CERTAIN SURFACES ADMITTING OF CONTINUOUS DEFORMATION WITH PRESERVATION OF CONJUGATE LINES.

BY DR. BURKE SMITH.

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BIANCHI has shown* that surfaces which admit of continuous deformation with preservation of conjugate lines are the associates of surfaces whose curvature in terms of the parameters of the asymptotic lines is of the form $K = -[\phi(u) + \chi(v)]^{-2}$, where ϕ and χ are arbitrary. Stäckel † has considered those surfaces on which the two families of lines of curvature fall together into a single family which are also at the same time minimal lines and asymptotic lines. The curvature of such surfaces may be written in the form $K = e^{-2m(v)}$ and consequently it is of the above type. We proceed to find the coordinates x_0, y_0, z_0 of certain of these surfaces.

The fundamental magnitudes of the Gauss sphere which corresponds to such a surface may be written

$$(1)\quad e = 0, \quad f = \frac{-2\phi'(u)\psi'(v)}{[\phi(u)+\psi(v)]^2}, \quad g = \frac{-2m'(v)\psi'(v)}{\phi(u)+\psi(v)} + n(v),$$

where the primes denote derivatives and ϕ, ψ, m and n are arbitrary. We consider only the case where $\phi \equiv u$ and $\psi \equiv v$. Upon substituting in the fundamental equations for the second derivatives of X and the first derivatives of x with regard to u and v we have

$$(2)\qquad \frac{\partial^2 X}{\partial u^2} = \frac{-2}{u+v}\frac{\partial X}{\partial u},$$

$$(3)\qquad \frac{\partial^2 X}{\partial u \partial v} = \frac{-m'}{2}\frac{\partial X}{\partial u} + \frac{2}{(u+v)^2}X,$$

$$(4)\quad \frac{\partial^2 X}{\partial v^2} = \left\{\begin{matrix}2\ 2\\ 1\end{matrix}\right\}'\frac{\partial X}{\partial u} + \tfrac{1}{2}\left[m' - \frac{4}{u+v}\right]\frac{\partial X}{\partial v} + \left[\frac{2m'}{u+v} - n\right]X,$$

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* Bianchi-Lukat : Vorlesungen über Differential-Geometrie, p. 337.
† *Leipziger Ber.*, vol. 48, p. 479.

$$
(5) \qquad \begin{cases} \dfrac{\partial x_0}{\partial u} = - e^m \dfrac{\partial X}{\partial u}, \\[2mm] \dfrac{\partial x_0}{\partial v} = \dfrac{u+v}{2}\left[2m' - (u+v)n\right] e^m \dfrac{\partial X}{\partial u} - e^m \dfrac{\partial X}{\partial v}, \end{cases}
$$

with similar equations for Y, Z, y_0, z_0, where

$$
\begin{Bmatrix} 2 & 2 \\ & 1 & \end{Bmatrix}' = \tfrac{1}{4}\Big(2(u+v)(m'' - m'^2) + m'\left[6 + (u+v)^2 n\right] \\ - (u+v)\left[(u+v)n' + 4n\right]\Big).
$$

The general integral of (2) is

$$
(6) \qquad X = \frac{F(v)}{u+v} + G(v).
$$

Now (3) has equal invariants, and by the substitution $\theta = X\sqrt{ie^m}$ becomes $\dfrac{\partial^2 \theta}{\partial u \partial v} = \dfrac{2\theta}{(u+v)^2}$, of which the general integral is

$$
\theta = \frac{2(U_1 + V_1)}{u+v} - U_1' - V_1',
$$

U_1 and V_1 being arbitrary functions of u and v respectively. Comparing with (6), we must have $U_1 \equiv 0$ and X can be written in the form

