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Hyperkähler prequantization of the Hitchin system and Chern–Simons theory with complex gauge group

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Abstract

Hitchin has shown that the moduli space $\mathcal{M}$ of the dimensionally reduced self-dual Yang–Mills equations has a hyperKähler structure. In this paper, we first explicitly show the hyperKähler structure, the details of which is missing in Hitchin's paper. We show here that $\mathcal{M}$ admits three prequantum line bundles, corresponding to the three symplectic forms. We use Quillen's determinant line bundle construction and show that the Quillen curvatures of these prequantum line bundles are proportional to each of the symplectic forms mentioned above. The prequantum line bundles are holomorphic with respect to their respective complex structures. We show how these prequantum line bundles can be derived from cocycle line bundles of Chern–Simons gauge theory with complex gauge group in the case when the moduli space is smooth.

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1 Introduction

Given a symplectic manifold $(\mathcal{M}, \Omega)$, geometric prequantization is a construction of a prequantum line bundle $\mathcal{L}$ on $\mathcal{M}$, whose curvature is proportional to the symplectic form. If $\mathcal{M}$ admits such a prequantum line bundle $\mathcal{L}$, then one can associate a Hilbert space, namely, the square integrable sections of $\mathcal{L}$ and a correspondence between functions on $\mathcal{M}$ to operators acting on the Hilbert space such that the Poisson bracket of two functions corresponds to the commutator of the corresponding operators. The latter is ensured by the fact that the curvature of the prequantum line bundle is precisely the symplectic form Ω [29]. Let $f \in C^\infty(\mathcal{M})$. Let X_f be the vector field defined by $\Omega(X_f, \cdot) = -df$. Let θ be the symplectic potential correponding to Ω. Then we can define the operator corresponding to the function f to be $\hat{f} = -\mathrm{i}\hbar[X_f - \mathrm{i}/\hbar\theta(X_f)] + f$. Then if $f_1, f_2 \in C^\infty(\mathcal{M})$ and $f_3 = \{f_1, f_2\}$, Poisson bracket of the two induced by the symplectic form, then $[\hat{f}_1, \hat{f}_2] = -\mathrm{i}\hbar\hat{f}_3$. When $\mathcal{M}$ has hyperKähler structure, there are three symplectic structures and hence three Possion brackets and one can often construct prequantum line bundles for each one of them. This is called hyperKähler prequantization.

A relevant example in our context would be geometric quantization of the moduli space of flat connections on a principal G-bundle P on a compact Riemann surface Σ [27, 2]. We describe this in some detail. Let $\mathcal{A}$ be the space of Lie-algebra valued connections on the principal bundle P. Let $\mathcal{N}$ be the moduli space of flat connections (i.e., the space of flat connections modulo the gauge group). The prequantum line bundle is the the determinant line bundle of the Cauchy–Riemann operator, namely, $\mathcal{L} = \wedge^{\mathrm{top}}(\mathrm{Ker}\,\bar{\partial}_A)^* \otimes \wedge^{\mathrm{top}}(\mathrm{Coker}\,\bar{\partial}_A)$. It carries the Quillen metric such that the canonical unitary connection has a curvature form which coincides with the natural Kähler form on the moduli space of flat connections on vector bundles over the Riemann surface of a given rank [20]. To elaborate, on the affine space of all connections, there is a natural symplectic form, proportional to $\mathrm{Tr}\int_\Sigma \alpha \wedge \beta$, where $\alpha, \beta \in T_A\mathcal{A} = \Omega^1(M, \mathrm{ad}\,P)$. It can be shown, using a moment map construction, that this symplectic form descends to the moduli space of flat connections. One can show that the determinant line bundle equipped with the Quillen metric has curvature proportional to this symplectic form [20].

Inspired by this construction, we constructed a prequantum line bundle on the moduli space of solutions to the vortex equations [9]. In this paper we geometrically quantize the hyperKähler structure in the Hitchin system. We elaborate on this.

The self-duality equations on a Riemann surface arise from a dimensional reduction of self-dual Yang–Mills equations from 4 to 2 dimensions [14]. They have been studied extensively in [14]. They are as follows. Let M be a compact Riemann surface of genus $g > 1$ and let P be a principal $U(n)$-bundle over M. Let A be a unitary connection on P, i.e., $A = A^{(1,0)} + A^{(0,1)}$ such that $A^{(1,0)} = -A^{(0,1)*}$, where $*$ denotes conjugate transpose [12, 18]. Thus we can identify the space of all unitary connections with its $(0,1)$-part, i.e., $A^{(0,1)}$. Let $\Phi^{1,0}$ be a complex Higgs field such that $\Phi^{1,0} \in \mathcal{H} = \Omega^{1,0}(M; \mathrm{ad}\, P \otimes \mathbb{C})$.

Note. In [14], this $\Phi^{1,0}$ is written as Φ. But we will be using the present notation since we will define $\Phi^{0,1}$ as well and our $\Phi = \Phi^{1,0} + \Phi^{0,1}$.

The pair $(A, \Phi^{1,0})$ will be said to satisfy the self-duality equations if,

$$F(A) = -[\Phi^{(1,0)}, \Phi^{(1,0)*}], \tag{1.1}$$

$$d''_A \Phi^{(1,0)} = 0. \tag{1.2}$$

Here $F(A)$ is the curvature of the connection A. The operator d''_A is the $(0,1)$ part of the extension of the covariant derivative operator to act on $\Omega^{1,0}(M, \mathrm{ad}\, P \otimes \mathbb{C})$. Also $\Phi^{(1,0)*} = \phi^* d\bar{z}$, where ϕ^* is taking conjugate transpose of the matrix of ϕ. There is a gauge group acting on the space of (A, Φ) which leave the equations invariant. If g is an $U(n)$ gauge transformation then (A_1, Φ_1) and (A_2, Φ_2) are gauge equivalent if $d_{A_2} g = g d_{A_1}$ and $\Phi_2 g = g \Phi_1$ [14, p. 69]. Taking the quotient by the gauge group of the solution space to (1.1) and (1.2) gives the moduli space of solutions to these equations and is denoted by $\mathcal{M}$. Hitchin shows that there is a natural metric on the moduli space $\mathcal{M}$ and further proves that the metric is hyperKähler [14].

This paper is a sequel of the paper [8], where we constructed the prequantum line bundle on $\mathcal{M}$ whose curvature is the first symplectic form of [14]. In [8], we had explicitly given this symplectic form , the metric and the moment map construction (details of which are missing in [14]). But the prequantum line bundle we constructed in [8] is a bit unnatural, since we used $\overline{\partial} + \overline{A_0^{(1,0)}} + \overline{\Phi^{(1,0)}}$ which gauge transforms like $\bar{g}(\overline{\partial} + \overline{A_0^{(1,0)}} + \overline{\Phi^{(1,0)}})\bar{g}^{-1}$. In this paper, we first rectify this and construct the prequantum bundle on $\mathcal{M}$ corresponding to the first symplectic form using $\bar{\partial} + A_0^{(0,1)} + \Phi^{(0,1)}$ which gauge transforms like $g(\bar{\partial} + A_0^{(0,1)} + \Phi^{(0,1)})g^{-1}$ which is more natural to use. Next, in this paper, we construct the prequantum line bundles on $\mathcal{M}$ corresponding to the other two symplectic forms which give rise to the hyperKähler structure. We show the metric, the three symplectic forms, the three complex structures, and the three prequantum lines bundles explicitly. In the next section we discuss the holomorphicity of the prequantum line bundles w.r.t. the three complex structures. In the last section, we show how the prequantum line bundles are related to cocycle line bundles in Chern–Simons gauge theory with complex gauge group at least in the case when the moduli space is smooth. In [28], Witten had quantized one of the symplectic forms from Chern–Simons gauge theory with complex gauge group. We find that by introducing a parameter λ we can obtain all three symplectic forms from this theory — though applications of this as in [11] is still a topic of research.

