EARLY HISTORY OF GALOIS' THEORY OF EQUATIONS.

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(Read before the American Mathematical Society at the Meeting of February 26, 1897.)

THE first part of this paper will treat of Galois' relations to Lagrange, the second part will sketch the manner in which Galois' theory of equations became public. The last subject being intimately connected with Galois' life will afford me an opportunity to give some details of his tragic destiny which in more than one respect reminds one of the almost equally unhappy lot of Abel.* Indeed, both Galois and Abel from their earliest youth were strongly attracted toward algebraical theories; both believed for a time that they had solved the celebrated equation of fifth degree which for more than two centuries had baffled the efforts of the first mathematicians of the age; both, discovering their mistake, succeeded independently and unknown to each other in showing that a solution by radicals was impossible; both, hoping to gain the recognition of the Paris Academy of Sciences, presented epoch making memoirs, one of which

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* Sources for Galois' Life are:

1° Revue Encyclopédique, Paris (1832), vol. 55.

a Travaux Mathématiques d Évariste Galois, p. 566–576. It contains Galois' letter to Chevalier, with a short introduction by one of the editors.

β *Ibid.* Nécrologie, Évariste Galois, p. 744–754, by Chevalier. This touching and sympathetic sketch every one should read.

2° Magasin Pittoresque. Paris (1848), vol. 16. Évariste Galois, p. 227–28. Supposed to be written by an old school comrade, M. Flaugergues, an ardent admirer of Galois.

3° Mes Mémoires, A. Dumas, (père). Paris (1869), vol. 8, p. 159–161 and p. 166–169. Gives an account of the notorious affair of the Vendanges de Bourgogne and of Galois' first trial.

4° Journal de Mathém tiques, (1846), v. 11. Oeuvres Mathématiques d'Évariste Galois, p. 381–444. The avertissement is by Liouville and contains some personal notes about Galois.

5° Nouvelles Annales de Mathématiques. Paris (1849), vol. 8, p. 452. A few lines only.

6° Annales de l'École Normale. Paris (1896), 3 Ser. vol. 13, p. 197–266. La Vie d'Évariste Galois, by P. Dupuy, professor of history in the École Normale. This is the most extensive biography of Galois. It contains all that is essential in the five preceding references and an immense amount of other matter. It goes without saying that I am indebted to it for many of the details here given.

7° Oeuvres Mathématiques d'Évariste Galois. Paris, 1897. The introduction by Picard contains interesting remarks.

was lost forever, the other published first twelve years after the death of its illustrious author; in the case of both the passionate feverish blood of genius led them to a premature death, one at the age of twenty, the other at twenty-six. Both died leaving behind them theories whose thresholds they had hardly crossed, theories which, belonging to the first creations of this century, will keep their memories perennially green. I shall have no occasion in this paper to speak of anything except Galois' algebraical theories. It is, however, only just to his memory to recall a fact too universally overlooked that in spite of his extreme youth he was perfectly familiar with the researches of Abel and Jacobi and that he was certainly in possession of the most essential results of Riemann in regard to Abelian integrals discovered twenty-five years later. A remark he makes regarding a generalization of Legendre's equation connecting the periods of elliptic integrals of the first and second species

$$FE' + EF' - FF' = \frac{\pi}{2}$$

leads us to suspect that he had also anticipated some of the results of Weierstrass and Fuchs in this field. All goes to show the justness of Picard's remark* that "if only a few years more had been given him to develop his ideas in this direction he would have been the glorious continuator of Abel and would have erected in its most essential parts the theory of algebraic functions of one variable as we know it to-day."

1. *Galois' relations to Lagrange.*

It is well known that Galois, like Abel and Ruffini, received inspiration from the writings of Lagrange regarding the algebraic solution of equations, in particular his mémoire "Sur la résolution algébrique des équations" and his "Traité de la résolution des équations, etc.," but no one, I believe, has remarked how well Lagrange had prepared the way for Galois.

Let us capitulate rapidly the principal facts of Lagrange's theory referring for more detail to my paper† on "Lagrange's place in the theory of substitutions." In the first place he gave the theory of the solution of general equations $f(x) = 0$ of degree n an entirely new basis in employing rational func-

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* Introduction to the new edition of Galois' Oeuvres, which has just appeared under the auspices of the Société Mathématique de France.

