# THE OCTOBER MEETING OF THE AMERICAN MATHEMATICAL SOCIETY.

At the meeting of the American Mathematical Society held on May 29, 1897, the By-Laws of the Society were amended to provide that regular meetings should thereafter be held on the last Saturdays of October, February and April, unless otherwise ordered by the Council. The Annual Meeting, held in the last week of December, is unaffected by this action. The reduction of the number of New York meetings to four annually, instead of eight as heretofore, was made with a view to securing greater prominence and interest for each meeting, and to affording the members of the Society a better opportunity for scientific and social intercourse. For this purpose, and in compensation for the reduction in the number of meetings, it was provided that each meeting should hereafter extend through two sessions, to be held in the morning and afternoon. The wisdom of the new departure was abundantly demonstrated at the first meeting held under it on Saturday, October 30, 1897, in New York City.

Forty-one persons were in attendance, of whom thirty-seven were members of the Society. Nine papers were presented at the two sessions. Both numbers greatly surpass those of any previous regular meeting, and the result is especially gratifying in view of the brief interval since the highly successful Summer Meeting of the Society at Toronto.

In the absence of President Newcomb, the Vice-President, Professor R. S. Woodward, occupied the chair. The register of members in attendance was as follows: Professor Cleveland Abbe, Professor E. W. Brown, Dr. J. B. Chittenden, Professor F. N. Cole, Professor E. S. Crawley, Dr. J. W. Davis, Professor T. W. Edmondson, Professor A. M. Ely, Professor T. S. Fiske, Mr. P. R. Heyl, Dr. G. W. Hill, Dr. J. E. Hill, Professor H. Jacoby, Mr. S. A. Joffe, Mr. C. J. Keyser, Professor P. Ladue, Professor G. Legras, Dr. G. H. Ling, Dr. E.McClintock, Mr. J. Maclay, Professor H. P. Manning, Professor M. Merriman, Professor G. D. Olds, Mr. J. C. Pfister, Professor M. I. Pupin, Professor J. P. Pierpont, Professor J. K. Rees, Mr. C. H. Rockwell, Professor C. A. Scott, Professor H. Taber, Professor H. D. Thompson, Professor C. L. Thornburg, Professor J. M. Van Vleck, Professor E. B. Van Vleck, Professor J. B. Webb, Mr. G. L. Wiley, Professor R. S. Woodward.

The Council announced the election of the following persons to membership in the Society: Mr. Paul Capron, Dummer Academy, South Byfield, Mass.; Mr. John Kinsey Gore, Prudential Insurance Company, Newark, N. J.; Professor Alfred George Greenhill, Artillery College, Woolwich, England. Ten applications for membership were reported.

In the interval between the sessions a large number of the members took advantage of the opportunity to lunch together. The same convenient arrangement will be afforded at all future meetings in New York.

The following papers were presented:—

(1) Dr. G. W. Hill: "Intermediary orbits in the lunar theory."

(2) Mr. P. R. Heyl: "Notes on the theory of light on the hypothesis of a fourth dimension."

(3) Professor C. A. Scott: "Note on linear systems of curves."

(4) Professor R. S. Woodward: "On the cubic equation defining the Laplacian envelope of the earth's atmosphere."

(5) Professor R. S. Woodward: "On the integration of a system of simultaneous linear differential equations."

(6) Professor Mansfield Merriman: "The probability of hit on a target when the probable error in aim is known; with a comparison of the probabilities of hit by the methods of independent and parallel fire from mortar batteries."

(7) Professor E. W. Brown: "Note on the steering of an eight-oared boat."

(8) Professor A. S. Chessin: "Note on hyperelliptic integrals."

(9) Dr. E. O. Lovett: "Note on the fundamental theorems of Lie's transformation groups."

Dr. Hill's paper, which will be published in the *Astronomical Journal*, discusses certain modern refinements in the planetary theory. The difficulty astronomers have met in endeavoring to push the approximation to the coördinates of a planet beyond the simple Keplerian theory, in that the time appeared as a multiplier outside of the trigonometrical functions, is at once removed if we divide the potential function otherwise than is commonly done. This modification has given rise to what are known as *intermediary orbits*. But the mode in which Gyldén and others have introduced the latter seems unnecessarily obscure, as they are often discussed without the slightest reference to the law of gravitation. In the lunar theory much will be gained if we can make the perigee and node have a uniform motion already in the first approximation. Now the disturbing function of the solar action may be considered as involving a term rigorously proportional to the square of the radius and another proportional to the square of the moon's perpendicular distance from the plane of the ecliptic. The first will produce a motion of the perigee while the second does the same for the node.