$$
X = \frac{2V}{u+v} - V' - \frac{m'}{2} V.
$$

Since also (4) must be satisfied, we have to determine V the equation

$$
(7) \quad V''' + \left(m'' - \frac{m'^2}{4} + n\right) V' + \left(\frac{m'''}{2} - \frac{m''m'}{4} + \frac{n'}{2}\right) V = 0,
$$

where the primes denote derivatives. To every solution of (7) we have a value for X, and similar equations hold for Y and Z. When $n = m'^2/4 - m'' + 1$, (7) takes the form $V''' + V' = 0$ and consequently, $V = a \cos v + b \sin v + c$. If the constants a, b, c, are so chosen that $X^2 + Y^2 + Z^2 = 1$ then (5) will give by quadrature the coördinates x_0, y_0, z_0. Take, for example, a, b, c, so that

$$X=\left(\frac{2}{u+v}-\frac{m'}{2}\right)\cos v+\sin v,$$

$$Y=\left(\frac{2}{u+v}-\frac{m'}{2}\right)\sin v-\cos v, \qquad Z=i\left(\frac{2}{u+v}-\frac{m'}{2}\right).$$

Then from (5) we have

$$(8)\begin{cases} x_0=\dfrac{-2e^m\cos v}{u+v}+\displaystyle\int e^m\left[\left(\frac{m'^2}{4}-\frac{m''}{2}\right)\cos v-\frac{m'}{2}\sin v\right]dv \\ y_0=\dfrac{-2e^m\sin v}{u+v}+\displaystyle\int e^m\left[\left(\frac{m'^2}{4}-\frac{m''}{2}\right)\sin v+\frac{m'}{2}\cos v\right]dv \\ iz_0=\dfrac{2e^m}{u+v}-\displaystyle\int e^m\left(\frac{m'^2}{4}-\frac{m''}{2}+1\right)dv, \end{cases}$$

where $m=m(v)$ may be chosen arbitrarily.

In another place the writer has shown that the equations of the surfaces $S(x, y, z)$ which are associate to a given surface $S_0(x_0, y_0, z_0)$ may be written in the form

$$(9)\qquad\begin{cases} x=\rho^{-\frac{3}{2}}(eg-f^2)^{-\frac{1}{2}}\left(\theta\dfrac{\partial^2 x_0}{\partial u\partial v}-\dfrac{\partial\theta}{\partial u}\dfrac{\partial x_0}{\partial v}-\dfrac{\partial\theta}{\partial v}\dfrac{\partial x_0}{\partial u}\right), \\ y=\rho^{-\frac{3}{2}}(eg-f^2)^{-\frac{1}{2}}\left(\theta\dfrac{\partial^2 y_0}{\partial u\partial v}-\dfrac{\partial\theta}{\partial u}\dfrac{\partial y_0}{\partial v}-\dfrac{\partial\theta}{\partial v}\dfrac{\partial y_0}{\partial u}\right), \\ z=\rho^{-\frac{3}{2}}(eg-f^2)^{-\frac{1}{2}}\left(\theta\dfrac{\partial^2 z_0}{\partial u\partial v}-\dfrac{\partial\theta}{\partial u}\dfrac{\partial z_0}{\partial v}-\dfrac{\partial\theta}{\partial v}\dfrac{\partial z_0}{\partial u}\right), \end{cases}$$

where $-1/\rho^2$ is the curvature of S_0 and θ is a solution of the equation

$$\frac{\partial^2\theta}{\partial u\partial v}=\left(-\frac{1}{4}\frac{\partial\rho}{\partial u}\frac{\partial\rho}{\partial v}-f\right)\theta.$$

