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T-duality for torus bundles with H-fluxes via noncommutative topology, II: the high-dimensional case and the T-duality group

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Abstract

We use noncommutative topology to study T-duality for principal torus bundles with H-flux. We characterize precisely when there is a "classical" T-dual, i.e., a dual bundle with dual H-flux, and when the T-dual must be "nonclassical," that is, a continuous field of noncommutative tori. The duality comes with an isomorphism of twisted K-theories, required for matching of D-brane charges, just as in the classical case. The isomorphism of twisted cohomology which one gets in the classical

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case is replaced in the nonclassical case by an isomorphism of twisted cyclic homology. An important part of the paper contains a detailed analysis of the classifying space for topological T-duality, as well as the T-duality group and its action. The issue of possible nonuniqueness of T-duals can be studied via the action of the T-duality group.

T-duality is a duality of type II string theories that involves exchanging a theory compactified on a torus with a theory compactified on the dual torus. The T-dual of a type II string theory compactified on a circle, in the presence of a topologically nontrivial NS 3-form H-flux, was analyzed in special cases in [1, 2, 19, 28, 30]. There it was observed that T-duality changes not only the H-flux, but also the spacetime topology. A general formalism for dealing with T-duality for compactifications arising from a free circle action was developed in [4]. This formalism was shown to be compatible with two physical constraints: (1) it respects the local Buscher rules [10, 11], and (2) it yields an isomorphism on twisted K-theory, in which the Ramond–Ramond charges and fields take their values [7, 32, 37, 39, 45, 46]. It was shown in [4] that T-duality exchanges the first Chern class with the fiberwise integral of the H-flux, thus giving a formula for the T-dual spacetime topology. The purpose of this paper is to extend these results to the case of torus bundles of higher rank.

It is common knowledge that noncommutative tori occur naturally in string theory and in M-theory compactifications [13, 44]. This paper derives another instance where they make a natural appearance, in the sense that if we start with a classical spacetime, that is a principal torus bundle with H-flux, then the T-dual is sometimes a continuous field of noncommutative tori, and we characterize exactly when this happens.

The first part of the paper forms a sequel to our earlier paper [33] on T-duality for principal torus bundles with an H-flux. In that paper, we dealt primarily with the case of fibers which are tori of dimension 2, in which case every principal bundle is T-dualizable, though not necessarily in the "classical" sense; in fact, if the integral of the H-flux over the torus fibers is nontrivial in cohomology, then the T-dual of such a principal torus bundle with H-flux is a continuous field of noncommutative tori. A similar phenomenon was also noticed in [31], in fact for one of the same examples studied in [33] (a trivial T^2-bundle over S^1, but with nontrivial flux). In this paper, we consider principal torus bundles of arbitrary dimension. Still another phenomenon appears: there are some principal bundles with H-flux which are not dualizable even in our more general sense (where the T-dual is allowed to be a C^*-algebra). The simplest case of this phenomenon is the 3-dimensional torus, considered as a 3-torus bundle over a point, with H-flux chosen to be a nonzero integer multiple of the volume 3-form on the torus.

More precisely, a given principal torus bundle with H-flux, over a base Z, is T-dualizable in our generalized sense if and only if the restriction of the H-flux to a torus fiber T is trivial in cohomology. Moreover, the T-dual of such a principal torus bundle with H-flux is "nonclassical" if and only if the push-forward of the flux in $H^1(Z, H^2(T))$ is nontrivial. In [6], the results of [33] and of this paper were applied and extended, and it was shown that every principal torus bundle is T-dualizable in an even more general sense, where the T-dual is a field of *nonassociative* tori, i.e., taking us out of the category of C^*-algebras.

The other main part of this paper contains the analysis of the classifying space for topological T-duality and of the T-duality group, as well as its action. The theory divides naturally into two cases, when there is a classical T-dual and when there is only a nonclassical T-dual, and the T-duality group respects these two cases. One somewhat unexpected result is that, when $n = 2$, the classifying space for torus bundles with H-flux splits as a product, one factor corresponding to the classical case and one factor corresponding to the nonclassical case. For a rank n torus bundle with H-flux, the T-duality group is $GO(n, n; \mathbb{Z})$ and acts by homotopy automorphisms of the classifying space for classically dualizable bundles. Study of this group enables us to understand puzzling instances of nonuniqueness of T-duals.

Some of the results of this paper were announced in [34].