† BULLETIN, 2 Series, vol 1, p. 196. (1895.)

tions of the roots of f to build his resolvents. Two rational functions of the roots of f invariant for the same permutations of the roots he calls similar and shows that they are roots of rational equations whose degrees are equal and a divisor of $n\,!$; furthermore, they are rational in one another. If φ be a rational function of the roots taking on r values for all possible permutations and ψ another function which changes its value for all the permutations which alter φ as well as for some of the permutations which leave φ unaltered taking on in all rs values, then φ is rational in ψ while ψ is root of an equation of degree s whose coefficients are rational in φ. This being so, a function t which changes its value for every permutation enjoys the remarkable property that the roots themselves, $x_1, x_2, \cdots, x_n$, of f are rational in t. Such a function, and Lagrange remarks that its simplest form is

$$t = ax_1 + bx_2 + \cdots + lx_n,$$

is root of a rational equation of degree $n\,!$, and all its roots are rational in any one of them.

This resolvent plays an important part in Lagrange's theory; I shall call it Lagrange's resolvent and denote it by $L(t) = 0$. The solution of $f = 0$ and $L(t) = 0$ are evidently equivalent problems. Let us see how a general scheme for solving the equation $f = 0$ may be deduced from these facts. Take a rational function φ of the roots of f and form its resolvent. Its solution gives its roots $\varphi_1, \varphi_2, \varphi_3, \cdots$ as known algebraic functions of the coefficients of f. That is, certain rational functions of the roots of f are now known which at the start were not. The effect of this is to make the Lagrangian resolvent $L = 0$ which before was irreducible split up into a certain number of equal factors whose coefficients are rationally known. In other words, the determination of t which before depended upon an equation of degree $n\,!$, depends now upon an equation of less degree say $L_1(t) = 0$. Take now another rational function ψ of the roots of f which is not rational in the φ's. This will give rise to a resolvent whose coefficients are rational in the φ's; its solution makes still other rational functions of the roots known which before were not and the effect of this is to make $L_1(t)$ reducible and thus t depends upon an equation of still less degree $L_2(t) = 0$ with known coefficients. Continuing in this way, the degree of the equation upon which t depends becomes less and less, at last it must depend upon an equation of first degree, and then t is rational in known quantities. But since the roots of f are rational in t, these roots are also rational in known quantities.

Now this is precisely the Galoisian scheme of solution. In other words, the task Galois had before him was: 1° to formulate the solution of *any* equation as here sketched and as indicated with considerable clearness in Lagrange's writings, and 2° to extend the theorems upon which this scheme of solution depends, from the limited case here considered, viz. equations whose coefficients are independent variables, to the case of any special equation and an arbitrary domain of rationality. The application Lagrange made of his theory to solve the special equations upon which the roots of unity of prime order depend was doubtless of great assistance to Galois in his work of generalization. Lagrange proves in this connection the cardinal fact that all cyclic functions of the roots are rationally known and uses this as the base of his solution. The cyclic group is precisely the group which here plays the same rôle as the symmetric group does for general equations.

2. *How Galois' algebraic theories became public.*

As already observed the manner in which this took place is so intimately connected with his short and agitated career that a few details of his life are indispensable.

Évariste Galois was born October 25, 1811, in the little town of Bourg-la-Reine, a few miles to the south of Paris. His father, Nicolas-Gabriel, as his grandfather before him, was proprietor of a flourishing school, established before the French Revolution and at Évariste's birth a corporate part of the University of France. His mother, Adélaïde Demante, was daughter and sister of eminent jurists of the Faculté de Droit at Paris. From her father she had received a careful and extensive education in the classics. Till Évariste was twelve years old she was his only teacher. At this stage, that is, in the autumn of 1823 he entered Louis-le-Grand, one of the celebrated lycées of Paris, a stone's throw from the Sorbonne and the Collége de France. At fifteen he first began the study of mathematics, entering the class known as *Mathématiques préparatoires*. A new world now opened up to him; the study of mathematics became the absorbing passion of his life. The Geometry of Legendre he is said to have read through as one reads a novel. Soon ordinary treatises did not satisfy him; they lacked, so he said, the stamp of the great inventors; he began the study of the original authors, especially Lagrange. In the Autumn of 1828 he entered the class of *Mathématiques spéciales*, under the charge of Professor

Richard, an excellent mathematician and a man of noble character. Under his intelligent and sympathetic direction Galois' progress became even more remarkable. His original solutions of problems proposed in the class were explained by Richard with just praise to his admiring classmates. In the trimestrial reports of the school we find Richard writing : " Cet élève a une supériorité marquée sur tous ses condisciples," and again " Cet élève ne travaille qu' aux parties supérieures des Mathématiques," while reports on his other studies are a series of lamentations caused by Galois' neglect. So, for example, this: " Il ne fait absoluement rien pour la classe. C'est la fureur des Mathematiques qui le domine." At this epoch, that is, when seventeen years old, Galois made his first discoveries in the theory of equations which can be solved by radicals. He drew up a short mémoire on the subject and Cauchy took it to present to the Academy of Sciences. The mémoire was never heard of again, although Galois reclaimed it several times at the secrétariat. At this time also his first paper, entitled " Démonstration d'un théorème sur les fractions continues périodiques" was published, appearing in the March number of the *Annales de Gergonne* for 1829.