This equation becomes, in the case of the surfaces defined by (8), $\dfrac{\partial^2\theta}{\partial u\partial v}=\dfrac{2\theta}{(u+v)^2}$, of which the general solution is

$$\theta=\frac{2(U+V)}{u+v}-U'-V'.$$

On substituting in (9), we have

$$x = e^{-m/2}(u+v)^2 \left\{ \left[A\left(\frac{m'^2}{4} - \frac{m''}{2}\right) + Bm' + D \right] \cos v - \left(A\frac{m'}{2} + B \right) \sin v \right\},$$

$$y = e^{-m/2}(u+v)^2 \left\{ \left[A\left(\frac{m'^2}{4} - \frac{m''}{2}\right) + Cm' + D \right] \sin v + \left(A\frac{m'}{2} + C \right) \cos v \right\},$$

$$-iz = e^{-m/2}(u+v)^2 \left\{ A\left(\frac{m'^2}{4} - \frac{m''}{2} + 1\right) + Cm' + D \right\},$$

where

$$A = \frac{U+V}{(u+v)^2} - \frac{U'}{u+v} + \frac{U''}{2}, \quad B = \frac{U'-V'}{(u+v)^2} + \frac{U''}{u+v},$$

$$C = \frac{U'-V'}{(u+v)^2} - \frac{U''}{u+v}, \quad D = \frac{U''+V''}{(u+v)^2}.$$

These equations, depending on three arbitrary functions, U, V and m, are therefore the equations of a group of surfaces which admit of continuous deformation with preservation of conjugate lines.

In the special case where $m = 0$, equations (8) reduce to

$$(10) \quad x_0 = \frac{-2\cos v}{u+v}, \quad y_0 = \frac{-2\sin v}{u+v}, \quad iz_0 = \frac{2}{u+v} - v,$$

which are the equations of a ruled surface whose generatrices are straight lines of zero length. The equations of the corresponding associated surfaces are

$$(11) \begin{cases} x = (U''+V'')\cos v - (U'-V'+(u+v)U''), \\ y = (U''+V'')\sin v + (U'-V'-(u+v)U''), \\ -iz = U''+V''+U+V-(u+v)U' + \dfrac{(u+v)^2}{2}U''. \end{cases}$$

The following relations exist between the symbols of Christoffel formed for any surface S_0 and its associate surface S and the Gauss sphere, provided S_0 is referred to its asymptotic lines and S is referred to conjugate lines : *

$$
(12)\quad
\begin{aligned}
&\left\{ {12 \atop 1} \right\}_{S_0} = -\left\{ {12 \atop 1} \right\}' = \frac{D''}{D}\left\{ {11 \atop 2} \right\}_S, \quad \left\{ {12 \atop 2} \right\}_{S_0} = -\left\{ {12 \atop 2} \right\}' = \frac{D}{D''}\left\{ {22 \atop 1} \right\}_S,\\
&\left\{ {22 \atop 1} \right\}_{S_0} = -\left\{ {22 \atop 1} \right\}' = \frac{D''}{D}\left\{ {12 \atop 2} \right\}_S, \quad \left\{ {11 \atop 2} \right\}_{S_0} = -\left\{ {11 \atop 2} \right\}' = \frac{D}{D''}\left\{ {12 \atop 1} \right\}_S.
\end{aligned}
$$

If we exclude developable surfaces, the second fundamental magnitudes D and D'' of S cannot be identically zero. Now by (1) the fundamental magnitudes of the sphere which corresponds to (10) and (11) are,

$$e = 0, \quad f = \frac{-2}{(u+v)^2}, \quad g = 1.$$

Hence in this case, $\left\{ {12 \atop 1} \right\}' = \left\{ {12 \atop 2} \right\}' = \left\{ {11 \atop 2} \right\}' = 0$. Thus taking account of (1) and (12), we have for (11), $\left\{ {11 \atop 2} \right\} = 0$ and $E \neq 0$. Hence the lines $v =$ const. on (11) are geodesics. Also $\left\{ {22 \atop 1} \right\} = 0$ and $G = 0$ and hence the lines $u =$ const. are minimal lines. Thus the surfaces (11) play a rôle between that of the minimal surfaces, on which the invariant conjugate system consists of two families of minimal lines, and the surfaces of Voss, on which the invariant conjugate system consists of two families of geodesics.