1 Notation and review of results from [33]

We begin by reviewing the precise mathematical framework from [33]. We assume X (which will be the spacetime of a string theory) is a (second-countable) locally compact Hausdorff space. In practice, it will usually be a compact manifold, though we do not need to assume this. But we assume that X is finite-dimensional and has the homotopy type of a finite CW-complex, in order to avoid some pathologies. We assume X comes with a free action of a torus T; thus (by the Gleason slice theorem [24]) the quotient map $p\colon X \to Z$ is a principal T-bundle.

All C^*-algebras and Hilbert spaces in this paper will be over $\mathbb{C}$. A *continuous-trace algebra A over X* is a particularly nice type I C^*-algebra with Hausdorff spectrum X and good local structure (the "Fell condition" [22] — that there are continuously varying rank-one projections in a neighborhood of any point in X). We will always assume A is separable; then a basic structure theorem of Dixmier and Douady [17] says that after stabilization (i.e., tensoring by $\mathcal{K}$, the algebra of compact operators on an infinite-dimensional separable Hilbert space $\mathcal{H}$), A becomes *locally* isomorphic to $C_0(X, \mathcal{K})$, the continuous $\mathcal{K}$-valued functions on X vanishing at infinity. However, A need not be *globally* isomorphic to $C_0(X, \mathcal{K})$, even after stabilization. The reason is that a stable continuous-trace algebra is the algebra of sections (vanishing at infinity) of a bundle of algebras over X, with fibers all isomorphic to $\mathcal{K}$. The structure group of the bundle is $\operatorname{Aut} \mathcal{K} \cong PU(\mathcal{H})$, the projective unitary group $U(\mathcal{H})/\mathbb{T}$. Since $U(\mathcal{H})$ is contractible and the circle group $\mathbb{T}$ acts freely on it, $PU(\mathcal{H})$ is an Eilenberg–Mac Lane $K(\mathbb{Z}, 2)$-space, and thus bundles of this type are classified by homotopy classes of continuous maps from X to $BPU(\mathcal{H})$, which is a $K(\mathbb{Z}, 3)$-space, or in other words by $H^3(X, \mathbb{Z})$. In this way, one can show that the continuous-trace algebras over X, modulo Morita equivalence over X, naturally form a group under the operation of tensor product over $C_0(X)$, called the *Brauer group* $\operatorname{Br}(X)$, and that this group is isomorphic to $H^3(X, \mathbb{Z})$ via the Dixmier–Douady class. Given an element $\delta \in H^3(X, \mathbb{Z})$, we denote by $CT(X, \delta)$ the associated stable continuous-trace algebra. (If $\delta = 0$, this is simply $C_0(X, \mathcal{K})$.) The (complex topological) K-theory $K_\bullet(CT(X, \delta))$ is called the *twisted K-theory* [43, Section 2] of X with twist δ, denoted $K^{-\bullet}(X, \delta)$. When δ is torsion, twisted K-theory had earlier been considered by Donovan and Karoubi [18]. When $\delta = 0$, twisted K-theory reduces to ordinary K-theory (with compact supports).

Now recall we are assuming X is equipped with a free T-action with quotient $X/T = Z$. (This means our theory is "compactified along tori" in a way reflecting a global symmetry group of X.) In general, a group action on X need not lift to an action on $CT(X, \delta)$ for any value of δ other than 0, and even when such a lift exists, it is not necessarily essentially unique. So one wants a way of keeping track of what lifts are possible and how unique they are. The *equivariant Brauer group* defined in [14] consists of equivariant Morita equivalence classes of continuous-trace algebras over X equipped with group actions lifting the action on X. Two group actions on the same stable continuous-trace algebra over X define the same element in the equivariant Brauer group if and only if they are outer conjugate. (This implies in particular that the crossed products are isomorphic. However, it is perfectly possible for the crossed products to be isomorphic even if the actions are *not* outer conjugate.) Now let G be the universal cover of the torus T, a vector group. Then G also acts on X via the quotient map $G \twoheadrightarrow T$ (whose kernel N can be identified with the free abelian group $\pi_1(T)$). In our situation there are three Brauer groups to consider: $\operatorname{Br}(X) \cong H^3(X, \mathbb{Z})$, $\operatorname{Br}_T(X)$, and $\operatorname{Br}_G(X)$, but $\operatorname{Br}_T(X)$ is rather uninteresting, as it is naturally isomorphic to $\operatorname{Br}(Z)$ [14, Section 6.2]. Again by [14, Section 6.2], the natural "forgetful map" (forgetting the T-action) $\operatorname{Br}_T(X) \to \operatorname{Br}(X)$ can simply be identified with $p^*\colon\ \operatorname{Br}(Z) \cong H^3(Z, \mathbb{Z}) \to H^3(X, \mathbb{Z}) \cong \operatorname{Br}(X)$.

