3. A mark for each face, and a list of the edges and vertices in their order upon the boundary of each face.

Such a notation must contain a mark of distinction for the two sides of an edge ; an easy matter if the direction of positive rotation be adopted uniformly in listing arrangements about the vertices and faces respectively.

These processes, and the proved existence of fundamental polygons, open a range of particular problems of considerable interest. But of even superior interest must be, at least until it is solved, *the problem of finding a method for constructing, a priori, upon a given surface the exceptional (Davis) special reticulations whose characteristics are given by the restrictive tables.*

NORTHWESTERN UNIVERSITY,
April, 1898.

SYSTEMS OF SIMPLE GROUPS DERIVED FROM THE ORTHOGONAL GROUP.

BY DR. L. E. DICKSON.

1. IN the February number of the BULLETIN I determined the order ω of the group G of orthogonal substitutions of determinant unity on m indices in the $GF[p^n]$ and proved that, for* $p^n > 5$, $p \neq 2$, the group is generated by the substitutions

$$O_{i,j}^{\alpha,\beta}: \quad \begin{array}{l} \xi_i' = \alpha\xi_i + \beta\xi_j, \\ \xi_j' = -\beta\xi_i + \alpha\xi_j, \end{array} \quad (\alpha^2 + \beta^2 = 1).$$

The structure of G was determined for the case $p = 2$. I have since proved† that for every $m > 4$ and every $p^n > 5$ of the form $8l + 3$ or $8l + 5$, the factors of composition of G

* The fact that $p^n = 3$ is an exception was not pointed out in the BULLETIN. In fact Jordan had not proven case 2° of § 211 when $-1 =$ square, so that the case $a^2 = b^2 = c^2 = \dots = 1$ was unsolved when $p = 3$, $m = 3k + 1$. The theorem is readily proven when $p^n = 3^n$, $n > 1$; but for $p^n = 3$ an additional generator is necessary and sufficient, viz.,

$$W = \begin{pmatrix} 1 & 2 & 2 & 2 \\ 1 & 2 & 1 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 2 \end{pmatrix}, \quad W^3 = 1.$$

† A preliminary account was presented before the Mathematical Conference at Chicago, December 30, 1897.

are 2 and $\omega/2$ for m odd, and 2, 2, $\omega/4$ for m even. [For later results on the cases thus excluded see §§ 11–15.]

For m odd, the orthogonal group G had the same order and factors of composition as the linear Abelian group on $m-1$ indices. Judging from the results for the corresponding continuous groups of Lie, the resulting triply-infinite systems of simple groups of the same order $\omega/2$ are probably isomorphic when $m=3$, but not when $m>3$. [See § 14.]

Excluding here the cases $m=5, 6, 7$, which require lengthy special investigations, I will now give a short, simple proof of the above result. The complete memoir will appear in the *Proceedings of the California Academy of Sciences.**

The substitutions $O_{1,2}^{\alpha,\beta}$ form a commutative group of order† $p^n \pm 1$. A subgroup of index 2 is formed by the substitutions

$$Q_{1,2}^{\alpha,\beta}: \quad \begin{aligned} \xi_1' &= (\alpha^2-\beta^2)\,\xi_1 - 2\alpha\beta\xi_2, \\ \xi_2' &= 2\alpha\beta\xi_1 + (\alpha^2-\beta^2)\xi_2. \end{aligned}$$

Indeed,

$$Q_{1,2}^{\gamma,\delta}\, Q_{1,2}^{\alpha,\beta} = Q_{1,2}^{\alpha\gamma-\beta\delta,\ \alpha\delta+\beta\gamma}.$$

Further, its order is $\frac{1}{2}(p^n \pm 1)$ since $Q_{1,2}^{\alpha,\beta}$ and $Q_{1,2}^{\gamma,\delta}$ are identical if and only if $\alpha = \pm\gamma$, $\beta = \pm\delta$.

If C_i denotes the substitution affecting only the index ξ_i, whose sign it changes, C_1C_2 is always contained among the substitutions $Q_{1,2}^{\alpha,\beta}$.

