CONCERNING SURFACES WHOSE FIRST AND SECOND FUNDAMENTAL FORMS ARE THE SECOND AND FIRST FUNDAMENTAL FORMS RESPECTIVELY OF ANOTHER SURFACE.

BY PROFESSOR ALEXANDER PELL.

(Read before the American Mathematical Society, August 20, 1901.)

IN the July number of the BULLETIN Dr. Eisenhart treated these surfaces and reached the following conclusions: "The ruled surfaces defined by the equations

$$y + \mu x = \sqrt{1 + \mu^2} + C_1\mu + C_1$$
$$z - ix\sqrt{1 + \mu^2} = (\mu + C_1\sqrt{1 + \mu^2} + C_3) \cdot \frac{1}{i}$$

are the only surfaces whose first and second fundamental forms can be taken for the second and first fundamental forms of a surface. Further the second surface is only the first to a translation près. And of these surfaces the only real one is the sphere of radius unity."

In fact the above equations represent the spheres of radius unity and nothing else. For we have

$$y + zi + (\mu + \sqrt{1 + \mu^2})\, x = (c_1 + 1)(\mu + \sqrt{1 + \mu^2}) + c_2 + c_3$$
$$y - zi + (\mu - \sqrt{1 + \mu^2})\, x = (c_1 - 1)(\mu - \sqrt{1 + \mu^2}) + c_2 - c_3$$

or

$$(y-c_2) + (zi - c_3) = -(x - c_1)(\mu + \sqrt{1+\mu^2}) + (\mu + \sqrt{1+\mu^2})$$
$$(y-c_2) - (zi - c_3) = -(x - c_1)(\mu - \sqrt{1+\mu^2}) - (\mu - \sqrt{1+\mu^2}),$$

i. e.,

$$(y - c_2)^2 + (z + c_3 i)^2 + (x - c_1)^2 = 1.$$

But there are two more surfaces of the kind, both imaginary, both spheres of the radii ω, ω^2, respectively, where $\omega = \sqrt[3]{1}$.

For let

$$x = R \sin u \cos v, \quad y = R \sin u \sin v, \quad z = R \cos u.$$

Then

$$E = R^2, \quad G = R^2 \sin^2 u, \quad F = 0,$$

$$D = R, \quad D' = 0, \quad D'' = R \sin^2 u.$$

If now (1) $R = \omega$,

$$E_\omega = \omega^2, \; G_\omega = \omega^2 \sin^2 u, \; D_\omega = \omega, \; D_\omega'' = \omega \sin^2 u \,;$$

and if (2) $R = \omega^2$,

$$E_{\omega 2} = \omega, \; G_{\omega 2} = \omega \sin^2 u, \; D_{\omega 2} = \omega^2, \; D_{\omega 2}'' = \omega^2 \sin^2 u.$$

No other surface of the kind exists. For, as is shown in the article referred to, for these surfaces (ρ_1 and ρ_2 being two principal radii of curvature)

$$\rho_1 = \rho_2 = \frac{E}{D} = \frac{G}{D''},$$

i. e.,

$$\frac{D}{E} = \frac{D''}{G} = \lambda.$$

Bianchi has shown (Differentialgeometrie, German translation, p. 93, footnote), that λ is constant, if D, D', D'', E, F, G are proportional.

Using the equations (1) and (9) of Dr. Eisenhart's article, we have

$$\lambda^2 \cdot \sqrt{EG} + \frac{\partial}{\partial u}\left[\frac{1}{\sqrt{E}} \cdot \frac{\frac{\partial G}{\partial u}}{2\sqrt{G}}\right] + \frac{\partial}{\partial v}\left[\frac{1}{\sqrt{G}} \cdot \frac{\frac{\partial E}{\partial v}}{2\sqrt{E}}\right] = 0,$$

$$\frac{\sqrt{EG}}{\lambda} + \frac{\partial}{\partial u}\left[\frac{1}{\sqrt{E}} \cdot \frac{\frac{\partial G}{\partial u}}{2\sqrt{G}}\right] + \frac{\partial}{\partial v}\left[\frac{1}{\sqrt{G}} \cdot \frac{\frac{\partial E}{\partial v}}{2\sqrt{E}}\right] = 0.$$

Hence, on subtraction,

$$\lambda^2 \sqrt{EG} = \frac{\sqrt{EG}}{\lambda}, \qquad \lambda^3 = 1,$$

i. e.,

$$\lambda = 1, \; \omega, \; \omega^2.$$

UNIVERSITY OF SOUTH DAKOTA.