The basic setup from [33] is as follows.

Basic Setup ([**33, 3.1**]). *A spacetime X compactified over a torus T will correspond to a space X (locally compact, finite-dimensional, homotopically finite) equipped with a free T-action. Without essential loss of generality, we may as well assume that X is connected. The quotient map $p\colon X \to Z$ is a principal T-bundle. The NS 3-form H on X has an integral cohomology class δ which corresponds to an element of $\mathrm{Br}(X) \cong H^3(X, \mathbb{Z})$. A pair (X, δ) will be a candidate for having a T-dual when the T-symmetry of X lifts to an action of the vector group G on $CT(X, \delta)$, or in other words, when δ lies in the image of the forgetful map $F\colon \mathrm{Br}_G(X) \to \mathrm{Br}(X)$.*

Recall from [33] that if T is a torus of dimension n, so that $G \cong \mathbb{R}^n$, we have the following facts. Here we denote by $H^\bullet_M(G, A)$ the cohomology of the topological group G with coefficients in the topological G-module A, as defined in [38]. This is sometimes called "Moore cohomology" or "cohomology with Borel cochains." We denote by $H^\bullet_{\mathrm{Lie}}(\mathfrak{g}, A)$ the Lie algebra cohomology of the Lie algebra $\mathfrak{g}$ of G with coefficients in a module A.

Theorem 1.1 ([**33, Theorem 4.4**]). *Suppose $G \cong \mathbb{R}^n$ is a vector group and X is a locally compact G-space (satisfying our finiteness assumptions). Then there is an exact sequence:*

$$H^2(X, \mathbb{Z}) \xrightarrow{d_2''} H^2_M(G, C(X, \mathbb{T})) \xrightarrow{\xi} \mathrm{Br}_G(X) \xrightarrow{F} H^3(X, \mathbb{Z}) \xrightarrow{d_3} H^3_M(G, C(X, \mathbb{T})).$$

Lemma 1.2 ([**33, equation (5)**]). *If X is as above, then*

$$H^\bullet_M(G, C(X, \mathbb{T})) \cong H^\bullet_M(G, C(X, \mathbb{R}))$$

for $\bullet > 1$.

Theorem 1.3 ([**26, Cor, III.7.5**], **quoted in** [**33, Corollary 4.7**]). *If G is a vector group with Lie algebra $\mathfrak{g}$, and if A is a G-module which is a complete metrizable topological vector space, then $H^\bullet_{\mathrm{cont}}(G, A) \cong H^\bullet_{\mathrm{Lie}}(\mathfrak{g}, A_\infty)$. (Here A_∞ is the submodule of smooth vectors for the action of G.) In particular, it vanishes for $\bullet > \dim G$.*

2 Main mathematical results

Now we are ready to start on the main results. First we begin with a lemma concerning the computation of a certain Moore cohomology group. This lemma is quite similar to [33, Lemma 4.9].

Lemma 2.1. *If G is a vector group and X is a G-space (with a full lattice subgroup N acting trivially) as in the Basic Setup above, then the maps*

$p^*\colon C(Z,\mathbb{R}) \to C(X,\mathbb{R})$ and "averaging along the fibers of p" $\int\colon C(X,\mathbb{R}) \to C(Z,\mathbb{R})$ (defined by $\int f(z) = \int_T f(g\cdot x)\,dg$, where dg is Haar measure on the torus T and we choose $x \in p^{-1}(z)$) induce isomorphisms

$$H^\bullet_M(G, C(X,\mathbb{R})) \leftrightarrows H^\bullet_M(G, C(Z,\mathbb{R})) \cong C(Z,\mathbb{R}) \otimes \bigwedge\nolimits^\bullet(\mathfrak{g}^*)$$

which are inverses to one another.

