will be absolutely convergent for a smaller region than u_0. The function u tends to a continuous limit on the boundary of a circle in both cases of convergence and the function u with its boundary function is uniformly defined by the boundary function of u_0. By conformal representation the theorem of convergence can be extended to a simply connected surface with any boundary whatever. Assuming that the boundary is composed of analytic curves with only a finite number of singular points, it is proved that u tends to a continuous limit on the boundary except at singular points where logarithmic discontinuities occur. Finally the boundary functions of u and u_0 correspond to each other uniformly.

In the *Proceedings of the London Mathematical Society* (Vol. 27, p. 385), Professor E. W. Brown gives a simple and interesting solution of Delaunay's canonical system of equations by means of Hamilton's principal function. In Professor Stone's solution Hamilton's function is avoided.

F. N. COLE.

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THE DECEMBER MEETING OF THE CHICAGO SECTION.

THE fourth regular meeting of the Chicago Section of the AMERICAN MATHEMATICAL SOCIETY was held at the University of Chicago on Thursday and Friday, December 29 and 30, 1898. The attendance and number of papers contributed indicate the interest, in the work of the Society and in the development of mathematical science, of those members who are accustomed to attend the meetings of this Section, and would seem to clearly justify its formation.

The total attendance numbered twenty-six including the following members of the Society:

Professor Henry Benner, Professor E. W. Davis, Dr. Harris Hancock, Professor T. F. Holgate, Mr. H. G. Keppel, Professor Malcolm McNeill, Professor H. Maschke, Professor E. H. Moore, Professor H. B. Newson, Professor James Pierpont, Professor J. B. Shaw, Dr. H. F. Stecker, Professor A. L. P. Wernicke, Professor H. S. White, Mr. Wm. H. Williams, Professor Mary F. Winston, Professor J. W. A. Young.

The first session opened at 10 o'clock on Thursday with Professor E. H. Moore, Vice-President of the Society, in the chair. Aside from the reading and discussion of papers the Section gave attention to a few items of necessary business. Professor T. F. Holgate was re-elected Secretary for the ensuing year and Professors J. B. Shaw and E. W. Davis were chosen as additional members of the Programme Committee. The next meeting of the Section will be held on Saturday, April 1, 1899, at Northwestern University, Evanston.

The following papers were read:

(1) Professor J. W. A. YOUNG: "Report on the teaching of mathematics in the higher schools of Prussia."

(2) Professor H. B. NEWSON: "On the Riemann-Helmholtz problem."

(3) Professor H. S. WHITE: "Preliminary report on certain new relations among the fundamental covariants of a ternary cubic."

(4) Professor E. H. MOORE: "The decomposition of modular systems connected with the doubly generalized Fermat theorem. (First communication.)"

(5) Professor E. H. MOORE: "Concerning Klein's group of $(n+1)!$ n-ary collineations."

(6) Professor J. B. SHAW: "Some quaternion integrals and their related classes of functions."

(7) Professor H. B. NEWSON: "Normal forms of projective transformations (second communication)."

(8) Professor H. MASCHKE: "Some general theorems concerning linear substitution groups of finite order."

(9) Dr. H. F. STECKER: "Non-Euclidean images of plane cubics on rotation surfaces of constant negative curvature."

(10) Professor H. B. NEWSON: "What constitutes a one parameter group?"

(11) Dr. L. E. DICKSON: "The determination of the structure of all linear homogeneous groups in a Galois field which are defined by a quadratic invariant, with the announcement of two new systems of simple groups."

(12) Mr. C. C. ENGBERG: "The Cartesian oval and the auxiliary parabola."

(13) Professor T. P. HALL: "An algebra of space and its relation to quaternions (preliminary communication)."

In the absence of Dr. Dickson, Mr. Engberg and Professor Hall their papers were presented and read by title.