Papers which may be of interest in this context are [3, 4, 6, 15, 23, 28]. These papers use algebraic geometry and algebraic topology and may provide alternative methods to quantizing the hyperKähler system, though ours is the only paper we have seen in which all three quantizations appear explicitly. Our method is very elementary and we explictly construct the prequantum line bundles. The only machinery we use is Quillen's construction of the determinant line bundle [20]. It would be interesting to see if there is any relation between the present quantizations and the ones appearing in the previous papers.

After writing the paper, the author found Kapustin and Witten's paper [17], where they have applied Beilinson and Drinfeld's quantization of one of the symplectic forms of the Hitchin system to study the geometric Langlands program. We should mention that though the metric on the moduli space is exactly the same their complex structure $\mathcal{J}$ is our $-\mathcal{K}$ and their $\mathcal{K}$ is our $\mathcal{J}$. In the end of section 2, we will explain the dictionary between the physicists' notation for the symplectic forms in [17] and our notation. It would be interesting to see if there is any relevance of all the three quantizations obtained here to geometric Langlands program.

Geometric prequantization of the moduli spaces of vector bundles (with fixed determinant) over a Riemann surface can be found in [24]. Some interesting applications of the determinant line bundles to geometry and physics can be found in [25, 26].

2 Symplectic and hyperKähler structures

Let the configuration space be defined as $\mathcal{C} = \{(A^{0,1}, \Phi^{1,0}) | A^{0,1} \in \mathcal{A}, \Phi^{1,0} \in \mathcal{H}\}$ where $\mathcal{A}$ is the space of unitary connections on P, identified with its $A^{(0,1)}$ part and $\mathcal{H} = \Omega^{(1,0)}(M, \mathrm{ad}\, P \otimes \mathbb{C})$ is the space of Higgs field. We can extend the Higgs field $\Phi^{(1,0)}$ by its $(0,1)$ part by defining $\Phi^{(0,1)} = -\Phi^{(1,0)*}$, where $*$ is conjugate transpose. But only the $(1,0)$ parts belong to $\mathcal{H}$ and appear in the equation (1.1) and (1.2). Unitary connections satisfy $A = A^{(1,0)} + A^{(0,1)}$, where $A^{(1,0)*} = -A^{(0,1)}$. Thus we can identify the space of unitary connections with its $(0,1)$ part and the tangent space is also $(0,1)$ part of a 1-form Let $\alpha^{(0,1)}, \beta^{(0,1)} \in T_A\mathcal{A} = \Omega^{(0,1)}(M, \mathrm{ad}\, P \otimes \mathbb{C})$ such that $\alpha^{(1,0)} = -\alpha^{(0,1)*}$ and $\beta^{(1,0)} = -\beta^{(0,1)*}$. Let $\gamma^{(1,0)}, \delta^{(1,0)} \in T_\Phi\mathcal{H} = \Omega^{(1,0)}(M, \mathrm{adP} \otimes \mathbb{C})$. Let us extend $\gamma^{(1,0)}, \delta^{(1,0)}$ by defining $\gamma^{(0,1)} = -\gamma^{(1,0)*}$ and $\delta^{(0,1)} = -\delta^{(1,0)*}$. Thus, $\alpha = \alpha^{(0,1)} + \alpha^{(1,0)}$, $\beta = \beta^{(0,1)} + \beta^{(1,0)}$, $\gamma = \gamma^{(0,1)} + \gamma^{(1,0)}$, and $\delta = \delta^{(0,1)} + \delta^{(1,0)}$, i.e., $\alpha, \beta, \gamma, \delta \in \Omega^1(M, \mathrm{ad}\, P)$. Let X, Y be two tangent vectors to the configuration space, given by $X = (\alpha^{(0,1)}, \gamma^{(1,0)})$, and $Y = (\beta^{(0,1)}, \delta^{(1,0)})$. As in [8], let us define a metric on the complex configuration space

$$\begin{aligned} g(X,Y) &= g((\alpha^{(0,1)}, \gamma^{(1,0)}), (\beta^{(0,1)}, \delta^{(1,0)})) \\ &= -\int_M \mathrm{Tr}(\alpha \wedge *_1\beta) - 2\,\mathrm{Im}\int_M \mathrm{Tr}(\gamma^{(1,0)} \wedge *_2\delta^{(1,0)\mathrm{tr}}) \\ &= -2\,\mathrm{Im}\int_M \mathrm{Tr}(\alpha^{(0,1)} \wedge \beta^{(1,0)}) - 2\,\mathrm{Im}\int_M \mathrm{Tr}(\gamma^{(1,0)} \wedge \delta^{(1,0)*}) \\ &= 2\,\mathrm{Im}\int_M \mathrm{Tr}(\alpha^{(0,1)} \wedge \beta^{(0,1)*}) - 2\,\mathrm{Im}\int_M \mathrm{Tr}(\gamma^{(1,0)} \wedge \delta^{(1,0)*}). \end{aligned} \tag{2.1}$$

Here $*_1$ denotes the Hodge star taking dx forms to dy forms and dy forms to $-dx$ forms (i.e., $*_1(\eta dz) = -i\eta dz$ and $*_1(\bar\eta d\bar z) = i\bar\eta d\bar z$) and $*_2$ denotes the operation (another Hodge star) such that $*_2(\eta dz) = \bar\eta d\bar z$ and $*_2(\bar\eta d\bar z) = -\eta dz$. To get the metric in its final form, we have used the fact that

$$\begin{aligned} 2\,\mathrm{Im}\int_M \mathrm{Tr}(\alpha^{(0,1)} \wedge \beta^{(1,0)}) &= \frac{1}{\mathrm{i}}\int_M \mathrm{Tr}\left[\alpha^{(0,1)} \wedge \beta^{(1,0)} - \overline{\alpha^{(0,1)}} \wedge \overline{\beta^{(1,0)}}\right] \\ &= \frac{1}{\mathrm{i}}\int_M \mathrm{Tr}\left[\alpha^{(0,1)} \wedge \beta^{(1,0)} - (-\overline{\alpha^{(0,1)\mathrm{tr}}}) \wedge (-\overline{\beta^{(1,0)\mathrm{tr}}})\right] \\ &= -\mathrm{i}\int_M \mathrm{Tr}\left[\alpha^{(0,1)} \wedge \beta^{(1,0)} - \alpha^{(1,0)} \wedge \beta^{(0,1)}\right] \\ &= \int_M \mathrm{Tr}(\alpha \wedge *_1\beta). \end{aligned}$$

Note that the metric can also be expanded and written in the form

$$g(X,Y) = \mathrm{i}\left[\int_M \mathrm{Tr}(\alpha^{0,1} \wedge \beta^{1,0} - \alpha^{1,0} \wedge \beta^{0,1}) - \int_M \mathrm{Tr}(\gamma^{1,0} \wedge \delta^{0,1} + \gamma^{0,1} \wedge \delta^{1,0})\right]$$

We check that this coincides with the metric on the moduli space $\mathcal{M}$ given by [14, p. 79, p. 88]. On $T_{(A,\Phi)}\mathcal{C} = T_A\mathcal{A} \times T_\Phi\mathcal{H}$ which is $\Omega^{(0,1)}(M, \mathrm{ad}\, P \otimes$