Proof. We apply Lemma 1.2 and Theorem 1.3. Note that the G-action on $C(Z,\mathbb{R})$ is trivial, so every element of $C(Z,\mathbb{R})$ is smooth for the action of G. But for any real vector space V with trivial G-action,

$$H^\bullet_M(G,V) \cong H^\bullet_{\mathrm{Lie}}(\mathfrak{g},V) \cong H^\bullet_{\mathrm{Lie}}(\mathfrak{g},\mathbb{R}) \otimes V \cong \bigwedge\nolimits^\bullet(\mathfrak{g}^*) \otimes V.$$

Clearly $\int \circ\, p^*$ is the identity on $C(Z,\mathbb{R})$, so we need to show $p^* \circ \int$ induces an isomorphism on cohomology of $C(X,\mathbb{R})$. The calculation turns out to be local, so by a Mayer–Vietoris argument, we can reduce to the case where p is a trivial bundle, i.e., $X = (G/N) \times Z$, with G acting only on the first factor. The smooth vectors in $C(X,\mathbb{R})$ for the action of G can then be identified with $C(Z, C^\infty(G/N))$. So we obtain

$$\begin{aligned} H^\bullet_M\left(G, C(X,\mathbb{R})\right) &\cong H^\bullet_{\mathrm{Lie}}\left(\mathfrak{g}, C\left(Z, C^\infty\left(\frac{G}{N}\right)\right)\right) \\ &\cong C\left(Z, H^\bullet_{\mathrm{Lie}}\left(\mathfrak{g}, C^\infty\left(\frac{G}{N}\right)\right)\right), \end{aligned}$$

with the cohomology moving inside since G acts trivially on Z. However, we have by [3, Chapter VII, Section 2] that

$$H^\bullet_{\mathrm{Lie}}\left(\mathfrak{g}, C^\infty\left(\frac{G}{N}\right)\right) \cong H^\bullet_M(N,\mathbb{R}) \cong \bigwedge\nolimits^\bullet(\mathfrak{g}^*).$$

□

The first main result generalizes the result in [33] that says that the forgetful map $F\colon \mathrm{Br}_G(X) \to \mathrm{Br}(X)$ is surjective when $\dim G \le 2$. In higher dimensions, F need not be surjective, but we characterize its image.

Theorem 2.2. *Let T be a torus, G its universal covering, and $p\colon X \to Z$ be a principal T-bundle, as in the Basic Setup of Section 1. Then the image of the forgetful map $F\colon \mathrm{Br}_G(X) \to H^3(X,\mathbb{Z})$ is precisely the kernel of the map $\iota^*\colon H^3(X,\mathbb{Z}) \to H^3(T,\mathbb{Z})$ induced by the inclusion $\iota\colon T \hookrightarrow X$ of a torus fiber into X.*

Proof. There are two distinct parts to the proof. First we show that if $\delta \in H^3(X,\mathbb{Z})$ and $\iota^*(\delta) \ne 0$, then δ cannot be in the image of F. For this (since we can always restrict to a G-invariant set), it suffices to consider the case where $X = T$, with T acting simply transitively. Then if δ lies in the image of F, that means the corresponding principal PU-bundle $E \to T$ (with classifying map δ) carries an action of G for which the bundle projection is G-equivariant. This action corresponds to an integrable way of choosing horizontal spaces in the tangent bundle of E, or in other words, to a flat connection on E. That means, since $T = G/N$ with N free abelian, that the bundle, together with its G-action, comes from a homomorphism $\rho\colon \pi_1(T) = N \to PU$, which we can regard as a projective unitary representation of N. In other words, if $A = CT(T, \delta)$ carries an action of G compatible with the transitive action of G on T, then the action is induced from the action of N on $\mathcal{K}$ associated to a projective unitary representation ρ of N. So we have a surjective "induction" homomorphism $H^2_M(N, \mathbb{T}) \cong \mathrm{Br}_N(\mathrm{pt}) \to \mathrm{Br}_G(G/N)$.