If the general solution θ of an equation of the form $\partial^2\theta/\partial u\partial v = M\theta$ is known, where M is an arbitrary function of u and v, and one can find three particular solutions, ξ, η, ζ, linearly independent of θ, the equations of a surface $S_0(x_0, y_0, z_0)$ referred to its asymptotic lines can be found by six quadratures from the formulas †

$$
(13)\quad
\left\{
\begin{aligned}
\frac{\partial x_0}{\partial u} &= \zeta\frac{\partial \eta}{\partial u} - \eta\frac{\partial \zeta}{\partial u}, & \frac{\partial x_0}{\partial v} &= \eta\frac{\partial \zeta}{\partial v} - \zeta\frac{\partial \eta}{\partial v},\\
\frac{\partial y_0}{\partial u} &= \xi\frac{\partial \zeta}{\partial u} - \zeta\frac{\partial \xi}{\partial u}, & \frac{\partial y_0}{\partial v} &= \zeta\frac{\partial \xi}{\partial v} - \xi\frac{\partial \zeta}{\partial v},\\
\frac{\partial z_0}{\partial u} &= \eta\frac{\partial \xi}{\partial u} - \xi\frac{\partial \eta}{\partial u}, & \frac{\partial z_0}{\partial v} &= \xi\frac{\partial \eta}{\partial v} - \eta\frac{\partial \xi}{\partial v},
\end{aligned}
\right.
$$

* Bianchi-Lukat, *l. c.*, pp. 127, 135.
† Bianchi Lukat, *l. c.*, p 132 ff.

where $K = -1/\rho^2$ is the curvature of S_0 and $\xi = \sqrt{\rho}X$, $\eta = \sqrt{\rho}Y$, $\zeta = \sqrt{\rho}Z$. If ξ, η, ζ are chosen such that $\xi^2 + \eta^2 + \zeta^2 = [\phi(u) + \chi(v)]^2$, the associate of such a surface will admit of continuous deformation with preservation of conjugate lines.

As an example, consider the equation $\partial^2\theta/\partial u \partial v = 0$, of which the general solution is $\theta = U + V$. As three particular solutions, assume $\xi = v$, $\eta = \psi(v)$, $\zeta = u$, where $\psi(v)$ is arbitrary. Then $\xi^2 + \eta^2 + \zeta^2 = u^2 + v^2 + \psi^2$. Formulas (13) give for S_0 the right conoid,

$$(14) \qquad x_0 = u\psi, \quad y_0 = uv, \quad z_0 = \int(\psi - v\psi')dv.$$

Then for the group of associated surfaces we have from (9),

$$(15) \qquad x = \frac{\psi'(uU' - U) + \psi V' - \psi' V}{\psi - v\psi'},$$

$$y = \frac{uU' - U + vV' - V}{\psi - v\psi'}, \quad z = U',$$

where the primes denote partial derivatives. These surfaces were first investigated by Mlodzieowski.*

For the fundamental magnitudes we find

$$E = \frac{U''^2}{(\psi - v\psi')^2}[u(1 + \psi'^2) + (\psi - v\psi')^2],$$

$$F = \frac{(v + \psi\psi')uU''}{(\psi - v\psi')^3}[V''(\psi - v\psi') + \psi''(uU' - U + vV' - V)],$$

$$G = \frac{\psi^2 + v^2}{(\psi - v\psi')^4}[V''(\psi - v\psi') + \psi''(uU' - U + vV' - V)],$$

$$D = \frac{U''}{\sqrt{u^2 + v^2 + \psi^2}}, \qquad D' = 0,$$

$$D'' = \frac{V''(\psi - v\psi') + \psi''(uU' - U + vV' - V)}{(\psi - v\psi')\sqrt{u^2 + v^2 + \psi^2}}.$$

* See Goursat. *Amer. Jour. of Math.*, vol. 14, pp. 1–8.