In Professor Newson's paper (No. 2) the attempt was made to find the properties common to the system of

Euclidean and the two systems of non-Euclidean motions and those by which these systems of motions are distinguished from all other possible systems. The motion of a rigid body in threefold space is a point to point transformation ; it is furthermore a projective transformation, since lines are transformed into lines and planes into planes ; it is also a conformal transformation, since all angles are transformed into equal angles. The transformations which are common to G_{15}, the general projective group, and G_{10}, the conformal group, in threefold space are either motions or transformations of similarity. These groups have in common three subgroups, viz. : G_7 the group of similarity, $G_6(H)$ and $G_6(E)$, where H is the absolute of hyperbolic space and E that of elliptic space. The invariant figure of G_7 is the absolute of parabolic space. The common and characteristic properties of these three groups can be found without difficulty.

The following is a brief outline of the preliminary report (No. 3) by Professor White : Gordan's reduced form system of the ternary cubic contains 34 forms, which can be arranged by order and class into 16 groups ; two are invariants, four covariants (in the restricted sense), four contravariants, one the identical covariant, and twenty-three mixed concomitants. It is proposed as desirable to tabulate the reduced form system of each of these considered as a new stem form, then to set up the relatively small simultaneous systems of these 34 forms taken in pairs, etc. As a contribution to this end, in addition to results published by Clebsch and Gordan in Volume 6 of the *Mathematische Annalen*, certain covariants are calculated and reduced which arise from the seven forms which are of equal order and class. The chief interest in these particular forms lies in their aspect as covariant operators with recurrent laws of low degree.

Paper No. 4 stated a converse of the theorem given by Professor Moore in the BULLETIN for April, 1896.

Paper No. 5, also by Professor Moore, effects the determination, for various geometrical interpretations, of the fundamental region in real space of the group of substitutions

$$y_i = y'_{a_i} \qquad (i = 0, \cdots, n)$$

where the indices $a_0, \cdots, a_n$ are in some order $0, \cdots, n$.

In Professor Newson's second communication on projective transformations (No. 7) all the different types of projective transformations in one, two, and threefold space are reduced to their normal forms in terms of their essential parameters, *i. e.*, the coördinates of their invariant elements and their characteristic cross ratios.

Professor Maschke gave in his paper (No. 8) : First, a short report of the theorem : "If in a finite linear substitution group G at least one coefficient occupying one and the same place (but not in the principal diagonal) in the matrices of the substitutions of G, vanishes in every substitution, then G is intransitive." This part of the paper has been offered to the *Mathematische Annalen* for publication. Second, a report of the theorem : "If at least one substitution of a finite linear substitution group possesses one root which is different from its other roots, then the group can be so transformed that every coefficient of every substitution of G is cyclotomic." This theorem is an extension of a similar theorem which was given in a paper read before the Society at the Toronto meeting 1897 (*Mathematische Annalen*, vol. 50, p. 452). Third, a geometrical proof of the intransitivity of every finite quaternary substitution group, each substitution of which possesses two double roots.

In Dr. Stecker's paper (No. 9) certain geometrical material was presented which the author hopes to make use of in his further endeavors to derive geometrically the non-Euclidean properties of a plane cubic. This material related to the hyperbolic case, similar material already existing for the elliptic case. By a non-Euclidean image of a plane cubic is understood a curve or part of a curve whose parabolic properties, properly interpreted, apply to the elliptic or hyperbolic plane. The equations and forms of portions of certain sextic curves (corresponding to the five types of ordinary plane cubics) were derived by stereographic projection of the parabolic plane upon the sphere, followed by an orthographic projection of the lower half sphere. These parts of sextic curves were then built upon the three types of rotation surfaces of constant negative curvature, giving the material sought.

