which is not a square; one may add to any set of postulates defining a field these two : (α) -1 is a not-square. (β) The sum of two not-squares is a not-square. On defining $a < b$ to mean that $a - b$ is a not-square, the usual propositions about the symbol $<$ follow at once. If a continuity axiom is added, the system of postulates so obtained defines the real number system. This note is to be published in the *Transactions*.

22. The third paper by Professor Dickson deals with the generalization of the concept of hypercomplex number systems and of the precise definition of this generalized concept by means of independent postulates. The elements are $a = (a_1, \cdots, a_n)$, where the a_i are marks of a given field F. A system of such elements together with n^3 fixed marks γ_{ijk} form a number system if the following six postulates hold : (1) if a and b are elements of the system then $(a_1 + b_1, \cdots, a_n + b_n)$ is also an element ; (2) the element $0 = (0, \cdots, 0)$ occurs in the system ; (3) if 0 occurs, then to any element a corresponds an element a' of the system such that $a_i + a_i' = 0$ $(i = 1, \cdots, n)$; (4) if a and b are any two elements, then $(p_1, \cdots, p_n)$ is an element of the system, where $p_i = \Sigma a_j b_k \gamma_{jki}$; (5) the usual relations between the γ's assuring associativity of multiplication ; (6) if $\tau_1, \cdots, \tau_n$ are marks of F such that $\tau_1 a_1 + \cdots + \tau_n a_n = 0$ for every a, then $\tau_1 = 0, \cdots, \tau_n = 0$. It is shown that n units linearly independent with respect to F can be determined. The paper is to appear in the July number of the *Transactions*.

<div style="text-align:right">J. W. YOUNG,
Secretary pro tem. of the Section.</div>

EVANSTON, ILL.,
   May 20, 1905.

<div style="text-align:center">

A GENERAL THEOREM ON ALGEBRAIC NUMBERS.

BY PROFESSOR L. E. DICKSON.[*]

</div>

1. LET $r_1, \cdots, r_n$ belong to a field F and let

$$(1) \qquad \rho^n = \sum_{i=1}^{n} r_i \rho^{n-i}$$

be irreducible in F. A multiplication table

$$(2) \qquad \rho^{n-i}\rho^{n-k} = \sum_{s=1}^{n} \gamma_{iks}\rho^{n-s} \qquad (i,\, k = 1,\, \cdots,\, n)$$

enables us to express the product of

$$(3) \qquad x = \sum_{i=1}^{n} x_i\rho^{n-i}, \quad y = \sum_{k=1}^{n} y_k\rho^{n-k} \qquad (x_i,\, y_k \text{ in } F)$$

in the form $xy = z$,

$$(4) \qquad z = \sum_{s=1}^{n} z_s\rho^{n-s}, \quad z_s \equiv \sum_{i,\,k}^{1,\,\ldots,\,n} \gamma_{iks}x_iy_k \qquad (s = 1,\, \cdots,\, n).$$

THEOREM. *Each of the $3n$ determinants**

$$(5) \qquad C_i \equiv |\gamma_{iks}|, \quad C'_k \equiv |\gamma_{iks}|, \quad C''_s = |\gamma_{iks}|$$

is not zero. We have the evaluations

$$(6) \qquad \begin{array}{c} C_i = (-1)^{i(n-i)}r_n{}^{n-i}, \quad C'_k = (-1)^{k(n-k)}r_n{}^{n-k}, \\ C''_1 = \theta_n, \quad C''_n = \theta_{n-1}r_n^{n-1}, \end{array}$$

where θ_m is $+1$ if m is of the form $4l$ or $4l+1$, but θ_m is -1 if m is of the form $4l+2$ or $4l+3$; while for $1 < s < n$, C''_s is, aside from sign, the resultant of the equations

$$(7) \qquad \rho^{s-1} = \sum_{i=1}^{s-1} r_i\rho^{s-1-i}, \quad 0 = \sum_{i=s}^{n} r_i\rho^{n-i},$$

obtained by splitting equation (1) into two parts.