$\mathbb{C}) \times \Omega^{(1,0)}(M, \operatorname{ad} P \otimes \mathbb{C})$, Hitchin defines a metric g_1 such that

$$g_1((\alpha^{(0,1)}, \gamma^{(1,0)}), (\alpha^{(0,1)}, \gamma^{(1,0)})) = 2\mathrm{i} \int_M \operatorname{Tr}(\alpha^{(0,1)*} \wedge \alpha^{(0,1)}) + 2\mathrm{i} \int_M \operatorname{Tr}(\gamma^{(1,0)} \wedge \gamma^{(1,0)*}).$$

$*$ denotes conjugate transpose as usual. Let $\gamma^{(1,0)} = c dz$, where c is a matrix. On $T_{(A,\Phi)}\mathcal{C}$, our metric

$$\begin{aligned}
&g((\alpha, \gamma^{(1,0)}), (\alpha, \gamma^{(1,0)})) \\
&= -\int_M \operatorname{Tr}(\alpha \wedge *_1 \alpha) - 2 \operatorname{Im} \int_M \operatorname{Tr}(\gamma^{(1,0)} \wedge *_2 \gamma^{(1,0)\mathrm{tr}}) \\
&= -\int_M \operatorname{Tr}\big[(\alpha^{(1,0)} + \alpha^{(0,1)}) \wedge (-\mathrm{i}\alpha^{(1,0)} + \mathrm{i}\alpha^{(0,1)})\big] - 2 \operatorname{Im} \int_M \operatorname{Tr}(c dz \wedge c^* d\bar{z}) \\
&= 2\mathrm{i} \int_M \operatorname{Tr}(\alpha^{(0,1)} \wedge \alpha^{(1,0)}) - 2 \operatorname{Im} \int_M (-2\mathrm{i}) \operatorname{Tr}(cc^*) dx \wedge dy \\
&= -2\mathrm{i} \int_M \operatorname{Tr}(\alpha^{(0,1)} \wedge \alpha^{(0,1)*}) + 4 \int_M \operatorname{Re}(\operatorname{Tr}(cc^*)) dx \wedge dy \\
&= 2\mathrm{i} \int_M \operatorname{Tr}(\alpha^{(0,1)*} \wedge \alpha^{(0,1)}) + 2\mathrm{i} \int_M (-2\mathrm{i}) \operatorname{Tr}(cc^*) dx \wedge dy \\
&= 2\mathrm{i} \int_M \operatorname{Tr}(\alpha^{(0,1)*} \wedge \alpha^{(0,1)}) + 2\mathrm{i} \int_M \operatorname{Tr}(cc^*) dz \wedge d\bar{z} \\
&= 2\mathrm{i} \int_M \operatorname{Tr}(\alpha^{(0,1)*} \wedge \alpha^{(0,1)}) + 2\mathrm{i} \int_M \operatorname{Tr}(\gamma^{(1,0)} \wedge \gamma^{(1,0)*}),
\end{aligned}$$

where we have used the fact that $\alpha^{(1,0)} = -\alpha^{(0,1)*}$ and that $\operatorname{Tr}(cc^*)$ is real. Thus we get the same metric as Hitchin does.

To give the complex structures of the hyperKähler structure explicitly, (which is missing in [14]), let us define three almost complex structures acting on the tangent space to the configuration space, i.e. acting on $T = \Omega^{(0,1)}(M, \operatorname{ad} P \otimes \mathbb{C}) \times \Omega^{(1,0)}(M, \operatorname{ad} P \otimes \mathbb{C})$,

$$\mathcal{I} = \begin{bmatrix} \mathrm{i} & 0 \\ 0 & \mathrm{i} \end{bmatrix}, \quad \mathcal{J} = \begin{bmatrix} 0 & \mathrm{i}\tilde{*}_2 \\ \mathrm{i}\tilde{*}_2 & 0 \end{bmatrix}, \quad \mathcal{K} = \begin{bmatrix} 0 & -\tilde{*}_2 \\ -\tilde{*}_2 & 0 \end{bmatrix},$$

where $\tilde{*}_2(\alpha) = *_2 \alpha^{\mathrm{tr}}$ is such that $\tilde{*}_2(\mathrm{i}\alpha) = -\mathrm{i}\tilde{*}_2\alpha$.

Note. A more detailed description of the three complex structures is given in the section (4), where we discuss holomorphicity of the prequantum line bundles. There we give it in physicists' notation commensurate with [17].

These three complex structures satisfy the quarternionic algebra of matrices acting on $T = \Omega^{(0,1)}(M, \mathrm{ad}\, P \otimes \mathbb{C}) \times \Omega^{(1,0)}(M, \mathrm{ad}\, P \otimes \mathbb{C})$:

$$\begin{aligned} \mathcal{I}^2 = \mathcal{J}^2 &= \mathcal{K}^2 = -1, \\ \mathcal{I}\mathcal{J} &= -\mathcal{J}\mathcal{I} = \mathcal{K}, \\ \mathcal{J}\mathcal{K} &= -\mathcal{K}\mathcal{J} = I, \\ \mathcal{K}\mathcal{I} &= -\mathcal{I}\mathcal{K} = \mathcal{J}. \end{aligned}$$

We define three symplectic forms as follows:

$$\begin{aligned} \Omega(X, Y) &= g(X, \mathcal{I}Y), \\ \mathcal{Q}_1(X, Y) &= g(X, \mathcal{J}Y), \\ \mathcal{Q}_3(X, Y) &= g(X, \mathcal{K}Y) \end{aligned}$$

Let $X = (\alpha^{(0,1)}, \gamma^{(1,0)})$, $Y = (\beta^{(0,1)}, \delta^{(1,0)})$ be two tangent vectors belonging to T.

$$\begin{aligned} &\Omega((\alpha^{(0,1)}, \gamma^{(1,0)}), (\beta^{(0,1)}, \delta^{(1,0)})) \\ &\quad = g((\alpha^{(0,1)}, \gamma^{(1,0)}), (\mathrm{i}\beta^{(0,1)}, \mathrm{i}\delta^{(1,0)})) \\ &\quad = 2\,\mathrm{Im} \int_M \mathrm{Tr}(\alpha^{(0,1)} \wedge (\mathrm{i}\beta^{(0,1)})^*) - 2\,\mathrm{Im} \int_M \mathrm{Tr}(\gamma^{(1,0)} \wedge (\mathrm{i}\delta^{(1,0)})^*) \\ &\quad = 2\mathrm{Re} \int_M \mathrm{Tr}(\alpha^{(0,1)} \wedge \beta^{(1,0)}) - 2\,\mathrm{Im} \int_M \mathrm{Tr}((\mathrm{i})\gamma^{(0,1)} \wedge (\delta^{(0,1)}) \\ &\quad = \int_M \mathrm{Tr}(\alpha \wedge \beta) - 2\mathrm{Re} \int_M \mathrm{Tr}(\gamma^{(1,0)} \wedge \delta^{(0,1)}) \\ &\quad = \int_M \mathrm{Tr}(\alpha \wedge \beta) - \int_M \mathrm{Tr}(\gamma \wedge \delta), \end{aligned}$$

where we have used the fact that

$$2\mathrm{Re} \int_M \mathrm{Tr}(\alpha^{(0,1)} \wedge \beta^{(1,0)}) = \int_M \mathrm{Tr}(\alpha \wedge \beta)$$

which follows from the fact that

$$\begin{aligned} \int_M \mathrm{Tr}(\overline{\alpha^{(0,1)}} \wedge \overline{\beta^{(1,0)}}) &= \int_M \mathrm{Tr}\left[(-\overline{\alpha^{(0,1)\mathrm{tr}}}) \wedge (-\overline{\beta^{(1,0)\mathrm{tr}}})\right] \\ &= \int_M \mathrm{Tr}(\alpha^{(1,0)} \wedge \beta^{(0,1)}). \end{aligned} \tag{2.2}$$