If in (14) we choose $\psi = 1$ we have the paraboloid, and from (15) as its associated surfaces

$$x = V', \qquad y = uU' - U + vV' - V, \qquad z = U',$$

which are the equations of surfaces of translation whose generating lines are plane and lie in perpendicular planes.

Putting $\psi = \sqrt{1 - v^2}$ and changing v to $\sin v'$, we have from (14), omitting the primes, the helicoid

$$x_0 = u \cos v, \quad y_0 = u \sin v, \quad z_0 = v$$

and, for its associated surfaces, the moulure surfaces

$$\begin{aligned} x &= -(uU' - U) - V' \cos v + V \sin v, \\ y &= uU' - U - V' \sin v - V \cos v, \\ z &= U'. \end{aligned}$$

Finally, putting $U = c\sqrt{1 + u^2}$, $V = 0$, $\psi = -1/b\sqrt{c^2 - a^2v^2}$, where a, b, c, are constants, and changing u to $\cot u'$, we have from (14), omitting the primes,

$$x_0 = -c/b \cot u \cos v, \quad y_0 = c/a \cot u \sin v, \quad z_0 = -c^2v/ab$$

and for the associated surfaces, the quadrics

$$x = a \sin u \sin v, \quad y = b \sin u \cos v, \quad z = c \cos u.$$

If S is any surface which admits of continuous deformation with preservation of a conjugate system, the different forms which S assumes during the deformation may be regarded as distinct surfaces, so that the deformation leads to a continuous system of applicable surfaces. To find the individual surfaces of such a system, let E, F, G, D, 0, D'' be the first and second fundamental magnitudes of S referred to a conjugate system of lines: then the corresponding magnitudes of any other surface of the system are E, F, G, λD, 0, D''/λ, where λ satisfies the equation *

$$(16) \qquad \frac{\partial \lambda}{\partial u} = \left\{ {12 \atop 2} \right\}'(\lambda - \lambda^3), \quad \frac{\partial \lambda}{\partial v} = \left\{ {12 \atop 1} \right\}'\left(\frac{1}{\lambda} - \lambda\right).$$

* Bianchi-Lukat, l. c., p. 336.

If the surface S_0 to which S is associated is referred to asymptotic lines, S and S_0 must have the same spherical representation and the symbols $\left\{ {12 \atop 1} \right\}'$ and $\left\{ {12 \atop 2} \right\}'$ will be the same functions of u, v whether calculated from the equations of S or from those of S_0. Using S_0, Codazzi's equations may be written in the form

$$(17) \qquad \left\{ {12 \atop 1} \right\}' = -\tfrac{1}{2}\frac{\partial}{\partial u}(\log \rho), \quad \left\{ {12 \atop 2} \right\}' = -\tfrac{1}{2}\frac{\partial}{\partial v}(\log \rho),$$

where

$$\rho = \frac{1}{\sqrt{-K_0}} = \phi(u) + \chi(v).$$

From (17) we have for (16),

$$\frac{\partial}{\partial u}\left[\log\left(\frac{\lambda}{1-\lambda^2}\right)\right] = -\frac{\partial}{\partial u}(\log \rho),$$

$$\frac{\partial}{\partial v}\left[\log(1-\lambda^2)\right] = \frac{\partial}{\partial v}(\log \rho).$$

Solving these equations for λ we find

$$\lambda = \sqrt{\frac{c+\chi(v)}{c-\phi(u)}},$$

where c is a constant. Then by Bonnet's theorem, to every value of c there corresponds a surface whose fundamental magnitudes are E, F, G, λD, 0, D''/λ and the determination of whose cartesian coordinates depends upon the integration of an equation of Riccati. In the case of the surfaces defined by (16) we have,

$$\lambda = \sqrt{\frac{c+v^2+\psi^2}{c-u^2}}.$$

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