Consider the Mackey obstruction $M(\rho) \in H^2_M(N, \mathbb{T}) \cong \bigwedge^2(N^*) \otimes \mathbb{T}$, which is the class of the N-action on $\mathcal{K}$ as an element of $\mathrm{Br}_N(\mathrm{pt})$. Since $H^2_M(N, \mathbb{T})$ is generated by product cocycles $\omega_1 \otimes \cdots \otimes \omega_k \otimes 1$, where $2k \leq n = \dim G$, N splits as $N_1 \times \cdots N_k \times N'$ with each N_j of rank 2, and with ω_j skew and nondegenerate on N_j, it is enough to show δ is trivial in such a case. But in this case, if we let $G_j = N_j \otimes_{\mathbb{Z}} \mathbb{R}$, $G' = N' \otimes_{\mathbb{Z}} \mathbb{R}$, then $A = \mathrm{Ind}_N^G(\mathcal{K}, \rho)$ evidently splits as an "external tensor product" $\mathrm{Ind}_{N_1}^{G_1}(\mathcal{K}, \omega_1) \otimes \cdots \otimes \mathrm{Ind}_{N_k}^{G_k}(\mathcal{K}, \omega_k) \otimes \mathrm{Ind}_{N'}^{G'}(\mathcal{K}, 1)$. Since $\mathrm{Ind}_{N'}^{G'}(\mathcal{K}, 1) \cong C(G'/N') \otimes \mathcal{K}$ has trivial Dixmier–Douady class, the Dixmier–Douady class of A is thus determined by the Dixmier–Douady classes of continuous-trace algebras over the 2-tori G_j/N_j. These of course have to be trivial, since $H^3(T^2) = 0$. This completes the first half of the proof.

For the second half of the proof, we have to show that if $\iota^*\delta = 0$, then δ is in the image of the forgetful map. We apply Theorem 1.1. This says it suffices to show that δ is in the kernel of the map

$$d_3\colon\ H^3(X, \mathbb{Z}) \to H^3_M(G, C(X, \mathbb{T})).$$

from that theorem. By Lemma 2.1, the calculation of the Moore cohomology group is local (in the base Z) and leads to a natural isomorphism

$$H^3_M(G, C(X, \mathbb{T})) \cong C(Z, \mathbb{R}) \otimes H^3(T, \mathbb{R}).$$

Thus d_3 must be a map $H^3(X, \mathbb{Z}) \to C(Z, \mathbb{R}) \otimes H^3(T, \mathbb{R})$ which is locally defined (in Z) and natural for all principal T-bundles. It must therefore factor through ι^*, and so everything in the kernel of ι^* is the image of F. □

Theorem 2.3. *In the basic setup with* $\dim G = n$ *arbitrary, there is a commutative diagram of exact sequences:*

$$
\begin{array}{ccccccc}
 & & H^0(Z, \bigwedge^2(\mathbb{Z}^n)) & & 0 & & \\
 & & \downarrow & & \downarrow & & \\
H^2(X,\mathbb{Z}) & \xrightarrow{d_2''} & H^2_M(G, C(X,\mathbb{T})) & \xrightarrow{\xi} & \ker F & \xrightarrow{\eta} & 0 \\
 & & \vdots\, a & & \downarrow & & \\
 & & C(Z, H^2_M(N,\mathbb{T})) & \xleftarrow{M} & \mathrm{Br}_G(X) & & \\
 & & h \downarrow & & \downarrow F & & \\
 & & H^1(Z, \bigwedge^2(\mathbb{Z}^n)) & \overset{p_!}{\leftarrow\cdots} & H^3(X,\mathbb{Z}) & & \\
 & & \downarrow & & \iota^* \downarrow & & \\
 & & 0 & & H^3(T,\mathbb{Z}) & &
\end{array}
$$

with M *the Mackey obstruction map defined in* [40] *and with*

$$
\begin{aligned}
h\colon\ C(Z, H^2_M(N,\mathbb{T})) \cong C(Z, \textstyle\bigwedge^2(\mathbb{Z}^n) \otimes \mathbb{T}) &\longrightarrow H^1(Z, \textstyle\bigwedge^2(\mathbb{Z}^n)) \\
&\cong H^1(Z, \mathbb{Z}^k), \quad k = \binom{n}{2},
\end{aligned}
$$

the map sending a continuous function $Z \to \bigwedge^2(\mathbb{Z}^n) \otimes \mathbb{T}$ *to its homotopy class. Here* $p_!$ *is defined on* $\ker \iota^*\colon\ H^3(X,\mathbb{Z}) \to H^3(T,\mathbb{Z})$ (*the dotted arrow indicates that it is not always defined on all of* $H^3(X,\mathbb{Z})$), *and there is given by the map to the subquotient* $E^{1,2}_\infty$ *of the Serre spectral sequence, or* (*more informally*) *by*

$$
H^3(X,\mathbb{Z}) \ni H \longmapsto \left(\int_{\mathbb{T}^2_1} H,\ \ldots,\ \int_{\mathbb{T}^2_k} H \right) \in H^1(Z, \mathbb{Z}^k),
$$

where $\mathbb{T}^2_j$, $j = 1, \ldots, k$ *run through a basis for the possible two-dimensional subtori in the fibers.*