Following the ideas in [14], we had shown in [8] by a moment map construction that this form descends to a symplectic form on the moduli space $\mathcal{M}$. (The explicit construction of this form is missing in [14]). The first equation, i.e., equation (1.1) gives the moment map for this symplectic form.

$$\begin{aligned}
\mathcal{Q}_1(X,Y) &= g(X,\mathcal{J}Y) = g((\alpha^{(0,1)},\gamma^{(1,0)}),(\mathrm{i}\tilde{*}_2\delta^{(1,0)},\mathrm{i}\tilde{*}_2\beta^{(0,1)}))\\
&= g((\alpha^{(0,1)},\gamma^{(1,0)}),(-\mathrm{i}\delta^{(0,1)},\mathrm{i}\beta^{(1,0)}))\\
&= -2\mathrm{Re}\int_M \mathrm{Tr}(\alpha^{(0,1)}\wedge\delta^{(1,0)}) - 2\mathrm{Re}\int_M \mathrm{Tr}(\gamma^{(1,0)}\wedge\beta^{(0,1)})\\
&= -\left[\int_M \mathrm{Tr}(\alpha\wedge\delta) + \int_M \mathrm{Tr}(\gamma\wedge\beta)\right],\\
\mathcal{Q}_2(X,Y) &= g(X,\mathcal{K}Y) = g((\alpha^{(0,1)},\gamma^{(1,0)}),(-\tilde{*}_2\delta^{(1,0)},-\tilde{*}_2\beta^{(0,1)}))\\
&= g((\alpha^{(0,1)},\gamma^{(1,0)}),(\delta^{(0,1)},-\beta^{(1,0)}))\\
&= -2\,\mathrm{Im}\int_M \mathrm{Tr}(\alpha^{(0,1)}\wedge\delta^{(1,0)}) - 2\,\mathrm{Im}\int_M \mathrm{Tr}(\gamma^{(1,0)}\wedge\beta^{(0,1)})\\
&= \int_M \mathrm{Tr}(\alpha\wedge\tilde{\delta} + \tilde{\gamma}\wedge\beta),
\end{aligned}$$

where $\tilde{\delta} = \mathrm{i}\,(\delta^{(1,0)} - \delta^{(0,1)})$ and $\tilde{\gamma} = \mathrm{i}\,(\gamma^{(1,0)} - \gamma^{(0,1)})$. Next, following [14, p. 90], we define a symplectic form

$$\mathcal{Q}(X,Y) = 2\mathrm{Tr}\int_M (\delta^{(1,0)}\wedge\alpha^{(0,1)} - \gamma^{(1,0)}\wedge\beta^{(0,1)}) = (\mathcal{Q}_1 + \mathrm{i}\mathcal{Q}_2)(X,Y).$$

In [14], Hitchin shows that this form $\mathcal{Q}$ descends as a symplectic form to the moduli space of solution of the self-duality equations. He proves this using a moment map construction, i.e., the second equation, equation (1.2), gives the moment map for this symplectic form. (Note that the factor of two does not alter the moment map construction). Using his method, it can be shown that the metric g descends to the moduli space $\mathcal{M}$ and is hyperKähler and the three symplectic forms are exactly Ω, $\mathcal{Q}_1$, $\mathcal{Q}_2$.

2.1 Exactness of forms $\mathcal{Q}_1$ and $\mathcal{Q}_2$

In this section, we first establish a link between the notation in [17, p. 40–44] and our notation. We will take the example of $\mathcal{Q}_1$ and $\mathcal{Q}_2$ and explain the notation.

In the Kapustin Witten notation, $\Phi = \phi_z dz + \phi_{\bar{z}} d\bar{z} = \Phi^{1,0} + \Phi^{0,1}$ and let $X = (\alpha,\gamma)$, $Y = (\beta,\delta)$. We will denote by ω_1 their ω_K and by ω_2 their ω_J.

Their J is our $-K$ and their K is our J, their ω_I is $(-1/2\pi)\Omega$ and we will just now show that their ω_J is $(1/2\pi)Q_2$ and ω_K is $(-1/2\pi)Q_1$.

$$\begin{aligned}\omega_1(X,Y) &= \frac{\mathrm{i}}{2\pi}\int_M \mathrm{i}(dz\wedge d\bar z)\mathrm{Tr}\Big[(\delta\phi_{\bar z}\otimes\delta A_z - \delta A_z\otimes\delta\phi_{\bar z})\\ &\quad - (\delta\phi_z\otimes\delta A_{\bar z} - \delta A_{\bar z}\otimes\delta\phi_z)\Big](X,Y)\\ &= \frac{-1}{2\pi}\Bigg[\int_M \mathrm{Tr}(-\gamma^{0,1}\wedge\beta^{1,0} - \alpha^{1,0}\wedge\delta^{0,1})\\ &\quad - \mathrm{Tr}(\gamma^{1,0}\wedge\beta^{0,1} + \alpha^{0,1}\wedge\delta^{1,0})\Bigg]\\ &= \frac{1}{2\pi}\int_M \mathrm{Tr}(\gamma\wedge\beta + \alpha\wedge\delta)\\ &= \frac{-1}{2\pi}\mathcal{Q}_1(X,Y).\end{aligned}$$

Similarly,

$$\begin{aligned}\omega_2(X,Y) &= \frac{1}{2\pi}\int_M \mathrm{i}(dz\wedge d\bar z)\mathrm{Tr}\Big[(\delta\phi_{\bar z}\wedge\delta A_z) + (\delta\phi_z\wedge\delta A_{\bar z})\Big](X,Y)\\ &= \frac{1}{2\pi}\int_M \mathrm{i}(dz\wedge d\bar z)\mathrm{Tr}\Big[(\delta\phi_{\bar z}\otimes\delta A_z - \delta A_z\otimes\delta\phi_{\bar z})\\ &\quad + (\delta\phi_z\otimes\delta A_{\bar z} - \delta A_{\bar z}\otimes\delta\phi_z)\Big](X,Y)\\ &= \frac{\mathrm{i}}{2\pi}\int_M \mathrm{Tr}\Big[(-\gamma^{0,1}\wedge\beta^{1,0} - \alpha^{1,0}\wedge\delta^{0,1}) + \gamma^{1,0}\wedge\beta^{0,1} + \alpha^{0,1}\wedge\delta^{1,0})\Big]\\ &= \frac{1}{2\pi}\mathcal{Q}_2(X,Y).\end{aligned}$$

Now as in [17, p. 44], $\mathcal{Q}_1 = \delta\theta_1$, where

$$\theta_1 = -\mathrm{i}\int_M \mathrm{i}(dz\wedge d\bar z)\mathrm{Tr}(\phi_{\bar z}\delta A_z - \phi_z\delta A_{\bar z}) = -\int_M \mathrm{Tr}(\Phi\wedge\delta A)$$

such that $\theta_1(\alpha,\gamma) = -\int_M \mathrm{Tr}(\Phi\wedge\alpha)$.

Similarly, $\mathcal{Q}_2 = \delta\theta_2$, where $\theta_2 = \int_M \mathrm{i}(dz\wedge d\bar z)\mathrm{Tr}(\phi_{\bar z}\delta A_z + \phi_z\delta A_{\bar z})$ such that $\theta_2(\alpha,\gamma) = \int_M \mathrm{Tr}(\Phi^{1,0}\wedge\alpha^{0,1} - \Phi^{0,1}\wedge\alpha^{1,0})$.