Galois ardently desired to enter the École Polytechnique, the first school of Mathematics in France. Twice he presented himself for examination but was rejected both times to the astonishment of all who knew him. Two reasons probably lead to this deplorable mistake. Galois had the habit of working almost exclusively with his head and to deal with the broad aspects of a subject. To his remarkable mind many things seemed trivial or self-evident, which required demonstration for those less gifted, and complaint had already been made on various occasions that he was somewhat obscure in the expression of his ideas. When therefore at the examination it became necessary to work out on the blackboard, explaining as he proceeded, before a numerous audience questions of pure detail he was greatly embarrassed. He did not have what one called *l' habitude du tableau.* The second reason is that the examinators were flagrantly incapable of appreciating the extraordinary talents of the youth they had before them. At the distance of twenty years the injustice of his examinators was still keenly felt, for Terquem in a short notice on Galois which appeared in the Nouvelles Annales* writes : " Nous répéterons ici et nous ne cesserons de répéter une réflexion que nous

* Vol. 8 (1849), p. 452.

avons déjà consignée mainte fois : Un candidat d'une intelligence supérieure est perdu chez un examinateur d'une intelligence inférieure. *Barbarus hic ego sum quia non intelligor illis.* Certes M. Liouville qui nous a fait connaître le génie de Galois ne l'aurait pas jugé incapable."

Nothing was left to Galois but to enter the École Préparatoire,* a fate which filled his heart with bitterness and despair. At this crisis Galois' father, whom he dearly loved, died a violent death. As mayor of his native town, M. Galois had defended for many years the interest of the liberals against the grasping power of the clerical party. Driven at last to desperation by their incessant and venomous attacks, his kindly nature succumbed and in a moment of despondency he committed suicide. Still Galois continued with his mathematics, adding to his already vast stores of erudition and maturing his own inventions. During the first year at the École, that is, when eighteen years old, he published four mémoires. One of these is a brief summary of some of his results regarding the algebraic solution of equations; another is devoted to the theory of those numbers which we now call Galois' imaginaries and which play an important part in the theory of groups to-day. In January of this year (1830) Galois presented to the Academy another mémoire containing an account of his researches written out with care and detail. He placed great hope in this. The manuscript was given to Fourier to read, who shortly after died, and the mémoire was lost. Six months later the Revolution broke out which drove Charles X. from the throne and installed Louis Philippe in his stead. In this movement Galois entered with heart and soul. The attitude which the director of the school, M. Guigniault, took toward the new government offended Galois so deeply that he was rashly led to take part in a polemic which the *Gazette des Écoles* was waging against Guigniault. The scandal that this raised produced Galois' expulsion (Dec. 9, 1830). Powerful friends however interested themselves in him. A minister of the Royal Council, M. Barthe, summoned the unfortunate youth and bade him not to be concerned for the future. Poisson invited him to make a new redaction of his researches and volunteered to present it himself to the Academy. Inspired by fresh hopes Galois wrote the only finished mémoire we have of his grand theory of the solution of equations. It bears the title "Sur les conditions de

*This took the place of the École Normale suppressed in 1822. Dupuy speaks of it as a faint and humble copy of this celebrated school. After the revolution of 1830 it became the École Normale again.

résolubilité des équations par radicaux." At the same time he opened a course on Higher Algebra which was held in a bookstore near the Sorbonne. The *Gazette des Écoles* announced it as follows : "This course will be held Tuesdays at quarter past one ; it is destined for those who, feeling how incomplete courses on algebra are, desire to penetrate more deeply into this science. Many of the theories to be given are entirely new, none have ever been developed in university lectures. We content ourselves to mention a new theory of imaginaries, a theory of equations which are soluble by radicals, the theory of numbers, and the theory of elliptic functions treated entirely by algebra." But his hopes were short lived ; the mémoire intrusted to Poisson was returned, being declared unintelligible.

Already associated with the most reactionary faction of the republican party, Galois now plunged deeper and deeper into the political turmoil which was shaking France at its foundations. Twice arrested for political offenses, he spent the greater part of the last year and a half of his life in the prison of Sainte Pélagie. On being released the second time he became engaged in an affair of honor which resulted in his death (May 31, 1832). The night before the duel he spent in drawing up with feverish haste an account of his discoveries. No one can read this scientific testament written in the form of a letter to his friend and schoolmate, Auguste Chevalier, without emotion ; it is one of the most touching documents in the history of science. Cut off in the very commencement of his career, conscious of the importance of his discoveries, he spent the last few hours of his life trying to save to posterity what his contemporaries had been loth to accept. He requests Chevalier to publicly beg Gauss and Jacobi to give their opinion, not on the truth but upon the importance of his theorems. "Après cela," he adds, "il y aura, j'espère, des gens qui trouveront leur profit à déchiffrer tout ce gâchis."

