the corresponding deformation of S leaves the lines of curvature unaltered and only in this case.

PRINCETON,
June, 1902.

ON INTEGRABILITY BY QUADRATURES.

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(Read before the American Mathematical Society, September 3, 1902.)

THE object of this note is to show that Vessiot's noted theorem that: "the necessary and sufficient condition that a linear differential equation shall be integrable by quadratures is that its group of rationality shall be integrable," * is a special case of the Jordan-Beke † theorem on reducibility of differential equations.

The Jordan-Beke theorem is to the effect that "if a linear differential equation is reducible in the sense of Frobenius ‡ then its group of rationality will transform a certain linear manifoldness of the solutions (which does not include the total n-dimensional manifoldness) into itself."

Analytically interpreted § this says that the group

$$
(1)\quad
\begin{aligned}
y_1 &= a_{11}y_1 + \cdots + a_{1k}y_k, \\
&\cdots\cdots\cdots \\
y_k &= a_{k1}y_1 + \cdots + a_{kk}y_k, \\
y_{k+1} &= a_{k+1,1}y_1 + \cdots + a_{k+1,k}y_k + a_{k+1,\,k+1}y_{k+1} + \cdots + a_{k+1,n}y_n, \\
&\cdots\cdots\cdots \\
y_n &= a_{n1}y_1 + \cdots + a_{nk}y_k + a_{n,\,k+1}y_{k+1} + \cdots + a_{nn}y_n,
\end{aligned}
$$

is isomorphic with the group of rationality. For convenience it is well to adopt Loewy's notation, writing for (1) simply the coefficients

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* Vessiot: *Ann. de l'Ec. nor. sup.*, 1892.

† C. Jordan. *Bull. de la Soc. Math. de France*, vol. 2; Beke: *Math. Annalen*, vol. 45, p. 279.

‡ Frobenius: *Crelle*, vol. 76.

§ A. Loewy: "Ueber die irreduciblen Factoren," etc., *Berichte der math.-phy. Classe der Königl. Sächs. Gesellschaft der Wissenschaften zu Leipzig*, vol. 54 (1902), pp. 1–13.

$$(2)\qquad \left.\begin{array}{lll|llll} a_{11} & \cdots & a_{1k} & 0 & & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{k1} & \cdots & a_{kk} & 0 & & \cdots & 0 \\ \hline a_{k+11} & \cdots & a_{k+1,\,k} & a_{k+1,\,k+1} & & \cdots & a_{k+1,\,n} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{n1} & \cdots & a_{nk} & a_{n,\,k+1} & & \cdots & a_{nn}. \end{array}\right.$$

When a linear differential equation

$$(3)\qquad \frac{d^n y}{dx^n} + p_1 \frac{d^{n-1} y}{dx^{n-1}} + \cdots + p_n y = 0$$

is given, we fix a domain of rationality and now if equation (3) reduces within this domain to

$$(4)\qquad \frac{d^k y}{dx^k} + q_1 \frac{d^{k-1} y}{dx^{k-1}} + \cdots + q_k y = 0 \qquad (k < n)$$

the group (2) will be isomorphic with the group of rationality of the equation (3).

If $k = n - 1$, then by effecting a quadrature equation (3) will reduce to

$$(5)\qquad \frac{d^{n-1} y}{dx^{n-1}} + r_1 \frac{d^{n-2} y}{dx^{n-2}} + \cdots + r_{n-1} y = 0$$

and the group (2) will become

$$(6)\qquad \begin{array}{llll} a_{11} & \cdots & a_{1,\,n-1} & 0 \\ \cdot & \cdot & \cdot & \cdot \\ a_{n-1,\,1} & \cdots & a_{n-1,\,n-1} & 0 \\ a_{n,\,1} & \cdots & a_{n,\,n-1} & a_{nn}. \end{array}$$

If now (5) reduces within the same domain of rationality to

$$(7)\qquad \frac{d^{n-2} y}{dx^{n-2}} + s_1 \frac{d^{n-3} y}{dx^{n-3}} + \cdots + s_{n-2} y = 0$$

the group (6) becomes

$$
(8) \qquad \begin{array}{cccc}
a_{11} & \cdots a_{1,\,n-2} & 0 & 0 \\
\cdot & \cdot \quad \cdot \quad \cdot & \cdot & \cdot \\
a_{n-2,\,1} & \cdots a_{n-2,\,n-2} & 0 & 0 \\
a_{n-1,\,1} & \cdots a_{n-1,\,n-2} & a_{n-1,\,n-1} & 0 \\
a_{n,\,1} & \cdots a_{n,\,n-2} & a_{n,\,n-1} & a_{nn}.
\end{array}
$$

When equation (3) is integrable by quadratures this process can be continued n times; the group we are considering (which is isomorphic with the group of rationality) takes the form

$$
(9) \qquad \begin{array}{lllll}
a_{11} & & & & \\
a_{21} & a_{22} & & & \\
a_{31} & a_{32} & a_{33} & & \\
\cdot & \cdot & \cdot & \cdot & \cdot \\
a_{n-1,\,1} & a_{n-1,\,2} & \cdots & a_{n-1,\,n-1} & \\
a_{n,\,1} & a_{n,\,2} & \cdots & a_{n,\,n-1} & a_{n,\,n}.
\end{array}
$$

But (Lie-Engel, Transformationsgruppen I, chapter 27) the group (9) is an integrable group.

The Jordan-Beke condition for reducibility, employed above, is both necessary and sufficient. We have thus deduced as a special case of the Jordan-Beke theorem the theorem of Vessiot, namely, "*the necessary and sufficient condition that a linear homogeneous differential equation shall be integrable by quadratures is that its group of rationality be integrable.*"

PHILADELPHIA, PA.
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THE CENTENARY OF THE BIRTH OF ABEL.

AT the close of the first week in September last, the Royal Frederick University at Christiania, Norway, celebrated the one hundredth anniversary of the birth of Niels Henrik Abel. The occasion was noteworthy as the first international academic celebration in Norway and was in every way instructive and enjoyable for the delegates and guests of the university. On their arrival at the station they were met by representatives of the university who had been previously instructed as to which language each of them spoke and who conducted them to the rooms to which they had been assigned in the various hotels,