Now θ_1 and θ_2 both descend as 1-forms on the moduli space $\mathcal{M}$ since they are gauge invariant. Recall Φ, $\tilde\Phi$, α transform by the adjoint representation of $U(n)$ (unlike A) and keeps the 1-forms θ_1 and θ_2 gauge invariant (since we are taking trace).

Thus $\mathcal{Q}_1$ and $\mathcal{Q}_2$ are both exact.

3 Prequantum line bundles

In this section, we review Quillen's determinant line bundle of Cauchy–Riemann operators which will enable us to construct prequantum line bundles corresponding to the hyperKähler structure in the Hitchin system.

First, let us note that a connection A on a principal bundle induces a connection on any associated vector bundle E. We will denote this connection also by A, since the same " Lie-algebra valued 1-form" A (modulo representations) gives a covariant derivative operator enabling you to take derivatives of sections of E [19, p. 348]. A very clear description of the determinant line bundle can be found in [5, 20]. Here we mention the formula for the Quillen curvature of the determinant line bundle $\wedge^{\text{top}}(\text{Ker}\,\bar{\partial}_A)^* \otimes \wedge^{\text{top}}(\text{Coker}\,\bar{\partial}_A) = \det(\bar{\partial}_A)$, where $\bar{\partial}_A = \bar{\partial} + A^{(0,1)}$, given the canonical unitary connection ∇_Q, induced by the Quillen metric [20]. Recall that the affine space $\mathcal{A}$ (notation as in [20]) is an infinite-dimensional Kähler manifold. Here each connection is identified with its $(0,1)$ part. Since the total connection is unitary (i.e., of the form $A = A^{(1,0)} + A^{(0,1)}$, where $A^{(1,0)} = -A^{(0,1)*}$) this identification is easy. In fact, for every $A \in \mathcal{A}$, $T'_A(\mathcal{A}) = \Omega^{0,1}(M, \text{ad}\, P \otimes \mathbb{C})$ and the corresponding Kähler form is given by

$$F(\alpha, \beta) = 2\text{Re}\int_M \text{Tr}(\alpha^{(0,1)} \wedge \beta^{(0,1)*}) = -2\text{Re}\int_M \text{Tr}(\alpha^{(0,1)} \wedge \beta^{(1,0)}),$$

where $\alpha^{(0,1)}$, $\beta^{(0,1)} \in \Omega^{0,1}(M, \text{ad}\, P \otimes \mathbb{C})$, and $\beta^{(1,0)} = -\beta^{(0,1)*}$. It is skew symmetric, if you interchange $\alpha^{(0,1)} = \text{Ad}\,\bar{z}$ and $\beta^{(0,1)} = Bd\bar{z}$ (follows from the fact that $\text{Im}(\text{Tr}(AB^*)) = -\text{Im}(\text{Tr}(BA^*))$ for matrices A and B, using once again $d\bar{z} \wedge dz$ is imaginary). Let $\alpha = \alpha^{(0,1)} + \alpha^{(1,0)}$, $\beta = \beta^{(0,1)} + \beta^{(1,0)}$. It is clear from the fact that $\alpha^{(1,0)} = -\alpha^{(0,1)*}$ and $\beta^{(1,0)} = -\beta^{(0,1)*}$ we have

$$F(\alpha, \beta) = -\int_M \text{Tr}(\alpha \wedge \beta).$$

(see for instance [15, p. 358]). Let ∇_Q be the connection induced from the Quillen metric and $\mathcal{F}(\nabla_Q)$ be the Quillen curvature. Then one has,

$$\mathcal{F}(\nabla_Q) = \frac{\mathrm{i}}{2\pi} F$$

3.1 Quantization of the moduli space $\mathcal{M}$

In this section, we will show that for each of the symplectic forms Ω, $\mathcal{Q}_1$ and $\mathcal{Q}_2$ there are prequantum line bundles $\mathcal{P}$, $\mathcal{E}$, $\mathcal{N}$ whose respective curvatures are these symplectic forms.

First, we note that to the connection A we can add any one form and still obtain a derivative operator.

On the principal bundle P on the Riemann surface, we can define new connections, $A \pm \Phi = A^{(0,1)} + A^{(1,0)} \pm \Phi^{(0,1)} \pm \Phi^{(1,0)}$ and $A \mp \tilde{\Phi} = A^{(0,1)} + A^{(1,0)} \pm \mathrm{i}\Phi^{(0,1)} \mp \mathrm{i}\Phi^{(1,0)}$, where $\tilde{\Phi} = i(\Phi^{(1,0)} - \Phi^{(0,1)})$, i.e., $\tilde{\Phi}^{(0,1)} = -\mathrm{i}\Phi^{(0,1)}$ and $\tilde{\Phi}^{(1,0)} = \mathrm{i}\Phi^{(1,0)}$. (Note that, as usual, $\Phi^{(1,0)*} = -\Phi^{(0,1)}$ and $\tilde{\Phi}^{(1,0)*} = -\tilde{\Phi}^{(0,1)}$).

Definitions

Let us denote by $\mathcal{L} = \det(\bar{\partial} + A^{0,1})$ a determinant bundle on $\mathcal{A}$.

Let $\mathcal{R} = \det(\bar{\partial} + A_0^{(0,1)} + \Phi^{(0,1)})$, where A_0 is a connection whose gauge equivalence class is fixed, i.e. A_0 is allowed to change only in the gauge direction.

Let $\mathcal{P} = \mathcal{L}^{-2} \otimes \mathcal{R}^2$ denote a line bundle over $\mathcal{C} = \mathcal{A} \times \mathcal{H}$.

(This combination will give the prequantum line bundle corresponding to Ω). Let us define $\mathcal{E}_\pm = \det(\bar{\partial} + A^{(0,1)} \pm \Phi^{(0,1)})$ on the affine space $\mathcal{B}_\pm = \{A^{(0,1)} \pm \Phi^{(0,1)} | A^{(0,1)} \in \mathcal{A}, \Phi^{(0,1)*} = -\Phi^{(1,0)} \in \mathcal{H}\}$ which is isomorphic to $\mathcal{C} = \{A^{(0,1)} \in \mathcal{A}\} \times \{\Phi^{(1,0)} \in \mathcal{H}\} = \mathcal{A} \times \mathcal{H}$.

Let $\mathcal{E} = \mathcal{E}_+ \otimes (\mathcal{E}_-)^{-1}$ (We take this combination because it will give the prequantum line corresponding to $\mathcal{Q}_1$).

Similarly let us define $\mathcal{N}_\pm = \det(\bar{\partial} + A^{(0,1)} \pm \mathrm{i}\Phi^{(0,1)})$ on the affine space $\mathcal{V}_\pm = \{A^{(0,1)} \pm \mathrm{i}\Phi^{(0,1)} | A^{(0,1)} \in \mathcal{A}, \Phi^{(0,1)*} = -\Phi^{(1,0)} \in \mathcal{H}\}$ which is isomorphic to $\mathcal{C} = \mathcal{A} \times \mathcal{H}$.

Let $\mathcal{N} = \mathcal{N}_+ \otimes (\mathcal{N}_-)^{-1}$ (Once, again this will be the prequantum line bundle corresponding to $\mathcal{Q}_2$).