The integration of the equations of motion naturally brings in the $\theta$-function of Jacobi. But for this it is better to substitute its periodic development, as tables of elliptic functions are necessarily to double entry and are consequently difficult in interpolation; moreover, they do not readily adapt themselves to processes of combination as the circular functions do. In all special cases, where the values of the arbitrary constants are involved, it is always much shorter and easier to follow methods numerical in character.

The essential features of Mr. Heyl's paper may be summarized as follows: Statement of the 4-space theory of light; 3-space analogy—dust on a drumhead. Differential equation for a solid vibrating in a 4th dimension. Identity of this equation with the equation for sound; the difference one of interpretation of the dependent variable. Mutually supplementary character of the two interpretations illustrated by a consideration of boundary conditions. Problems in rectangular coördinates; problems in hypercylindrical coördinates. The functions $\frac{\sin mr}{r}$ and $\frac{\cos mr}{r}$ and the development of functions in terms of these as modified Fourier's series and Fourier's integrals. Slight departure from the law of inverse squares. Curious apparent retroactive effect in the æther.

Professor Scott's paper deals with two questions that present themselves when a twofold linear system of curves, a *net*, is used for the rational transformation of the plane. Divergence from the usual relation of points and curves is exhibited when the net $(\varphi_1 \varphi_2 \varphi_3)$ has a fundamental curve (*i. e.*, a curve met by the general curve of the net only at the fixed points) represented by a common factor $B$ in two distinct $\varphi$'s, and the modified form of the usual relation depends upon the power to which this factor occurs. The determination of the highest power of $B$ that can occur without any specialization in the position of the fundamental points is the principal object of the paper; the theorem is proved that the $k + 1$ determining curves of the $k$-fold system from which the net is derived by imposing extrinsic conditions can be written in a certain order, viz.: *one* not containing $B$, *two* containing $B^1$, *three* containing $B^2$, etc.; and that for the determination of the net the first of these, any other one, and any other before a certain one, may be chosen without affecting the fundamental points. Divergence from the usual relation occurs also when one or more of the $\varphi$'s have at one or more of the fundamental points a multiplicity greater than that assigned; and the determination of the limits within which this irregularity can occur without specialization in the fundamental points is the second matter considered.

In Chapter VII. of vol. 2 of the Mécanique Céleste Laplace determines from mechanical principles a limiting envelope to the atmosphere of a rotating planet. This envelope is symmetrical with respect to the axis of rotation of the planet, and the curve of intersection of a meridian plane with the envelope is defined by the following cubic equation:

$$x^3 - \frac{2}{a\,x_0 \sin^2\theta}\,x + \frac{2}{a\sin^2\theta} = 0,$$

where $x$ is the ratio of the radius vector, measured from the planet's center, to the radius of the planet; $\theta$ is the polar distance of the radius vector; $a$ is the ratio of centrifugal acceleration to the acceleration of gravity along the equator of the planet; and $x_0$ is the value of $x$ for $\theta = 0$.

Professor Woodward's first paper alludes first to the interesting way in which Laplace discussed this equation without the aid of modern methods; secondly, to the ease with which the least and greatest roots of the equation are discovered from mechanical considerations; thirdly, to the elegant solution of the equation afforded by the goniometric method, which indicates at once that one of the three real roots applicable to the envelope; and fourthly, to the special fitness of the latter method in supplying for the root used an expression which it is practicable to apply in computing the volume, mass, and mass distribution of the terrestrial atmosphere.

Professor Woodward's second paper will be published in the *Astronomical Journal.* The differential equations considered in this paper are the special forms of the equations of rotation of a non-rigid mass subject to no external forces and to the condition that its principal moments of inertia are equal. The equations are

$$\xi' - \gamma\eta + \beta\zeta = 0,$$
$$\eta' - \alpha\zeta + \gamma\xi = 0,$$
$$\zeta' - \beta\xi + \alpha\eta = 0,$$

where $\xi, \eta, \zeta$ are direction cosines; $\alpha, \beta, \gamma$ are axial components of an angular velocity $\sigma$; and the accent denotes differentiation with respect to the time. The paper as presented deals only with the case of constant values of $\alpha$, $\beta$, $\gamma$. For this case one form of the general solution is the following:

