resulting parabola is parallel to the two parallel asymptotes.

An extension to space of three dimensions is easy; thus the analogue to the first theorem gives

The three paraboloids contained in the family $S + \lambda S' = 0$ are all real, if either of the quadrics S, S' is an ellipsoid. So too, we find:

If the two quadrics S, S' are hyperboloids, two of the paraboloids will be imaginary if (and only if) two cones with a common vertex, parallel to their asymptotic cones, intersect in *two* real generators.

There are five possibilities when two (or three) of the paraboloids coincide; without enumerating them all, it may be noted that when S or S' is an ellipsoid, the coincidence implies degeneration of the paraboloids. All the other cases may be obtained by suitable interpretations of Weierstrass's algebra ("Zur Theorie der bilinearen und quadratischen Formen," *Monatsberichte d. k. Akad. z. Berlin*, 1868; Werke, volume 2, page 19).

Slightly digressing from the line of thought just indicated, and reverting to Huntington and Whittemore's paper, I note that their result, that the eccentricity is wholly indeterminate (*l. c.*, page 123), *suffices* to specify the conics considered by them. For, in orthogonal cartesians, the eccentricity is determined by the ratio $(a+b)^2/(ab-h^2)$, which is only indeterminate if

$$ab - h^2 = 0, \quad a + b = 0,$$

i. e., if

$$b = -a, \quad h = \pm ia,$$

and then the conic reduces to

$$a(x \pm iy)^2 + \text{linear terms} = 0.$$

QUEEN'S COLLEGE, GALWAY,
February 22, 1902.

A SECOND DEFINITION OF A GROUP.

BY DR. E. V. HUNTINGTON.

(Read before the American Mathematical Society, April 26, 1902.)

THE following note contains a definition of a group expressed in four independent postulates, suggested by the definition given in W. Burnside's Theory of Groups of Finite Order (1897). The definition presented by the writer at the February meeting contained three independent postulates,* and the definition just proposed by Professor Moore † contains five independent postulates. The comparison of these three definitions is therefore very striking.

Definition.

We consider here an assemblage or set of elements in which a rule of combination* denoted by $\circ$ is so defined as to satisfy the following four postulates :

1. *If a and b belong to the assemblage, then $a \circ b$ also belongs to the assemblage.*

2. $(a \circ b) \circ c = a \circ (b \circ c)$, *whenever* $a \circ b$, $b \circ c$, $(a \circ b) \circ c$ *and* $a \circ (b \circ c)$ *belong to the assemblage.*

3. *For every two elements a and b there is an element a' such that* $(a \circ a') \circ b = b$.

4. *For every two elements a and b there is an element a'' such that* $b \circ (a'' \circ a) = b$.

From 1, 2, 3 it follows that for any two elements a and b there is an element x such that $a \circ x = b$. For by 3 take a' so that $(a \circ a') \circ b = b$ and by 1 take $x = a' \circ b$; then by 2 $a \circ x = b$.

Similarly, from 1, 2, 4 follows the existence, for every two elements a and b, of an element y such that $y \circ a = b$.

Therefore every assemblage which satisfies the postulates 1, 2, 3, 4 is a group, according to the writer's previous definition.

If we wish to distinguish between finite and infinite groups we may add a fifth postulate, either :

5a. *The assemblage contains n elements, where n is a positiv integer ;*‡ or

5b. *The assemblage contains an infinitude of elements.*

Independence of Postulates 1, 2, 3, 4 and 5a, when $n > 2$.

The mutual independence of postulates 1, 2, 3, 4, 5a for finite groups may be established, when $n > 2$, by use of the following systems :

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* See BULLETIN, pp. 296–300.

† An abstract of Professor Moore's paper is given on p. 373 of the present number of the BULLETIN.

‡ The number of elements in a finite group is called the *degree* of the group by H. Weber, Algebra, Vol. II (1899), p. 4, or the *order* of the group by most other writers. Cf. W. Burnside, *loc. cit.*, p. 380.

M_1. If n is odd, $n = 2k + 1$, let M_1 be the system of al integers from $-k$ to $+k$, while $a \circ b = a + b$.