Lemma 3.1. *$\mathcal{P}$, $\mathcal{E}_\pm$ and $\mathcal{N}_\pm$ are well-defined line bundles over $\mathcal{M} \subset \mathcal{C}/\mathcal{G}$, where $\mathcal{G}$ is the gauge group.*

Proof. Let us consider the Cauchy–Riemann operator $D = \bar{\partial} + A^{(0,1)} + \Phi^{(0,1)}$ which appears in $\mathcal{E}_+$. All the other cases are analogous. Under gauge transformation $D = \bar{\partial} + A^{(0,1)} + \Phi^{(0,1)} \to D_g = g(\bar{\partial} + A^{(0,1)} + \Phi^{(0,1)})g^{-1}$, since it is the $(0,1)$ part of the connection operator $d + A + \Phi$ which transforms in the same way. We can show that the operators D and D_g have isomorphic kernel and cokernel and their corresponding Laplacians have the same spectrum and the eigenspaces are of the same dimension. Let Δ denote the Laplacian corresponding to D and Δ_g that corresponding to

D_g.The Laplacian is $\Delta = \tilde{D}D$, where $\tilde{D} = \partial + A^{(1,0)} + \Phi^{(1,0)}$, where recall $A^{(1,0)*} = -A^{(0,1)}$ and $\Phi^{(1,0)*} = -\Phi^{(0,1)}$. Note that $\tilde{D} \to \tilde{D}_g = g\tilde{D}g^{-1}$ under gauge transformation since it is the $(1,0)$ part of the connection operator $d + A + \Phi$ which transforms in the same way. Thus $\Delta_g = g\Delta g^{-1}$. Thus the isomorphism of eigenspaces of Δ and Δ_g is $s \to gs$. We describe here how to define the line bundle on the moduli space. Let $K^a(\Delta)$ is the direct sum of eigenspaces of the operator Δ of eigenvalues $< a$, over the open subset $U^a = \{A^{(0,1)} + \Phi^{(0,1)} | a \notin \operatorname{Spec}\Delta\}$ of the affine space $\mathcal{B}_+$. The determinant line bundle is defined using the exact sequence

$$0 \longrightarrow \operatorname{Ker} D \longrightarrow K^a(\Delta) \longrightarrow D(K^a(\Delta)) \longrightarrow \operatorname{Coker} D \longrightarrow 0$$

Thus one identifies $\det D = \wedge^{\text{top}}(\operatorname{Ker} D)^* \otimes \wedge^{\text{top}}(\operatorname{Coker} D)$ with $\wedge^{\text{top}}(K^a(\Delta))^* \otimes \wedge^{\text{top}}(D(K^a(\Delta)))$ (see [5] for more details) and there is an isomorphism of the fibers as $D \to D_g$. Thus one can identify

$$\wedge^{\text{top}}(K^a(\Delta))^* \otimes \wedge^{\text{top}}(D(K^a(\Delta))) \equiv \wedge^{\text{top}}(K^a(\Delta_g))^* \otimes \wedge^{\text{top}}(D(K^a(\Delta_g))).$$

By extending this definition from U^a to $V^a = \{(A^{(0,1)}, \Phi^{(1,0)}) | a \notin \operatorname{Spec}\Delta\}$, an open subset of $\mathcal{C}$, we can define the fiber over the quotient space $V^a/\mathcal{G}$ to be the equivalence class of this fiber. Covering $\mathcal{C}$ by open sets of the type V^a enables us to define it on $\mathcal{C}/\mathcal{G}$. Then we restrict it to the moduli space $\mathcal{M} \subset \mathcal{C}/\mathcal{G}$.

Similarly one can deal with the other terms in $\mathcal{E}$, $\mathcal{P}$ and $\mathcal{N}$. $\square$

3.2 Curvatures and symplectic forms

Recall, $\alpha \in \Omega^1(M, \operatorname{ad} P)$ has the decomposition $\alpha = \alpha^{(1,0)} + \alpha^{(0,1)}$, where $\alpha^{(1,0)} = -\alpha^{(0,1)*}$. Similar decomposition holds for $\beta, \gamma, \delta \in \Omega^1(M, \operatorname{ad} P)$.

Let $p = (A, \Phi) \in S$, where S is the space of solutions to Hitchin equations (1.1) and (1.2). Let $X, Y \in T_{[p]}\mathcal{M}$. We write $X = (\alpha, \gamma)$ and $Y = (\beta, \delta)$, where $\alpha^{(0,1)}, \beta^{(0,1)} \in T_A(\mathcal{A}^{(0,1)}) = \Omega^{(0,1)}(M, \operatorname{ad} P \otimes \mathbb{C})$ and $\gamma^{(1,0)}, \delta^{(1,0)} \in T_\Phi\mathcal{H} = \Omega^{(1,0)}(M, \operatorname{ad} P \otimes \mathbb{C})$. Since $T_{[p]}\mathcal{M}$ can be identified with a subspace in T_pS orthogonal to T_pO_p (the tangent space to the gauge orbit) then X, Y can be said to satisfy (a) $X, Y \in T_pS$ i.e., they satisfy linearization of (1.1) and (1.2) and (b) X, Y are orthogonal to T_pO_p, the tangent space to the gauge orbit.

Let $\mathcal{F}_{\mathcal{L}^{-2}}$, $\mathcal{F}_{\mathcal{R}^2}$, denote the Quillen curvatures of the determinant line bundles $\mathcal{L}^{-2}$, $\mathcal{R}^2$, respectively. Then, by the Quillen formula in the previous section,

$$\begin{aligned}\mathcal{F}_{\mathcal{L}^{-2}}((\alpha,\gamma),(\beta,\delta)) &= -2\mathcal{F}_{\mathcal{L}}((\alpha,\gamma),(\beta,\delta))\\ &= -2\frac{\mathrm{i}}{\pi}\mathrm{Re}\,\mathrm{Tr}\int_M(\alpha^{(0,1)}\wedge\beta^{(0,1)*})\\ &= \frac{\mathrm{i}}{\pi}\mathrm{Tr}\int_M\alpha\wedge\beta\end{aligned}$$

(Since there is no Φ-term in $\mathcal{L}$, γ and δ donot contribute).

$$\begin{aligned}\mathcal{F}_{\mathcal{R}^2}((\alpha,\gamma),(\beta,\delta)) &= 2\mathcal{F}_{\mathcal{R}}((\alpha,\gamma),(\beta,\delta))\\ &= 2\frac{\mathrm{i}}{\pi}\mathrm{Re}\int_M\mathrm{Tr}(\gamma^{(0,1)}\wedge\delta^{(0,1)*})\\ &= -2\frac{\mathrm{i}}{\pi}\mathrm{Re}\int_M\mathrm{Tr}(\gamma^{(0,1)}\wedge\delta^{(1,0)})\\ &= -2\frac{\mathrm{i}}{\pi}\mathrm{Re}\int_M\mathrm{Tr}(\overline{(-\gamma^{(0,1)tr})}\wedge\overline{(-\delta^{(1,0)tr})})\\ &= -2\frac{\mathrm{i}}{\pi}\mathrm{Re}\int_M\mathrm{Tr}(\gamma^{(1,0)}\wedge\delta^{(0,1)})\\ &= 2\frac{\mathrm{i}}{\pi}\mathrm{Re}\int_M\mathrm{Tr}(\gamma^{(1,0)}\wedge\delta^{(1,0)*})\\ &= -\frac{\mathrm{i}}{\pi}\int_M\mathrm{Tr}(\gamma\wedge\delta).\end{aligned}$$

Note. $\gamma^{(0,1)}$ and $\delta^{(0,1)*}$ contributes because of the term $\Phi^{(0,1)}$ in the C-R operator in $\mathcal{R}$. α, β do not contribute to this curvature because in the definition of $\mathcal{R}$ the gauge equivalence class of A_0 is fixed.

It is easy to check that the curvature of $\mathcal{P}$ is

$$\mathcal{F}_{\mathcal{L}^{-2}}+\mathcal{F}_{\mathcal{R}^2}=\frac{\mathrm{i}}{\pi}\Omega.$$

The line bundles $\mathcal{E}_\pm$ are determinant of C-R operators of connections $A^{(0,1)}\pm\Phi^{(0,1)}$. Hence, by the formula is the previous section, in the Quillen curvature of $\mathcal{E}_\pm$ terms like $\alpha\pm\gamma$ and $\beta\pm\delta$ will appear.