Proof. Exactness of the top horizontal sequence comes from Theorem 1.1, and exactness of the right-hand vertical sequence comes from Theorem 2.2. Exactness of the left-hand vertical sequence comes from the isomorphism $H^2_M(G, C(X,\mathbb{T})) \cong H^2_M(G, C(X,\mathbb{R}))$ of Lemma 1.2, the calculation of H^2_M $(G, C(X,\mathbb{R}))$ in Lemma 2.1, and the exact sequence

$$
0 \longrightarrow H^0(Z,K) \longrightarrow C(Z, K \otimes \mathbb{R}) \longrightarrow C(Z, K \otimes \mathbb{T}) \longrightarrow H^1(Z,K) \longrightarrow 0
$$

for K a free abelian group. (In this case, $K = \bigwedge^2(\mathbb{Z}^n)$.) So it remains to verify commutativity of the two squares. The proof is very similar to that of

[33, Theorem 4.10]. Commutativity of the upper square amounts to showing that $M \circ \xi$ is the same as the map

$$a\colon\ H^2_M(G, C(X,\mathbb{T})) \xrightarrow{\cong} C(Z, {\textstyle\bigwedge}^2(N)\otimes\mathbb{R}) \xrightarrow{\exp} C(Z, {\textstyle\bigwedge}^2(N)\otimes\mathbb{T}).$$

This is immediate from the definition of ξ in [14, Theorem 5.1]. The harder part of the proof is commutativity of the lower square, i.e., showing that if α of G on $CT(X,\delta)$, representing an element of $\mathrm{Br}_G(X)$, then $h \circ M(\alpha) = p_!(\delta)$. As in [33, Theorem 4.10], $h \circ M(\alpha)$ only depends δ, not on the specific choice of α, since any two actions (of the sort we are considering) on $CT(X,\delta)$ differ by an element of $\ker F$, which by the rest of the diagram is in the image of $H^2_M(G, C(X,\mathbb{T}))$, and thus only changes $M(\alpha)$ within its homotopy class.

Next we show that the map $H^3(X,\mathbb{Z}) \to H^1(Z, \bigwedge^2(\mathbb{Z}^n))$ induced by $h \circ M$ vanishes on the subquotients $E^{3,0}_\infty$ and $E^{2,1}_\infty$ of $H^3(X,\mathbb{Z})$ for the Serre spectral sequence of the fibration $T \to X \xrightarrow{p} Z$. First of all, $E^{3,0}_\infty = p^*\big(H^3(Z,\mathbb{Z})\big)$. If $\delta = p^*(\eta)$ with $\eta \in H^3(Z,\mathbb{Z})$, then by [14, Section 6.2], there is a T-action on $CT(X,\delta)$ corresponding to the T-action on X. This lifts to a G-action with the discrete subgroup N acting trivially, so the Mackey obstruction class for N is certainly trivial. Secondly, $E^{2,1}_\infty$ consists (modulo $E^{3,0}_\infty$) of classes pulled back from some intermediate space Y, where $X \xrightarrow{p_1} Y \xrightarrow{p_2} Z$ is some factorization of the T-bundle $p\colon X \to Z$ as a composite of two principal torus bundles, with p_1 having $(n-1)$-dimensional fibers T' and with p_2 having one-dimensional, i.e., S^1, fibers. But given such a factorization and a class $\delta_Y \in Y$, there is an essentially unique action of $\mathbb{R}$ on $CT(Y, \delta_Y)$ compatible with the S^1-action on Y with quotient Z, because of the results of [33, Section 4.1]. Pulling back from Y to X, we get an action of $\mathbb{R} \times T'$ on $CT(X, p_1^*\delta_Y)$ or in other words an action of G factoring through $\mathbb{R} \times T'$. Such an action necessarily has trivial Mackey obstruction.