Dr. Dickson's paper (No. 11) is intended for publication in the *American Journal of Mathematics*. The following abstract will serve to characterize it :

Every quadric form in m variables with coefficients belonging to the Galois field of order $p^n (p > 2)$ and having a determinant not zero in the field, can be reduced by a linear homogeneous m-ary substitution in the field, to one of the two canonical forms

$$\sum_{i=1}^{m} \xi_i^2, \quad \sum_{i=1}^{m-1} \xi_i^2 + \nu \xi_m^2,$$

where ν is a not-square in the field. The second form is reducible to the first if m be even. The first form defines the orthogonal group, the structure of which was announced by the writer in the BULLETIN for May, 1898. For the case $m = 4$ the quotient groups of composite order are isomorphic to the simple group of linear fractional substitutions of determinant unity in the Galois field of order p^n. The second form defines a new group leading to a system of simple groups of order

$$\tfrac{1}{2}[p^{nk} + (\pm 1)^k] p^{n(k-1)} (p^{n(2k-2)} - 1) p^{n(2k-3)} \cdots (p^{2n} - 1) p^n,$$

where $m = 2k > 2$ and the sign $\pm$ depends upon the form $4l \pm 1$ of p^n. For $m = 2k = 4$, this simple group of order $\frac{1}{2}(p^{4n} - 1)p^{2n}$ and belonging to the $GF(p^n)$ is isomorphic to the simple group of linear fractional substitutions of determinant unity in the $GF(p^{2n})$.

Every m-ary quadratic form in the Galois field of order 2^n, not expressible in terms of fewer than m variables in the field, can be reduced by a linear homogeneous m-ary substitution in the field, to one of the three canonical forms

$$\xi_0^2 + \sum_{i=1}^{M} \xi_i \eta_i, \quad \sum_{i=1}^{M} \xi_i \eta_i, \quad \sum_{i=1}^{M} \xi_i \eta_i + \lambda \xi_1^2 + \lambda \eta_1^2,$$

there being one type if m is odd and two types if m is even. The group defined by the first form is simply isomorphic to the Abelian group of $2M$ indices in the $GF(2^n)$ and is *simple* if $M > 1$, the case $M = 2$, $n = 1$ being an exception. The group defined by the second and third forms are respectively the generalized first and second hypoabelian groups. In the third form λ is any mark such that $\xi_1 \eta_1 + \lambda \xi_1^2 + \lambda \eta_1^2$ is irreducible in the field. The generalization of Jordan's second hypoabelian group is here first announced. The simple groups obtained by its decomposition are of order

$$(2^{nm} + 1)[(2^{2n(m-1)} - 1) 2^{2n(m-1)}] \cdots [(2^{2n} - 1) 2^{2n}],$$

m being any even integer and n any integer whatever.

Mr. Engberg's paper (No. 12) was a minute and detailed discussion of the Cartesian oval and its accompanying parabola.

Professor Hall's paper (No. 13) was a continuation of the subject treated in his paper read before the Society at the Boston summer meeting.

THOMAS F. HOLGATE,
Secretary of the Section.

EVANSTON, ILLINOIS.

REPORT ON RECENT PROGRESS IN THE THEORY OF THE GROUPS OF A FINITE ORDER.*

BY DR. G. A. MILLER.

(Read before Section A of the American Association for the Advancement of Science, Boston, August 25, 1898.)

DURING the last decade the general theory of groups has been made much more accessible by means of the publication of a number of treatises. Among these the six volumes by Lie, assisted by Engel and Scheffers (1888–1896), stand out preëminently. The other important treatises that were published in this period are: Cole's translation of a revised edition of Netto's "Theory of Substitutions" (1892); Kantor, "Theorie der endlichen Gruppen von eindeutigen Transformationen in der Ebene" (1895);† Vogt, "Leçons sur la résolution algébrique des équations" (1895); Weber, "Lehrbuch der Algebra" (1895–1897); Burnside, "Theory of Groups of a Finite Order" (1897); Klein-Fricke, "Automorphe Functionen (Die gruppen-theoretischen Grundlagen)" (1897); Bianchi, "Teoria dei gruppi di sostituzioni e delle equazioni algebriche secondo Galois" (1897).

A number of other books published during this period

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* The paper was prepared on the invitation of the officers and committee of Section A, "with a view to obtaining at this anniversary meeting such a survey of the field as may lead to a possible coöperation of effort."

† Cf. Wiman, *Math. Annalen*, vol. 48 (1896), pp. 195–240. This article has for its object the complete enumeration of the finite groups of birational transformations in the plane. The results do not agree with those at which Kantor arrived in the treatise cited.