The Quillen curvature of $\mathcal{E}_\pm$ is

$$\begin{aligned}\mathcal{F}_{\mathcal{E}_\pm}((\alpha,\gamma),(\beta,\delta)) &= \frac{-\mathrm{i}}{2\pi}\left(\int_M\mathrm{Tr}[(\alpha\pm\gamma)\wedge(\beta\pm\delta)]\right)\\ &= \frac{-\mathrm{i}}{2\pi}\left(\int_M\mathrm{Tr}[\alpha\wedge\beta\pm\gamma\wedge\beta\pm\alpha\wedge\delta+\gamma\wedge\delta]\right)\end{aligned}$$

Thus curvature of $\mathcal{E} = \mathcal{E}_+ \otimes (\mathcal{E}_-)^{-1}$ is

$$(\mathcal{F}_{\mathcal{E}_+} - \mathcal{F}_{\mathcal{E}_-})((\alpha,\gamma),(\beta,\delta)) = \frac{-\mathrm{i}}{\pi}\int_M \mathrm{Tr}(\alpha\wedge\delta + \gamma\wedge\beta) = \frac{\mathrm{i}}{\pi}\mathcal{Q}_1((\alpha,\gamma),(\beta,\delta))$$

Define $\tilde{\gamma} = \mathrm{i}\gamma^{(1,0)} - \mathrm{i}\gamma^{(0,1)}$. Similarly define $\tilde{\delta}$.

$\mathcal{N}_\pm$ are determinant lines of Cauchy–Riemann operators of connections $A^{(0,1)} \mp \tilde{\Phi}^{(0,1)}$. Thus by the formula of the Quillen curvature, we will have terms like $\alpha \mp \tilde{\gamma}$ and $\beta \mp \tilde{\delta}$. The Quillen curvature of $\mathcal{N}_\pm$ is

$$\begin{aligned}\mathcal{F}_{\mathcal{N}_\pm}((\alpha,\gamma),(\beta,\delta)) &= \frac{-\mathrm{i}}{2\pi}\left(\int_M \mathrm{Tr}[(\alpha\mp\tilde{\gamma})\wedge(\beta\mp\tilde{\delta})]\right)\\ &= \frac{-\mathrm{i}}{2\pi}\int_M \mathrm{Tr}[\alpha\wedge\beta \mp \tilde{\gamma}\wedge\beta \mp \alpha\wedge\tilde{\delta} + \tilde{\gamma}\wedge\tilde{\delta}])\end{aligned}$$

The Quillen curvature of the line bundle $\mathcal{N} = \mathcal{N}_+ \otimes (\mathcal{N}_-)^{-1}$ is

$$(\mathcal{F}_{\mathcal{N}_+} - \mathcal{F}_{\mathcal{N}_-})((\alpha,\gamma),(\beta,\delta)) = \frac{\mathrm{i}}{\pi}\int_M \mathrm{Tr}(\alpha\wedge\tilde{\delta} + \tilde{\gamma}\wedge\beta) = \frac{\mathrm{i}}{\pi}\mathcal{Q}_2((\alpha,\gamma),(\beta,\delta))$$

Thus we have proved the following theorem.

Theorem 3.2. *The moduli space of solutions $\mathcal{M}$ admits three prequantum line bundles $\mathcal{P}$, $\mathcal{E}$ and $\mathcal{N}$ such that their Quillen curvatures are respectively the three sympletic forms,* $(\mathrm{i}/\pi)\Omega$, $(\mathrm{i}/\pi)\mathcal{Q}_1$ *and* $(\mathrm{i}/\pi)\mathcal{Q}_2$ *which correspond to the hyperKähler structure in $\mathcal{M}$.*

4 Holomorphicity and Polarization

Proposition 4.1. *$\mathcal{P}^{-1}$ is a $\mathcal{I}$-holomorphic, $\mathcal{E}^{-1}$ is a $\mathcal{J}$-holomorphic and $\mathcal{N}^{-1}$ is a $\mathcal{K}$-holomorphic prequatum line bundle with curvature* $-(\mathrm{i}/\pi)\Omega$, $-(\mathrm{i}/\pi)\mathcal{Q}_1$ *and* $-(\mathrm{i}/\pi)\mathcal{Q}_2$*, respectively.*

Proof. Recall that

$$\begin{aligned}\mathcal{I}(\alpha^{(0,1)}) &= \mathrm{i}\alpha^{(0,1)},\\ \mathcal{I}(\gamma^{(1,0)}) &= \mathrm{i}\gamma^{(1,0)},\\ \mathcal{I}(\alpha^{(1,0)}) &= -\mathrm{i}\alpha^{(1,0)},\\ \mathcal{I}(\gamma^{(0,1)}) &= -\mathrm{i}\gamma^{(0,1)}.\end{aligned}$$

Thus w.r.t. $\mathcal{I}$, $A^{(0,1)}$ is holomorphic and $\Phi^{(0,1)}$ is anti holomorphic. Thus $\mathcal{L}$ is holomorphic and $\mathcal{R}$ is anti-holomorphic, by the same argument as in [20].

But $\mathcal{P}^{-1} = \mathcal{L}^2 \otimes \mathcal{R}^{-2}$ has the $A^{(0,1)}$-term as it is and the $\Phi^{(0,1)}$-term in the inverse bundle. Thus $\mathcal{P}^{-1}$ is $\mathcal{I}$-holomorphic.

Secondly,

$$\begin{aligned}
\mathcal{J}(\alpha^{(0,1)}) &= -\mathrm{i}\gamma^{(0,1)},\\
\mathcal{J}(\gamma^{(1,0)}) &= \mathrm{i}\alpha^{(1,0)},\\
\mathcal{J}(\alpha^{(1,0)}) &= \mathrm{i}\gamma^{(1,0)},\\
\mathcal{J}(\gamma^{(0,1)}) &= -\mathrm{i}\alpha^{(0,1)}.
\end{aligned}$$

Thus w.r.t. $\mathcal{J}$, the $A^{(0,1)} - \Phi^{(0,1)}$-term is holomorphic and the $A^{(0,1)} + \Phi^{(0,1)}$-term is anti-holomorphic. Thus $\mathcal{E}^{-1} = \mathcal{E}_+^{-1} \otimes \mathcal{E}_-$ is holomorphic since the anti-holomorphic term comes in the inverse.

Thirdly,

$$\begin{aligned}
\mathcal{K}(\alpha^{(0,1)}) &= \gamma^{(0,1)},\\
\mathcal{K}(\gamma^{(1,0)}) &= -\alpha^{(1,0)},\\
\mathcal{K}(\alpha^{(1,0)}) &= \gamma^{(1,0)},\\
\mathcal{K}(\gamma^{(0,1)}) &= -\alpha^{(0,1)}.
\end{aligned}$$

Thus w.r.t. $\mathcal{K}$, the $A^{(0,1)} + \mathrm{i}\Phi^{(0,1)}$-term is anti-holomorphic and the$A^{(0,1)} - \mathrm{i}\Phi^{(0,1)}$-term is holomorphic. Thus $\mathcal{N}^{-1} = \mathcal{N}_+^{-1} \otimes \mathcal{N}_-$ is holomorphic. □

4.1 Polarization

Since the symplectic forms are all Kähler, we can take square integrable $\mathcal{I}$-holomorphic sections of $\mathcal{P}^{-1}$, $\mathcal{J}$-holomorphic sections of $\mathcal{E}^{-1}$ and $\mathcal{K}$-holomorphic sections of $\mathcal{N}^{-1}$ as our Hilbert spaces. But we are still not guaranteed finite dimensional Hilbert spaces.