Since we suppose that $p^n = 8l \pm 3$, 2 is a not-square. Thus, with $\alpha^2+\beta^2=1$, we cannot have $\alpha^2-\beta^2=0$. Hence if T_{12} denotes the transposition $(\xi_1\xi_2)$, $T_{12}C_1$ is not of the form $Q_{1,2}^{\alpha,\beta}$ but serves to extend the group of the latter to the group of the $O_{1,2}^{\alpha,\beta}$. Furthermore, if $j>2$, $T_{12}C_1$ transforms $Q_{1j}^{\alpha,\beta}$ into $Q_{2,j}^{\alpha,\beta}$ and $Q_{2j}^{\alpha,\beta}$ into $Q_{1,j}^{\alpha,-\beta}$. Hence if we extend the alternating group on the m letters ξ_i by the substitutions $Q_{1,2}^{\alpha,\beta}$ we obtain a group H of index 2 under G.

3. Theorem: *For $p^n > 5$, $m > 7$, the maximal invariant subgroup of H is of order 2 or 1 according as m is even or odd.*

For m even, H contains an invariant subgroup of order 2 generated by the substitution

$$N: \quad \xi_i' = -\xi_i \quad (i = 1, \cdots, m).$$

Suppose H has an invariant subgroup I containing a substitution

* Third Series, vol. 1, No. 4; the later results in No. 5.

† "Orthogonal Group in a Galois Field," § 3; BULLETIN, February, 1898, pp. 196–200.

$$S: \qquad \xi_i' = \sum_{j=1}^{m} a_{ij}\xi_j \quad (i=1,\cdots,m),$$

neither the identity nor N. By suitably transforming S, we can suppose that $a_{11}{}^2 + a_{21}{}^2 \neq 1$. Then S is not commutative with $C_1 C_2$; for, if so, it would be merely a product of a substitution affecting only ξ_1 and ξ_2 by a substitution affecting only $\xi_3, \cdots, \xi_m$. Hence the group I contains the substitution, not the identity,

$$S^{-1} C_1 C_2 S C_1 C_2 = S_a C_1 C_2,$$

where $S_a \equiv S^{-1} C_1 C_2 S$, of period two, is found to be

$$\xi_i' = \xi_i - 2a_{i1}\sum_{j=1}^{m} a_{j1}\xi_j - 2a_{i2}\sum_{j=1}^{m} a_{j2}\xi_j \quad (i = 1, \cdots, m).$$

4. *Lemma:* The orthogonal substitution

$$O_{ijk}: \quad \begin{aligned} \xi_i' &= \lambda\ \xi_i + \mu\ \xi_j + \nu\ \xi_k, \\ \xi_j' &= \lambda'\ \xi_i + \mu'\ \xi_j + \nu'\ \xi_k, \\ \xi_k' &= \lambda''\xi_i + \mu''\xi_j + \nu''\xi_k, \end{aligned}$$

where $\lambda^2 + \mu^2 + \nu^2 = 1$, $\lambda\lambda' + \mu\mu' + \nu\nu' = 0$, etc., transforms S_a into $S_{a'}$ where

$$\begin{aligned} a_{il}' &= \lambda\ a_{il} + \mu\ a_{jl} + \nu\ a_{kl}, \\ a_{jl}' &= \lambda'\ a_{il} + \mu'\ a_{jl} + \nu'\ a_{kl}, \qquad\qquad (l = 1, 2) \\ a_{kl}' &= \lambda''a_{il} + \mu''a_{jl} + \nu''a_{kl}, \\ a_{sl}' &= a_{sl}, \qquad (s = 1, \cdots, m,\ s \neq i, j, k). \end{aligned}$$

If $a_{i1} \neq 0$, we can choose λ, μ, ν such that $a_{i1}' = 0$. For if $a_{i1}^2 + a_{j1}^2 = 0$ and therefore $a_{j1} \neq 0$, we may take

$$\lambda = \frac{-a_{k1}}{2a_{i1}}, \quad \mu = \frac{-a_{k1}}{2a_{j1}}, \quad \nu = 1.$$

If $a_{i1}^2 + a_{j1}^2 \neq 0$, we derive the equivalent condition,

$$\{\mu(a_{i1}^2 + a_{j1}^2) + \nu a_{j1} a_{k1}\}^2 + \nu^2 a_{i1}^2 (a_{i1}^2 + a_{j1}^2 + a_{k1}^2) = a_{i1}^2 (a_{i1}^2 + a_{j1}^2),$$

which has solutions* for μ and ν in the $GF[p^n]$ except when $a_{i1}^2 + a_{j1}^2 + a_{k1}^2 = 0$. In the latter case the condition $\lambda^2 + \mu^2 + \nu^2 = 1$ may be written

$$a_{i1}^2 = -(\mu a_{k1} - \nu a_{j1})^2,$$

having solutions if and only if -1 be a square.