Chevalier was thus appointed Galois' scientific executor. Conformably to Galois' wishes the letter we have been speaking of was published the following September in the Revue Encyclopédique and an editorial note states that all Galois' manuscripts would shortly appear in the same journal under the editorship of Chevalier. This was not to be the case. Nothing more was published until 1846 when a part of them, edited by Liouville, appeared in Vol. XI. of the Journal de Mathématiques. In the preface which accompanies this collection Liouville remarks : "When, at the desire of Évariste's friends, I began an attentive study of all his published mémoires and the manuscript he left behind, I considered it my sole task to disentangle, as far as possible, what there was new in these productions in order to make its value more evident. My zeal was soon rewarded and it gave me the greatest pleasure when, on filling some slight gaps, I recognized the entire correctness of the method whereby Galois proves this beautiful theorem: In order that an irreducible equation of prime degree be soluble by radicals it is necessary and sufficient that all its roots are rational in any two of them. This method, eminently worthy of the attention of geometers, will alone suffice to assure our compatriot a place among the few mathematicians who merit the title of inventor." Liouville's purpose was to publish all Galois' works and to add a commentary in which he intended to complete certain passages and elucidate various delicate points. Unfortunately this plan was never executed, only two new papers were published, the mémoire written at Poisson's invitation already mentioned and a fragment of a second mémoire entitled, "Des équations primitives qui sont solubles par radicaux." A note at the end of the preface informs us that the press of matter for publication as well as the extent of Galois' manuscript makes it necessary to give only a part in the current number, the rest following in the next volume. As we just said, this promise was not fulfilled.

The next reference I find to Galois' theory for the solution of algebraic equations is in a footnote, p. 344 of the first edition of Serret's Algèbre Supérieure, published in 1849: "My friend, M. Liouville," he remarks, "has announced to me his intention to publish one day some developments regarding this remarkable work. It is only by aid of these developments, a part of which M. Liouville has kindly communicated to me, that I have succeeded in comprehending certain points of Galois' Mémoire, whose study only those geometers can undertake who have occupied themselves quite specially with the theory of equations. For these reasons I have been prevented from presenting here the discoveries of Galois." The cause of Liouville's silence can only be conjectured. It seems probable that, on a more careful revision, preparatory to publishing, certain points in his demonstrations did not seem to be rigorously established. At any rate, Betti, having asked him in a paper published in the Annali of Tortolini, in 1851, not to deprive the public any longer of the results of his study, proceeded to publish his own commentary in the same journal the following year. Thus in 1852, twenty years after Galois' death, his theory of the resolution of algebraic equations was for the first time made intelligible to the general public and established with complete rigor. A few words more on the further history of Galois' theory will complete my account. According to Weber,* Kronecker probably first became acquainted with it during his visit to Paris in 1853 where he was associated intimately with Hermite, Bertrand, and other leading French mathematicians. The first mention Kronecker makes of Galois' name is in a letter to Dirichlet in March, 1856. Dedekind also became very early acquainted with Galois' theory since it is known that he lectured in the winter of 1857–58 on higher algebra and in particular on Galois' theory. According to Weber † this was probably the first extensive account of Galois' theory given at a German university. The first account of it given in a text-book on algebra is in the third edition of Serret's algebra (1866). This, together with Jordan's classic treatise which appeared in 1870, made a knowledge of Galois' theory possible to all the world.

Perfectly just was Galois' estimate of his own discoveries when he said shortly before his death: "*J'ai fait des recherches qui arrêteront bien des savants dans les leurs.*"

YALE UNIVERSITY,
February, 1898.

LOVE'S THEORETICAL MECHANICS.

Theoretical Mechanics, an Introductory Treatise on the Principles of Dynamics. By A. E. H. LOVE, M.A., F.R.S., Fellow and Lecturer of St. John's College, Cambridge. Cambridge, The University Press, 1897. 8vo, xiv + 379 pp.

THIS is a text-book on dynamics intended for the useof students who have some knowledge of differential and integral calculus and coördinate geometry. The statements of first principles necessarily relate to motion in three dimensions, but the systematic development of the subject is for the most part confined to the motion in two dimensions of particles and rigid bodies. A notable feature of the book is the careful attention which is paid to the statement of the theory of dynamics, not merely as a basis for mathematical problems, but also as a branch of science. In this respect it stands in marked contrast to most other textbooks of similar scope.

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* *Mathematische Annalen*, vol. 43, p. 1.

† Algebra, vol. 1, Einleitung, p. 7.