every such direct product satisfies the condition in question. The result of this section may be expressed as follows: *Whenever G contains six operators of order 2 which constitute a complete set of conjugates, its thirteen operators of order 2 generate either a group of order 24 or a group of order 72 and G is the direct product of one of these groups and some group of odd order.* Every such direct product contains exactly six conjugates of order 2, and involves 13 operators of this order.

§ 4. *Proof that there is no Additional Group in which the Number of Operators of Order 2 in a Complete Set of Conjugates is Seven, Eight, Nine, Ten, or Eleven.*

If there were a complete set of either seven, eight or nine conjugates of order 2, the remaining operators of this order could not be invariant, since G cannot contain the group o order 8 which contains 7 operators of order 2 as an invariant subgroup. These remaining operators could not occur in sets of conjugates, as these sets have been considered above. If G contained three invariant operators of order 2, while the remaining 10 formed a single set of conjugates, the three invariant operators would generate the four group. As any one of the remaining operators of order 2 and this four group would generate the group of order 8 which has 7 operators of order 2, the remaining operators could be arranged in distinct sets of four operators, which is evidently impossible.

If G contained two invariant operators, their product would also be invariant and hence there could not be a complete set of eleven conjugates. It only remains therefore to consider the case where G contains a single invariant operator of order 2 while the remaining 12 constitute a complete set of conjugates. It will be found that there are groups which come under this case. Hence there are groups containing exactly 13 operators of order 2 in which these operators form sets of conjugates containing any of the following numbers of operators: 13; 1, 2, 10; 1, 6, 6; 1, 12.

§ 5. *Groups in which there is a Complete Set of Twelve Conjugates of Order* 2.

Since such a G contains an invariant operator of order 2, it contains six conjugate four groups and transforms them according to a transitive substitution group T of degree 6. If the operators H of G which correspond to the identity of T included more than one operator of order 2, these six subgroups of order 4 would generate a group of order 2^m involving thirteen operators of order 2. As this is impossible, T contains a complete set of either three or six conjugate substitutions of order 2 and of degree < 6, corresponding to the operators of order 2 in G.

In the former case these three conjugate substitutions of T cannot be commutative, since all the operators of order 2 in G cannot be contained in a Sylow subgroup. Hence T is the group of order 12 which contains seven substitutions of order 2. The four operators of G which correspond to the same substitution in T generate the octic group. As the operators of order 2 in the other divisions transform this octic group into its conjugates, these three conjugates have the cyclic subgroup of order 4 in H in common. Hence this cyclic subgroup is invariant under G. From this it follows directly that H cannot contain any operator of order 8; for if H contained such an operator, its square would generate the only invariant cyclic subgroup of order 4, but this square could not be in the given octic groups.

We shall now prove that the Sylow subgroups of H could not include the quaternion group. If this group occurred in H, a Sylow subgroup S of G would be of order 32 and would include two non-cyclic subgroups of order 16 which would involve only one operator of order 2. As it would contain another subgroup of order 16 involving a cyclic subgroup of order 8, it would contain just three cyclic subgroups of order 8. This is impossible since the number of such subgroups in a group of order 2^m is even.* The order of H is therefore divisible by 4 but not by 8. As all of the operators of odd order in H transform each operator of order 2 in G into itself but some of its operators of even order do not have this property, the operators of odd order generate a subgroup invariant under G, and H is the direct product of this subgroup and its cyclic subgroup of order 4.

Since the product of two non-invariant operators of order 2 in G is commutative with every operator of H, these two operators may be so chosen that their product is of order 12 and hence *the operators of order* 2 *in* G *must generate the dihedral rotation group of order* 24 *whenever four of the operators of* G *correspond to the same substitution in* T. As this invariant subgroup has only identity in common with the largest group of odd order contained in H, G includes the direct product of these two groups and its order is twice the order of this direct product.

If an operator of order 4 corresponds to the invariant substitution of T, all the operators of this divisor whose order is a power of two must be of order 4. Such an operator transforms each of the factors of the given direct product into itself. It transforms the subgroup of order 24 according to an operator of order 2, while it may be either commutative with the other factor or transform it according to an operator of order 2. In the former case G is the direct product of a group of odd order and the group of order 48 which involves just thirteen operators of order 2 and in which all the operators of order 4 are commutative with the two operators of order 3. This

* *Transactions Amer. Math. Society*, vol. 6 (1905), p. 59.

group of order 48 may also be defined by the fact that it contains the direct product of the quaternion group and the group of order 3, and is generated by this direct product and an operator of order 2 which transforms the operators of order 3 into their inverses and the quaternion group into a contragredient isomorphism with itself. In the latter of the two cases mentioned above G may be obtained by establishing an isomorphism between this group and a group whose order is twice an odd number.