---

* BULLETIN, l.c. § 3.

5. *Lemma:* O_{ijkl} transforms S_α into $S_{\alpha'}$, where

$$a_{i1}' = \lambda a_{i1} + \mu a_{j1} + \nu a_{k1} + \sigma a_{l1}.$$

If $a_{i1}^2 + a_{j1}^2 + a_{k1}^2 = 0$, the values

$$\lambda = \frac{-a_{l1}}{a_{i1}}, \quad \mu = \frac{a_{k1}a_{l1}}{a_{i1}^2}, \quad \nu = \frac{-a_{l1}a_{j1}}{a_{i1}^2}, \quad \sigma = 1$$

make $$a_{i1}' = 0, \quad \lambda^2 + \mu^2 + \nu^2 + \sigma^2 = 1$$

6. The invariant subgroup I of H was shown to contain the substitution $S' \equiv S_\alpha C_1 C_2$ not the identity. Transforming S' successively by

$$O_{i345} \text{ or } T_{12} C_1 O_{i345} \qquad (i = m, m-1, \cdots, 6),$$

according as the one or the other belongs to H, we obtain, by §§ 4–5, a substitution $S_{\alpha'} C_1 C_2$ belonging to I and having $a_{m1}' = a_{m-1}'{}_1 = \cdots = a_{61}' = 0$. Also, by § 4, we may make $a_{51}' = 0$; for we have

$$a_{11}'^2 + a_{21}'^2 = a_{11}^2 + a_{21}^2 \neq 1,$$

so that $$a_{51}'^2 + a_{41}'^2 + a_{31}'^2 \neq 0.$$

Next we transform $S_{\alpha'} C_1 C_2$ successively by

$$O_{j567} \quad (j = m, m-1, \cdots, 8)$$

and obtain in I a substitution $S_{\alpha''} C_1 C_2$ having

$$a_{j1}'' \equiv a_{j1}' = 0, \quad a_{j2}'' = 0 \quad (j = m, \cdots, 8).$$

The group I thus contains a substitution

$$S: \qquad \xi_i' = \sum_{j=1}^{7} \beta_{ij}\xi_j \quad (i = 1, \cdots, 7)$$

neither the identity nor $N \equiv C_1 C_2 \cdots C_m$.

7. If S be commutative with every C_i, it is merely a product of an even number of the C_i, in which certain ones as C_k are lacking. But if

$$S = C_i C_j C_r C_s C_t \cdots,$$

the group I contains

$$T_{ij} T_{ik} S T_{ik} T_{ij} S^{-1} = C_k C_j,$$

and hence, by transforming by suitable even substitutions, every product of two C's. But H contains either $O_{1,2}^{\alpha,\beta}$ or $O_{1,2}^{\alpha,\beta} T_{12} C_1$, which transform $C_1 C_3$ into $Q_{1,2}^{\alpha,\beta} C_1 C_3$ and $Q_{1,2}^{\alpha,\beta} C_2 C_3$

respectively. Hence the group I contains every $Q^{\alpha,\beta}_{1,2}$, among which, if $p^n > 5$, occurs one different from the identity and from C_1C_2.

8. We may thus assume that S is not commutative with C_1, for example. Supposing $m \equiv 8$, S is commutative with C_8. Hence the group I contains the substitution not the identity

$$S^{-1}C_1C_8SC_1C_8 = R_\beta C_1,$$

where $R_\beta \equiv S^{-1}C_1S$ is seen to be

$$R_\beta: \quad \xi_i' = \xi_i - 2\beta_{i1}\sum_{j=1}^{7}\beta_{j1}\xi_j \quad (i = 1, \cdots, 7).$$

Transforming $R_\beta C_1$ by O_{i234} for $i = 7, 6, 5$ successively, we may suppose that $\beta_{71} = \beta_{61} = \beta_{51} = 0$.