2. That $C_i \neq 0$ and $C'_k \neq 0$ follows readily from the theory of higher complex numbers. For any y and z in $F(\rho)$, $y \neq 0$, there is an unique solution x of $xy = z$; for any x and z in $F(\rho)$, $x \neq 0$, there is an unique solution y of $xy = z$. Hence of the determinants

$$\triangle \equiv \left| \sum_{i=1}^{n} \gamma_{iks}x_i \right|_{k,\,s=1,\,\ldots,\,n}, \quad \triangle' \equiv \left| \sum_{k=1}^{n} \gamma_{iks}y_k \right|_{i,\,s=1,\,\ldots,\,n},$$

$\triangle$ vanishes only if $x_1 = \cdots = x_n = 0$, $\triangle'$ vanishes only if $y_1 = \cdots = y_n = 0$. But C_i is the coefficient of x_i^n in $\triangle$, C'_k is the coefficient of y_k^n in $\triangle'$.

3. From (1) we obtain

$$(8) \qquad \rho^{n+j} = \sum_{s=1}^{n} b_{sj}\rho^{n-s}, \quad b_{sj} \equiv \sum_{i=0}^{j} r_{s+j-i}f_i,$$

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* Sections of the cubic array $|\gamma_{iks}|$ by planes parallel to the faces.

where $r_t \, (t > n)$ is to be replaced by zero, and where

$$f_0 = 1, \quad f_1 = r_1, \quad f_2 = r_2 + r_1^2, \quad f_3 = r_3 + 2r_2r_1 + r_1^3,$$

$$f_4 = r_4 + 2r_3r_1 + r_2^2 + 3r_2r_1^2 + r_1^4,$$

$$f_5 = r_5 + 2r_4r_1 + 2r_3r_2 + 3r_3r_1^2 + 3r_2^2r_1 + 4r_2r_1^3 + r_1^5,$$

f_i being the sum of all terms of weight i in the r's with the number of arrangements of the r's as coefficients. Then

$$(9) \qquad f_{j+1} = \sum_{i=0}^{j} r_{1+j-i} \, f_i, \quad f_{j+1} = b_{1j}.$$

A comparison of $(8)_1$ with (2) gives

$$(10) \quad \gamma_{iks} = b_{s\,n-i-k} \quad (k \leqq n - i), \quad \gamma_{iks} = \delta_{k\,n-i+s} \quad (k > n - i),$$

where

$$\delta_{kk} = 1, \; \delta_{kt} = 0 \text{ if } t \neq k.$$

4. Applying $(10)_1$ for the first $n - i$ rows and $(10)_2$ for the last i rows, we obtain for $C_i = |\,\gamma_{iks}\,|$ the expression

$$
\begin{vmatrix}
b_{1\,n-i-1} & b_{2\,n-i-1} & \cdots & b_{n\,n-i-1} \\
\cdot & \cdot & \cdot & \cdot \\
b_{11} & b_{21} & \cdots & b_{n1} \\
b_{10} & b_{20} & \cdots & b_{n0} \\
1 & 0 & \cdots & 0 \\
0 & 1 & \cdots & 0 \\
\cdot & \cdot & & \cdot
\end{vmatrix}
= \sigma_i
\begin{vmatrix}
b_{i+1\,n-i-1} & \cdots & b_{n\,n-i-1} \\
\cdot & \cdot & \cdot \\
b_{i+1\,1} & \cdots & b_{n\,1} \\
b_{i+1\,0} & \cdots & b_{n\,0}
\end{vmatrix},
$$

where $\sigma_i = (-1)^{i(n-i)}$. Now by $(8)_2$, $b_{s0} = r_s$, $b_{s1} - f_1 r_s = r_{s+1}$, $b_{s2} - f_1 r_{s+1} - f_2 r_s = r_{s+2}$, $\cdots$ Thus,

$$
C_i = \sigma_i
\begin{vmatrix}
r_n & 0 & 0 & \cdots & 0 & 0 \\
r_{n-1} & r_n & 0 & \cdots & 0 & 0 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
r_{i+2} & r_{i+3} & r_{i+4} & \cdots & r_n & 0 \\
r_{i+1} & r_{i+2} & r_{i+3} & \cdots & r_{n-1} & r_n
\end{vmatrix}
= \sigma_i r_n^{\,n-i}.
$$

Since $\gamma_{iks} = \gamma_{kis}$, we have $C'_k = |\,\gamma_{kis}\,|_{s=1,\,\ldots,\,n} = C_k = \sigma_k r_n^{\,n-k}$.