When an operator of order 8 corresponds to the invariant substitution in T, all the operators of this divisor whose order is a power of two are of order 8. The remarks of the preceding paragraph apply directly to this case, with the exception that the given group of order 48 is replaced by the one which is generated by the cyclic group of order 24 and an operator which transforms each operator of this cyclic group into its eleventh power. Hence *there are two groups of order* 48 *which contain exactly* 13 *operators of order* 2, *twelve being conjugate, and transform their six four groups involving these operators according to just three distinct substitutions.* All the other possible groups which have this property can be obtained by dimidiating one of these two groups and a group whose order is twice an odd number and all such dimidiations give rise to groups having the required property.

We shall now consider the case where these six four groups correspond to six distinct substitutions of T. As these substitutions form a complete set of conjugates and are of degree less than six, T is either the positive group of order 24 or the group of order 48. If it were the latter, it would contain an invariant subgroup of order 8 times the order of H. As this subgroup would contain only one operator of order 2, it would include a Sylow subgroup having this property. The order of this S could not be less than sixteen. Hence it would have operators of order 8. This is impossible. In fact it is at once evident that such an S could not have an $(\alpha, 1)$ isomorphism with a group involving no operator of order 4 unless the order of this group is less than eight. The groups in question must, therefore, transform the six conjugate four groups according to the positive group of order 24, which is simply isomorphic with the symmetric group of degree 4.

The order of H is not divisible by four, since the operators which correspond to the three conjugate substitutions in T

must be all of a higher order than those of H. This follows also from the fact that the quaternion group is the only group of order 2^m involving only one operator of order 2 which has an operator of odd order in its group of isomorphisms. Hence H is the direct product of a group of order 2 and some group of odd order. Every operator of H is commutative with each of the operators of the group generated by all the operators of order 2 in G. Two of these operators corresponding to two commutative substitutions in T generate the octic group. Hence all the operators of H are commutative with the operators of order 4 corresponding to the three conjugate substitutions of T. That is, *G contains an invariant quaternion group and this subgroup includes all its operators of order* 4.

Two operators of order 2 can be so selected that their product is of order 3. This product and the given quaternion group generate a group of order 24 which is transformed into itself by any one of the operators of order 2. Hence *the thirteen operators of order* 2 *generate a group of order* 48, *and G is the direct product of this group and a group of odd order.* Moreover, any such direct product satisfies the given conditions. The given group is completely defined by the facts that it is of order 48 and is generated by thirteen operators of order 2. The main results of this section are as follows: If a group contains just thirteen operators of order 2 of which twelve are conjugate, these operators generate either the dihedral rotation group of order 24 or the group of order 48 which may be represented as a transitive group of degree 8 and contains the quaternion group invariantly. In the latter case G is the direct product of this group of order 48 and some group of odd order. In the former case G contains one of two groups of order 48 and may be constructed by dimidiating this group and a group whose order is twice an odd number.

If a group contains exactly thirteen operators of order 2 these operators must, therefore, generate one of the following five groups: The dihedral rotation group of order 24, the dihedral rotation group of order 26; the group of order 40 which contains thirteen operators of order 2; the group of order 48 which can be represented as a transitive group of degree 8 having four but not two systems of imprimitivity; or the group of order 72 which is generated by the direct product of the dihedral rotation group of order 12, an operator of order 3, and an operator of order 2 which transforms this operator of order 3 into its inverse and the non-invariant operators of order 2 in the dihedral rotation group of order 12 into themselves multiplied by the invariant operator of order 2. In case of the groups of order 40, 48 and 72 all the possible groups are obtained by forming the direct product of these groups and some group of odd order.

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TYPES OF SERIAL ORDER.

*The Continuum as a Type of Order: An Exposition of the Modern Theory. With an Appendix on the Transfinite Numbers.** By EDWARD V. HUNTINGTON. Cambridge, Mass., The Publication Office of Harvard University, 1905. 4to. 63 pp. Price, 50 cents.

THE *Annals of Mathematics* has for some time followed the plan of printing articles expository of subjects which are little known or not easily accessible in the English language. Reprints of these articles are then placed on sale with the double and laudable purpose of making the circulation of the article wider than it would otherwise be and of helping solve the difficult problem of financing a mathematical journal.

The plan can hardly fail to succeed if all the articles are as clear in style and as just in the balance between generality and detail as is that of Professor Huntington. In point of readableness, we are inclined to think that the only other exposition of subjects connected with the foundations of mathematics which can be compared with Huntington's is that (in French) of L. Couturat.

The principal contents of the paper are the ordinal theory of integers, rational numbers, and the continuum, together with an appendix on the transfinite numbers of Cantor. It is intended for non-mathematical readers as well as for mathematicians, and therefore presupposes very little in the way of detailed knowledge, though of course it requires for complete comprehension a considerable maturity in abstract reasoning. We have noticed only one error of any consequence. It is

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