It is readily seen that a substitution R affecting only $\xi_1, \cdots, \xi_4$, is not commutative with every T_{ij} $(i, j = 1, \cdots, 5)$ for example not with T_{12}. Then I contains the substitution, not the identity,

$$R^{-1}T_{12}T_{67}RT_{67}T_{12} = T_\delta T_{12},$$

where $T_\delta \equiv R^{-1}T_{12}R$ has the form

$$T_\delta: \quad \xi_i' = \xi_i - \delta_i\sum_{j=1}^{4}\delta_j\xi_j \quad (i = 1 \cdots 4),$$

where $$\delta_1^2 + \delta_2^2 + \delta_3^2 + \delta_4^2 = 2.$$

9. If $\delta_3 = \delta_4 = 0$, $T_\delta T_{12}$ becomes $Q^{\alpha,\beta}_{2,1}$ if we set

$$\alpha = \tfrac{1}{2}(\delta_1 - \delta_2), \quad \beta = \tfrac{1}{2}(\delta_1 + \delta_2).$$

Having $Q^{\alpha,\beta}_{2,1}$, I contains also $Q^{\alpha,\beta}_{2,3}$ and $Q^{\alpha,\beta}_{3,1}$ and thus the product of the two, which reduces to $T_\sigma T_{12}$, having

$$\sigma_1 = -1, \quad \sigma_2 = \alpha^2 - \beta^2, \quad \sigma_3 = 2\alpha\beta.$$

If $\alpha\cdot\beta = 0$, $Q^{\alpha,\beta}_{2,1} = C_1C_2$, not being the identity. Then, by §7, I contains every $Q^{\alpha,\beta}_{i,j}$ and therefore, if $p^n > 5$, a substitution $Q^{\alpha,\beta}_{2,3}Q^{\alpha,\beta}_{3,1} = T_\sigma T_{12}$ in which $\alpha\beta \neq 0$.

10. Thus I contains a substitution $T_\delta\ T_{12}$ having δ_3 and δ_4 not both zero, say $\delta_4 \neq 0$. Transforming it by O_{346}, we can make the resulting substitution $T_\delta'\ T_{12}$ commutative with T_{16}. Indeed, the conditions

$$\delta'_6 \equiv \alpha\delta_3 + \beta\delta_4 = \delta_1, \quad \alpha^2 + \beta^2 + \gamma^2 = 1,$$

combine into the single condition

$$\alpha^2 (\delta_3^2 + \delta_4^2) - 2\alpha\delta_1 \delta_3 + \delta_4^2 \gamma^2 = \delta_4^2 - \delta_1^2.$$

For $\delta_3^2 + \delta_4^2 = 0$, a solution is given by $\gamma = 0$ when $\delta_1 \neq 0$, and by $\gamma = 1$ when $\delta_1 = 0$. For $\delta_3^2 + \delta_4^2 \neq 0$, there exist* solutions α, γ in the $GF[p^n]$ of the equivalent equation of condition

$$\{\alpha(\delta_3^2 + \delta_4^2) - \delta_1 \delta_3\}^2 + \delta_4^2(\delta_3^2 + \delta_4^2)\gamma^2 = \delta_4^2(\delta_3^2 + \delta_4^2 - \delta_1^2).$$

Hence I contains the substitution, not the identity,

$$\begin{aligned}(T'_{\delta'}\ T_{12}) &= T_{26}\, T_{78}\, (T_{\delta'}\ T_{12})^{-1}\, T_{78}\, T_{26} \\ &= T_{\delta'}\ T_{16}\, T_{\delta'}^{-1}\, T_{26} = T_{16}\, T_{26}.\end{aligned}$$

The alternating group on $m > 4$ letters being simple, the group I, containing $T_{16}\, T_{26}$, contains the whole alternating group. Further, $C_1\, C_2$ transforms $T_{16}\, T_{26}$ into $T_{16}\, T_{26}\, C_1\, C^9$, so that I contains $C_1\, C_6$ and therefore every $C_i\, C_j$. Hence, by § 7, I contains every $Q_{1,2}^{\alpha,\beta}$. Thus the group I coincides with H.

ADDENDA † OF APRIL 18.

11. For $p^n = 3$ or 5, the maximal invariant sub-group of H is of order 2 or 1, according as $m > 4$ is even or odd. For $p^n = 3$, $m = 4$, the order of H is $2^5 \cdot 3^2$, and its factors of composition are all primes.