5. Similarly, C''_s equals

$$\begin{vmatrix}
b_{sn-2} & b_{sn-3} & b_{sn-4} & \cdot & \cdot & \cdot & \cdot & b_{s1} & b_{s0} & 0 \\
b_{sn-3} & b_{sn-4} & \cdot & \cdot & \cdot & \cdot & b_{s1} & b_{s0} & 0 & 0 \\
b_{sn-4} & \cdot & \cdot & \cdot & \cdot & b_{s1} & b_{s0} & 0 & 0 & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & b_{s1} & b_{s0} & 0 & 0 & \cdot & 0 & 1 \\
\cdot & \cdot & b_{s1} & b_{s0} & 0 & 0 & \cdot & 0 & 1 & 0 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
b_{s1} & b_{s0} & 0 & 0 & \cdot & 0 & 1 & 0 & \cdot & \cdot \\
b_{s0} & 0 & 0 & \cdot & 0 & 1 & 0 & \cdot & \cdot & 0 \\
0 & 0 & \cdot & 0 & 1 & 0 & \cdot & \cdot & 0 & 0
\end{vmatrix}$$

All the elements in a parallel to the left-hand diagonal are equal. The parallel with the elements 1 joins the sth element of the last row with the sth element of the last column.

In C_1'' the elements of the left-hand diagonal are all 1, and every element below this diagonal is zero. Hence $C_1'' = \theta_n = \pm 1$. In C_n'' every element in and below the left-hand diagonal is zero except the element 1 at the intersection of the last row and last column. Hence $C_n'' = \theta_{n-1} b_{n0}^{n-1}$.

For $1 < s < n, C_s'' = (-1)^{n-s} D_{sn}$, where D_{sn} is the minor of the element 1 in the last column. We proceed to the proof that D_{sn} is the eliminant by Sylvester's dialytic method of equations (7). The nature of the general proof will be clear from the following two illustrations:

For $n = 5, s = 2, D_{sn}$ is

$$\begin{vmatrix}
b_{23} & b_{22} & b_{21} & b_{20} \\
b_{21} & b_{20} & 0 & 1 \\
b_{20} & 0 & 1 & 0 \\
0 & 1 & 0 & 0
\end{vmatrix}, \quad
\begin{aligned}
& b_{20} = r_2, \\
& b_{21} = r_3 + r_2 f_1, f_1 = r_1, \\
& b_{22} = r_4 + r_3 f_1 + r_2 f_2, \\
& b_{23} = r_5 + r_4 f_1 + r_3 f_2 + r_2 f_3.
\end{aligned}$$

Multiply the fourth column by $-f_1, -f_2, -f_3$ and add to the third, second, first columns, respectively. In the resulting determinant, multiply the third column by $-f_1$ and $-f_2$, and add to the second and first columns, respectively. The first and second elements of the second row are now (§ 3)

$$r_3 + r_2 r_1 - f_3 + f_2 r_1 = 0, \; r_2 - f_2 + r_1^2 = 0.$$

Multiply the new second column by $-f_1$ and add to the first. There results

$$D_{25} = \begin{vmatrix} r_5 & r_4 & r_3 & r_2 \\ 0 & 0 & -r_1 & 1 \\ 0 & -r_1 & 1 & 0 \\ -r_1 & 1 & 0 & 0 \end{vmatrix},$$

the eliminant of $r_5 + r_4 \rho + r_3 \rho^2 + r_2 \rho^3 = 0, \; -r_1 + \rho = 0.$

For $n = 4$, $s = 3$, D_{sn} is

$$\begin{vmatrix} b_{32} & b_{31} & b_{30} \\ b_{31} & b_{30} & 0 \\ 0 & 0 & 1 \end{vmatrix}, \quad \begin{aligned} b_{30} &= r_3, \\ b_{31} &= r_4 + r_3 f_1, \\ b_{32} &= r_4 f_1 + r_3 f_2, \end{aligned}$$

the term r_5 in b_{32} being dropped since $5 > n$ (§ 3). Multiply the third column by $-f_1$ and $-f_2$ and add to the second and first columns, respectively. Multiply the new second column by $-f_1$ and add to the first. There results

$$D_{34} = \begin{vmatrix} 0 & r_4 & r_3 \\ r_4 & r_3 & 0 \\ -r_2 & -r_1 & 1 \end{vmatrix},$$

the eliminant of $r_4 + r_3 \rho = 0, \; -r_2 - r_1 \rho + \rho^2 = 0.$

CHICAGO, *December 8, 1904.*

ON THE DEFORMATION OF SURFACES OF TRANSLATION.

BY DR. L. P. EISENHART.

IN the January number of the BULLETIN * Dr. Burke Smith states the following theorem : The only non-developable surfaces of translation which may be deformed so that their gener-

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* "On the deformation of surfaces of translation," p. 187.