Since we are assuming that δ lies in the kernel of $\iota^*\colon\ H^3(X,\mathbb{Z}) \to H^3(T,\mathbb{Z})$ (otherwise, by Theorem 2.2, there is nothing to prove), the map induced by $h \circ M$ factors through the remaining subquotient of $H^3(Z,\mathbb{Z})$, viz., $E^{1,2}_\infty$. That says exactly that the map factors through $p_!$. We can now compute it as we did in the proof of [33, Theorem 4.10] by calculating in specific cases and using naturality. Thus the proof is concluded using Proposition 4.1 in Section 4 below. □

3 Applications to T-duality

Now we are ready to apply Theorem 2.3 to T-duality in type II string theory.

The following is the second main result of this paper.

Theorem 3.1. *Let* $p\colon X \to Z$ *be a principal* T*-bundle as in the Basic Setup, where* $n = \dim T$ *is arbitrary. Let* $\delta \in H^3(X, \mathbb{Z})$ *be an "H-flux" on* X *that is the kernel of* $\iota^*\colon H^3(X, \mathbb{Z}) \to H^3(T, \mathbb{Z})$*, where* ι *is the inclusion of a fiber. Let* $k = \binom{n}{2}$*. Then:*

(1) *If* $p_!\delta = 0 \in H^1(Z, \mathbb{Z}^k)$*, then there is a classical T-dual to* (p, δ)*, consisting of* $p^\#\colon X^\# \to Z$*, which is another principal* $\mathbb{T}^n$*-bundle over* Z*, and* $\delta^\# \in H^3(X^\#, \mathbb{Z})$*, the "T-dual H-flux" on* $X^\#$*. One obtains a picture of the form*

$$\begin{array}{ccccc} & & X \times_Z X^\# & & \\ & \swarrow^{p^*(p^\#)} & & \searrow^{(p^\#)^*(p)} & \\ X & & & & X^\# \\ & \searrow_{p} & & \swarrow_{p^\#} & \\ & & Z & & \end{array} \qquad (3.1)$$

There is a natural isomorphism of twisted K*-theory*

$$K^\bullet(X, \delta) \cong K^{\bullet+n}(X^\#, \delta^\#).$$

(*Because of Bott periodicity, the dimension shift can be ignored if and only if* $\dim T$ *is even. The shift in odd dimensions can be understood physically in terms of a duality between type IIA and type IIB string theories.*)

(2) *If* $p_!\delta \neq 0 \in H^1(Z, \mathbb{Z}^k)$*, then a classical T-dual as above does* not *exist. However, there is a "nonclassical" T-dual bundle of noncommutative tori over* Z*. It is not unique, but the nonuniqueness does not affect its* K*-theory, which is isomorphic to* $K^\bullet(X, \delta)$ *with a dimension shift of* $\dim G \bmod 2$*.*

Proof. By Theorem 2.2, the assumption that $\iota^*\delta = 0$ is equivalent to saying that δ lies in the image of F.

First consider the case when $p_!\delta = 0 \in H^1(Z, \mathbb{Z}^k)$. By commutativity of the lower square in Theorem 2.3, we can lift $\delta \in H^3(X, \mathbb{Z})$ to an element $[CT(X, \delta), \alpha]$ of $\mathrm{Br}_G(X)$ with $M(\alpha)$ homotopically trivial. Then by using commutativity of the upper square in Theorem 2.3, we can perturb α, without changing δ, so that $M(\alpha)$ actually vanishes. Once this is done, the element we get in $\mathrm{Br}_G(X)$ is actually unique modulo the discrete group $H^0(Z, \bigwedge^2(\mathbb{Z}^n))/d_2''(H^2(X, \mathbb{Z}))$. On the one hand, this can be seen from [40, Lemma 1.3] and [40, Corollary 5.18]. Alternatively, it can be read off from Theorem 2.3, since any two classes in $\ker M$ mapping to the same $\delta \in H^3(X,\mathbb{Z})$ differ by the image under ξ of something in $\ker a$. The nonuniqueness of the element of $\mathrm{Br}_G(X)$ will be addressed in the following Proposition. Finally, if $[CT(X,\delta),\alpha]$ has trivial Mackey obstruction, then as explained in [40, Section 1] and [20, Section 3], $CT(X,\delta)\rtimes_\alpha G$ has continuous trace and has spectrum which is another principal torus bundle over Z (for the dual torus, $\widehat{G}$ divided by the dual lattice). The "Thom isomorphism" of Connes [12] gives an isomorphism between the K-theory of $CT(X,\delta)$ and that of $CT(X,\delta)\rtimes_\alpha G$, with a dimension shift of $\dim G$. Thus we obtain the desired isomorphism in twisted K-theory.