12. Suppose $p^n = 8l \pm 1$, so that 2 is a square. Let $O_{1,2}^{\alpha,\beta}$ denote a definite orthogonal substitution not in $Q_{1,2}$, so that $1 \pm \alpha$ are not-squares. Denote by H_1 the group obtained by extending the group of the $Q_{i,j}$ by all the products $O_{i,j}^{\alpha,\beta}\ O_{k,l}^{\alpha,\beta}$.

THEOREM: *H_1 contains half of the substitutions of G.* For every substitution of G is of the form

$$S = h_1\ O_{i,j}^{\alpha,\beta}\ h_2\ O_{k,l}^{\alpha,\beta} \cdots,$$

h_1, h_2, $\cdots$, h denoting substitutions of H_1. Now $O_{i,j}^{\alpha,\beta}$ can be carried to the right of every $Q_{i,j}^{\lambda,\mu}$ and every $Q_{k,l}^{\lambda,\mu}$ $(k, l \neq i, j)$. Further, since $(O_{1,2}^{\alpha,\beta})^2 = Q_{1,2}^{\alpha,-\beta}$,

$$\begin{aligned}O_{i,j}^{\alpha,\beta}\ Q_{i,k}^{\lambda,\mu} &= O_{i,j}^{\alpha,\beta}\ (O_{i,k}^{\alpha,\beta})^2\ Q_{i,k}^{\alpha,\beta} \cdot Q_{i,k}^{\lambda,\mu}, \\ &= O_{i,j}^{\alpha,\beta}\ O_{i,k}^{\alpha,\beta} \cdot Q_{i,k}^{\alpha,\beta}\ Q_{i,k}^{\lambda,\mu} \cdot O_{i,k}^{\alpha,\beta} = h\ O_{i,k}^{\alpha,\beta}.\end{aligned}$$

---

* BULLETIN, l. c. § 3.

† The results here announced will be proven in full in the *Proceedings of the California Academy of Sciences*, Third Series, vol. 1, No. 5.

Thus S finally takes the form

$$h' \; O_{r,s}^{\alpha,\beta} = h' \; O_{r,s}^{\alpha,\beta} \; O_{2,1}^{\alpha,\beta} \; O_{1,2}^{\alpha,\beta} = h'' \; O_{1,2}^{\alpha,\beta}.$$

13. THEOREM: *For* $m > 4$, $p \neq 2$, *the maximal invariant sub group of* H_1 *is of order* 2 *or* 1 *according as* m *is even or odd.*

For $m > 7$, the group is similar to that for the group H as given above. In §6 we replace $T_{12}\, C_1\, O_{i345}$ by $O_{1,2}^{\alpha,\beta}\, O_{i345}$. We replace §7 by the

Lemma: If $p^n = 8l \pm 1, m > 3$, *an invariant sub-group* I *containing every* $C_i C_j$ *coincides with* H_1.

Indeed, $O_{2,4}^{\alpha,\beta}\, O_{1,2}^{\alpha,\beta}$ transforms $C_1\, C_3$ into $Q_{1,2}^{\alpha,\beta}\, C_1\, C_3$, so that I contains every $Q_{i,j}^{\alpha,\beta}$. Having $T_{23} C_3$, I contains every $O_{i,j}^{\alpha,\beta}\, O_{k,l}^{\alpha,\beta}$. Thus, for example,

$$(T_{23} C_3)\,(O_{1,2}^{\alpha,\beta}\, O_{1,4}^{\alpha,\beta})\,(T_{23} C_3)^{-1}\,(O_{1,2}^{\alpha,\beta}\, O_{1,4}^{\alpha,\beta})^{-1} = O_{1,3}^{\alpha,\beta}\, O_{2,1}^{\alpha,\beta}.$$

14. THEOREM: *For* $p > 2$, *the ternary orthogonal group in the* $GF[p^n]$ *has a sub-group* H' *of index two and of order* $\frac{1}{2}p^n(p^{2n}-1)$ *which is simply isomorphic to the group of linear fractional substitutions of determinant unity on a single index.*