5 Cherns–Simons gauge theory with complexified gauge group

We introduce here the Chern–Simons gauge theory with complexified gauge group since flat connections on a principal bundle with complexified gauge group are essentially solutions of self-duality equations, as we shall elaborate below. In [28], Witten had explored this — quantizing one of the symplectic forms. By introducing a parameter λ we wish to get all three the symplectic forms and all the prequantum line bundles from the Chern–Simons cocycle line bundles using the method of [21].

We take up the case of an $SU(2)$ principal bundle P and denote by P^c when the group is complexified, i.e., $SL(2,\mathbb{C})$. Then any flat connection on P^c is (upto gauge transformation) is a $sl(2,\mathbb{C})$ connection of the form

$$B_\lambda = A + \lambda\Phi^{(1,0)} + \frac{1}{\lambda}\Phi^{(1,0)*} = A + \lambda\Phi^{(1,0)} - \frac{1}{\lambda}\Phi^{(0,1)},$$

where $|\lambda|^2 = 1$. This is because B_λ is of the form $A + \mathrm{i}\Psi$, where A and Ψ are unitary. This decomposition is always possible since $\mathrm{ad}\,P^c = \mathrm{ad}\,P + \mathrm{i}\,\mathrm{ad}\,P$. Papers which use similar decomposition are [10, 7]. (Note Ψ is unitary since $|\lambda|^2 = 1$ and $\Phi = \Phi^{1,0} + \Phi^{0,1}$ is unitary). Flatness of B_λ for all $\lambda \in S^1$ is equivalent to the fact that $(A, \Phi^{(1,0)})$ satisfy the self-duality equations (which is easy to check). Thus the moduli space of connections B_λ which are flat for all λ is the moduli space of Hitchin systems, namely $\mathcal{M}$.

We consider now the Chern–Simons integral

$$Z = \int D\mathbf{A}\, \exp(ik\mathrm{CS}(\mathbf{A}))$$

where

$$\mathrm{CS}(\mathbf{A}) = \frac{1}{4\pi}\int_N \mathrm{Tr}\left(\mathbf{A}\wedge d\mathbf{A} + \frac{2}{3}\mathbf{A}\wedge\mathbf{A}\wedge\mathbf{A}\right).$$

Here N is a 3-manifold such that $\partial N = M$, where M is our original compact Riemann surface.

In what follows we will take $\mathbf{A} = \mathbf{B}_\lambda$ which is an extension of the $sl(2,\mathbb{C})$ connection B_λ mentioned before on M to N. $\tilde{g}$ is an extenstion of the $SL(2,\mathbb{C})$ gauge transformation to N. As in [21] we define the Chern–Simons cocycle to be

$$\Theta(B_\lambda, g) = \exp\mathrm{i}(\mathrm{CS}({\mathbf{B}_\lambda}^{\tilde{g}}) - \mathrm{CS}(\mathbf{B}_\lambda))$$

with which we define a line bundle on $\mathcal{M}$ (which is identified with flat B_λ connections as mentioned before) in what follows.

$$\mathcal{L}_\lambda = \mathcal{M}\times_\Theta \mathbb{C},$$

where there is a quotient by means of the equivalence relation:

$$(B_\lambda, z) \equiv (B_\lambda^g, \Theta(B_\lambda, g)z).$$

As in [21], the curvature of this line bundle can be computed to be

$$F_\lambda(\tilde{\alpha}, \tilde{\beta}) = \frac{\mathrm{i}}{2\pi}\mathrm{Tr}\int_M \tilde{\alpha}\wedge\tilde{\beta}$$

where

$$\tilde{\alpha} = \alpha^{1,0} + \alpha^{0,1} + \lambda\gamma^{1,0} - \frac{1}{\lambda}\gamma^{0,1},$$
$$\tilde{\beta} = \beta^{1,0} + \beta^{0,1} + \lambda\delta^{1,0} - \frac{1}{\lambda}\delta^{0,1}.$$

Thus, $F_\lambda = \mathrm{i}/2\pi(\omega_1 + \lambda\omega_2 + 1/\lambda\omega_3)$, where $\omega_1(X,Y) = \Omega(X,Y) = \int_M \mathrm{Tr}(\alpha \wedge \beta - \gamma \wedge \delta)$, $\omega_2(X,Y) = \int_M \mathrm{Tr}(\alpha^{0,1} \wedge \delta^{1,0} + \gamma^{1,0} \wedge \beta^{0,1})$, $\omega_3(X,Y) = -\int_M \mathrm{Tr}(\alpha^{1,0} \wedge \delta^{0,1} + \gamma^{0,1} \wedge \beta^{1,0})$. Thus we obtain a whole S^1 worth of line bundles whose curvatures are parametrised by λ.

To construct the prequantum line bundles $\mathcal{P}$, $\mathcal{E}$ and $\mathcal{N}$ from this family $\mathcal{L}_\lambda$ we note that if $\lambda = \pm\mathrm{i}$, $F_\lambda = \mathrm{i}/2\pi(\Omega \mp \mathrm{i}Q_1)$ and if $\lambda = \pm 1$, $F_\lambda = \mathrm{i}/2\pi$ $(\Omega \mp \mathrm{i}Q_2)$.

Thus $\tau = \mathcal{L}_i \otimes \mathcal{L}_{-i}$ has curvature $(\mathrm{i}/\pi)\Omega$. Thus τ and $\mathcal{P}$ have the same curvature. We say can that the Chern classes of these two line bundles are same as well when $H^2(\mathcal{M},\mathbb{Z})$ has no torsion. When the associated bundle V to P in [14] is of rank 2 and degree odd , the Hitchin moduli space $\mathcal{M}$ is smooth and $H^2(\mathcal{M},\mathbb{Z})$ has no torsion [16]. Since at least in this situation the Chern class cannot be torsion, curvature will determine the Chern class, this line bundle τ is topologically equivalent to $\mathcal{P}$.

Also, $\mathcal{L}_i^2 \otimes \tau^{-1}$ has curvature $(1/\pi)\mathcal{Q}_1$ which is exact and thus since the moduli space has no torison the Chern class is zero and hence the bundle is trivial. Since the curvature of $\mathcal{E}$ is $(i/\pi)Q_1$ it is also trivial. Since these are trivial bundles, they are isomorphic and one can put i-times the connection on $\mathcal{L}_i^2 \otimes \tau^{-1}$ to get the connection on $\mathcal{E}$.

Similarly, $\mathcal{L}_1^2 \otimes \tau^{-1}$ has curvature $(1/\pi)Q_2$ which is exact and thus since the moduli space has no torsion the Chern class is zero and hence the bundle is trivial. $\mathcal{N}$ has curvature $(\mathrm{i}/\pi)Q_2$ and hence it is also trivial. Since these are trivial bundles they are isomorphic and one can put i-times the connection on $\mathcal{L}_1^2 \otimes \tau^{-1}$ to get the connection on $\mathcal{N}$.

Thus it is possible to get the prequantum line bundles from these cocycle line bundles $\mathcal{L}_\lambda$ in the special case when the moduli space is smooth. In general, the moduli space is an orbifold, and perhaps, there could be torsion in $H^2(\mathcal{M},\mathbb{Z})$. Then it is not clear that having the same curvature would imply the lines bundles are topologically isomorphic.

Further work. It would be interesting to see if these cocycle line bundles $\mathcal{L}_\lambda$ can be used to get some topological or geometrical invariants of 3-manifolds, as in perhaps [11].

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