$$\xi = \alpha r_0 + r_1(\alpha\gamma - \beta i\sigma)\, e^{+i\sigma t} + r_2(\alpha\gamma + \beta i\sigma)\, e^{-i\sigma t},$$
$$\eta = \beta r_0 + r_1(\beta\gamma + \alpha i\sigma)\, e^{+i\sigma t} + r_2(\beta\gamma - \alpha i\sigma)\, e^{-i\sigma t},$$
$$\zeta = \gamma r_0 + r_1(\gamma\gamma - \sigma^2\ )\, e^{+i\sigma t} + r_2(\gamma\gamma - \sigma^2\ )\, e^{-i\sigma t}.$$

Excluding the imaginary parts, these equations give

$$\xi = \alpha r_0 + \alpha\gamma r \cos\sigma t + \beta\sigma r \sin\sigma t,$$
$$\eta = \beta r_0 + \beta\gamma r \cos\sigma t - \alpha\sigma r \sin\sigma t,$$
$$\zeta = \gamma r_0 + (\gamma^2 - \sigma)^2 r \cos\sigma t,$$

where $r = r_1 + r_2$.

The differential equations also give

$$\alpha\xi' + \beta\eta' + \gamma\zeta' = 0,$$

whence

$$\alpha\xi + \beta\eta + \gamma\zeta = \sigma \cos\psi,$$

where $\psi$ is the constant angle between the axis of $\sigma$ and the axis to which $\xi$, $\eta$, $\zeta$ refer. Hence, since

$$\xi^2 + \eta^2 + \zeta^2 = 1,$$

the constants $r_0$ and $r$ may be expressed in terms of $\psi$ alone by

$$r_0 = \frac{\cos\psi}{\sigma}, \qquad r = \frac{\sin\psi}{\sigma\sqrt{\alpha^2 + \beta^2}},$$

Professor Merriman's paper will be published in the *Journal of the United States Artillery*. The question of the comparative efficiency of the methods of independent and parallel fire from mortar batteries has recently excited great interest in military circles. The case of independent fire is that where each mortar of the battery is aimed with the intention of hitting the center of a target floating on the water. If $R_1$ and $r_1$ are the probable errors of shot dispersion in range and in azimuth, and $R_2$ and $r_2$ are the probable errors of aim in range and azimuth, it is shown that $R = \sqrt{R_1^2 + R_2^2}$ and $r = \sqrt{r_1^2 + r_2^2}$ are the resultant probable errors by which computations on the probable number of hits are to be made. If $2A$ be the length of the rectangular target in range and $2a$ its width in azimuth, and if $\rho$ denote the constant $0.4769 \cdots$, the expression

$$N\left(\frac{2}{\pi^{\frac{1}{2}}}\int_0^{\rho A/R} e^{-t^2}\,dt\right)\left(\frac{2}{\pi^{\frac{1}{2}}}\int_0^{\rho a/r} e^{-t^2}\,dt\right)$$

gives the probable number of hits when $N$ shots are fired. The case of parallel fire where each mortar is aimed parallel to an imaginary mortar at the center of the battery is then discussed. For the rectangular battery whose length is $2B$ parallel to the plane of fire and whose width is $2b$, it is shown that the expression for the probable number of hits from a volley of $N$ shots is

$$N\left(\frac{1}{\pi^{\frac{1}{2}}}\int_{\rho u_1}^{\rho u_2} e^{-t^2}\,dt\right)\left(\frac{1}{\pi^{\frac{1}{2}}}\int_{\rho v_1}^{\rho v_2} e^{-t^2}\,dt\right)$$

where the limits have the values $u_1 = (B - A)/R$, $u_2 = (B + A)/R$, $v_1 = (b - a)/r$, and $v_2 = (b + a)/r$. The conclusion follows that the probable number of hits under independent fire is greater than that under parallel fire. In the long run the advantage of the independent method appears to be about twenty-five per cent.

Professor Brown's "Note on the steering of an eight-oared boat" is an extract from the author's review of Lamb's Hydrodynamics, published in the November number of the BULLETIN (see pp. 77, 78).

Professor Chessin's paper is published in full in this number of the BULLETIN. Dr. Lovett's paper appeared in the November number.

F. N. COLE.

COLUMBIA UNIVERSITY.