Indeed, the orthogonal substitution

$$S:\; \xi_i' = \sum_{j=1}^{3} a_{ij}\,\xi_j \quad (i = 1, 2, 3),$$

expressed in terms of the new indices

$$\eta_1 = -i\xi_1, \quad \eta_2 = \xi_2 - i\xi_3, \quad \eta_3 = \xi_2 + i\xi_3,$$

leaves $\eta_1^2 - \eta_2\eta_3$ invariant and has the form

$$S_1:\; \left\{ \begin{array}{ccc} a_{11} & \frac{1}{2}(a_{13}-ia_{12}) & -\frac{1}{2}(a_{13}+ia_{12}) \\ a_{31}+ia_{21} & \frac{1}{2}(a_{22}-ia_{32}+ia_{23}+a_{33}) & \frac{1}{2}(a_{22}-ia_{32}-ia_{23}-a_{33}) \\ -a_{31}-ia_{21} & \frac{1}{2}(a_{22}+ia_{32}+ia_{23}-a_{33}) & \frac{1}{2}(a_{12}+ia_{32}-ia_{23}+a_{33}) \end{array} \right\}$$

Understanding by H' the group H or H_1 according as $p^n = 8l \pm 3$ or $p^n = 8l \pm 1$, we may verify that for every substitution of H' the coefficient $\frac{1}{2}(a_{22} - ia_{32} + ia_{23} + a_{33})$ is the square of a complex α of the form $\rho + \sigma i$, where ρ and σ are marks of the $GF[p^n]$. It readily follows* that S_1 may be written in the form :

$$S_1:\; \left\{ \begin{array}{ccc} \alpha\delta + \beta\gamma & \alpha\gamma & \beta\delta \\ 2\alpha\beta & \alpha^2 & \beta^2 \\ 2\gamma\delta & \gamma^2 & \delta^2 \end{array} \right\} \qquad [\alpha\delta - \beta\gamma = 1]$$

where α is conjugate to δ, β to γ. Further two such ternary

---

*Compare Klein-Fricke : Automorphic Functions I., p. 14 ; Weber: Algebra, II., p. 190.

substitutions have the same composition formula as linear fractional substitutions. Hence, according as -1 is a square or a not-square, H' is simply isomorphic to the "real" or the "imaginary" form* of the group of linear fractional substitutions of determinant unity. Thus, for $p^n > 3$, H' is simple.

15. Observing that the squares of the substitutions

$$O_{1,2}^{\alpha,\beta}, \quad O_{1,2}^{\alpha,\beta}\, T_{13} C_1 C_2 C_3, \quad O_{1,2}^{\alpha,\beta}\, T_{13} T_{24}$$

are respectively $Q_{1,2}^{\alpha,-\beta}$, $\quad O_{1,2}^{\alpha,\beta}\, O_{3,2}^{\alpha,\beta}$, $\quad O_{1,2}^{\alpha,\beta}\, O_{3,4}^{\alpha,\beta}$,
we may unite our results into the following

THEOREM: *The squares of the linear substitutions on m indices in the $GF[p^n]$, $p \neq 2$, which leave invariant the sum of the squares of the m indices, generate a group, which for $m = 2k + 1$ has the order*

$$\tfrac{1}{2}(p^{2nk} - 1)\, p^{2nk-n}\, (p^{2nk-2n} - 1)\, p^{2nk-3n} \cdots (p^{2n} - 1)\, p^n$$

and is simple except when $p^n = 3$, $m = 3$; while for $m = 2k > 4$ it has the factors of composition 2 *and*

$$\tfrac{1}{4}[p^{nk} - (\pm 1)^k]\, p^{nk-n}\, (p^{2nk-2n} - 1)\, p^{2nk-3n} \cdots (p^{2n} - 1)\, p^n,$$

the sign $\pm$ depending upon the form $4l \pm 1$ of p^n.

UNIVERSITY OF CALIFORNIA,
February 10, 1898.

A PROOF OF THE THEOREM:

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}$$

BY MR. J. K. WHITTEMORE.

(Read before the American Mathematical Society at the Meeting of April 30, 1898.)

THEOREM: *Let $u = f(x, y)$ denote a function of the two independent variables x and y which, together with its first derivatives and the two second derivatives in question, is continuous in the neighborhood of the point (x, y); then* $\dfrac{\partial^2 u}{\partial x \partial y} = \dfrac{\partial^2 u}{\partial y \partial x}$

Let $\dfrac{\partial^2 f(x, y)}{\partial x \partial y}$ denote $\dfrac{\partial}{\partial x}\left(\dfrac{\partial f(x, y)}{\partial y}\right)$

---

*Moore: Mathematical Papers of the Chicago Congress (1893), "A doubly-infinite system of simple groups," §§ 5–6.