If n is even, $n = 2k + 2$, let M_1 be the system of all integers from $-k$ to $+k$ with an additional element z, while the rule of combination is defined as follows: When $a \neq z$ and $b \neq z$, $a \circ b = a + b$; further, $z \circ 0 = 0 \circ z = z$, and $z \circ z = 0$; but when $a \neq 0$, $a \circ z = z \circ a = k + 1$ which does not belong to the assemblage.

M_2. Let M_2 be the system of positive integers from 1 to n, with the rule of combination defined as follows:

$$\begin{aligned} a \circ b &= a + b && \text{when } a + b \leqq n, \\ &= a + b - n && \text{when } a + b > n, \end{aligned}$$

except that $a \circ b = 2$ when $a + b = 1$ or $n + 1$,
and $a \circ b = 1$ when $a + b = 2$ or $n + 2$.

M_3. The system of positive integers from 1 to n, with $a \circ b = a$.

M_4. The system of positive integers from 1 to n, with $a \circ b = b$.

M_5. Any infinite group.

Since the system M_k is found to satisfy all the other postulates but not the kth ($k = 1, 2, 3, 4, 5$) we see that no one of these five postulates is a consequence of the remaining four.

Independence of Postulates 1, 2, 3, 4 and 5b.

Similarly, the mutual independence of postulates 1, 2, 3, 4, 5b for infinite groups may be established by the use of the following systems:

N_1. The system of all integers except $+1$ and -1, with $a \circ b = a + b$.

N_2. The system of all rational numbers, with $a \circ b = (a + b)/2$.

N_3. The system of all positive integers, with $a \circ b = a$.

N_4. The system of all positive integers, with $a \circ b = b$.

N_5. Any finite group.

Thus no one of these five postulates is a consequence of the remaining four.

Weber's Definition of a Finite Group.

In conclusion we may notice that if, in the definition of a *finite* group, we replace postulates 3 and 4 by the following:

3′. *If* $a \circ b = a \circ b'$ *then* $b = b'$;

4′. *If* $a \circ b = a' \circ b$ *then* $a = a'$;

we shall have the definition given by H. Weber, *loc. cit.* That these postulates 1, 2, 3′, 4′, 5*a* are mutually independent (when $n > 2$) has already been shown in the writer's previous paper (page 300).

It should be noticed, however, that postulates 1, 2, 3′, 4′, 5*b* would not be sufficient to define an *infinite* group, since the system of positive integers, with $a \circ b = a + b$, satisfies them all, and is not a group.

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DETERMINATION OF ALL THE GROUPS OF ORDER p^m, p BEING ANY PRIME, WHICH CONTAIN THE ABELIAN GROUP OF ORDER p^{m-1} AND OF TYPE (1, 1, 1, ⋯).

BY PROFESSOR G. A. MILLER.

(Read before the San Francisco Section of the American Mathematical Society, May 3, 1902.)

LET $t_1, t_2, \cdots, t_{m-1}$ represent a set of independent generators of the abelian group H of type (1, 1, 1, ⋯). It is well known that the order of the group of isomorphisms ϑ of H is $p^{\frac{(m-1)(m-2)}{2}}(p-1)(p^2-1)\cdots(p^{m-1}-1)$. One of its subgroups ϑ_1 of order $p^{\frac{(m-1)(m-2)}{2}}$ is composed of all the operators of ϑ which correspond to the holomorphisms of H in which t_α $(\alpha = 2, 3, \cdots, m-1)$ corresponds to itself multiplied by some operator in the group generated by $t_1, t_2, \cdots, t_{\alpha-1}$. The number of conjugates of ϑ_1 under ϑ is clearly equal to the order of ϑ divided by $p^{\frac{(m-1)(m-2)}{2}}(p-1)^{m-1}$. We shall first determine the number of sets of subgroups of ϑ_1 which are conjugate under ϑ. It may be observed that even characteristic subgroups of ϑ_1 may be conjugate under ϑ. For instance, the octic group has a characteristic subgroup of order two and four other subgroups of this order, yet all of these subgroups are conjugate under ϑ when the latter is the simple group of order 168.

All the holomorphisms of H may be obtained by establishing isomorphisms between H and its subgroups and letting the product of two corresponding operators in these isomorphisms correspond to the original operator of H.*

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* BULLETIN, vol. 6 (1900), p. 337.