Now consider the case when

$$p_!\delta \neq 0 \in H^1(Z,\mathbb{Z}^k). \tag{3.2}$$

It is still true as before that we can find an element $[CT(X,\delta),\alpha]$ in $\mathrm{Br}_G(X)$ corresponding to δ. But there is no classical T-dual in this situation since the Mackey obstruction *can't* be trivial, because of Theorem 2.3. In fact, since any representative $f\colon Z\to\mathbb{T}$ of a nonzero class in $H^1(Z,\mathbb{Z})$ must take on all values in $\mathbb{T}$, there are necessarily points $z\in Z$ for which the Mackey obstruction in $H^2(\mathbb{Z}^n,\mathbb{T})\cong\mathbb{T}^k$ is irrational, and hence the crossed product $CT(X,\delta)\rtimes_\alpha G$ cannot be type I. Nevertheless, we can view this crossed product as a *nonclassical* T-dual to (p,δ). The crossed product can be viewed as the algebra of sections of a bundle of algebras (not locally trivial) over Z, in the sense of [15]. The fiber of this bundle over $z\in Z$ will be $C(p^{-1}(z),\mathcal{K}(\mathcal{H}))\rtimes G\cong C(G/\mathbb{Z}^n,\mathcal{K}(\mathcal{H}))\rtimes G\cong A_{f(z)}\otimes\mathcal{K}(\mathcal{H})$, which is Morita equivalent to the twisted group C^*-algebra $A_{f(z)}$ of the stabilizer group $\mathbb{Z}^n$ for the Mackey obstruction class $f(z)$ at that point. In other words, the T-dual will be realized by a bundle of (stabilized) *noncommutative tori* fibered over Z (figure 1).

The bundle is not unique since there is no *canonical* representative f for a given nonzero class in $H^1(X,\mathbb{Z})$. However, any two choices are homotopic, and the resulting bundles will be in some sense homotopic to one another. The isomorphism of the K-theory of the crossed product with the twisted K-theory of (X,δ) again follows from the "Thom isomorphism" of Connes [12]. □

Remark 3.2. An almost identical argument, with Connes' Theorem replaced by its analogue in cyclic homology [21], can be used to get an isomorphism of twisted cohomology groups in the classical case and of twisted cohomology with cyclic homology of the (smooth) dual in the nonclassical case.

The next proposition corrects one small problem in Theorem 3.1, which also appeared in [33, Theorem 4.13]. Namely, it would appear that the "classical T-dual" in Theorem 3.1 is not unique, since when it exists, the

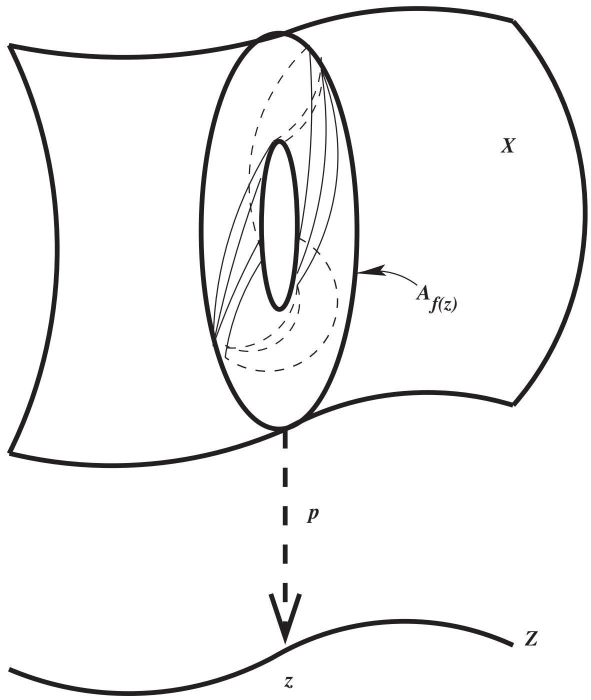


Figure 1: In the diagram, the fiber over $z \in Z$ is the noncommutative torus $A_{f(z)}$, which is represented by a foliated torus, with foliation angle equal to $f(z)$.

lifting of δ to a element of $\mathrm{Br}_G(X)$ with vanishing Mackey obstruction is not necessarily unique. We thank Bunke et al. [9] for noticing this. However, it turns out that the nonuniqueness does not matter after all.

Proposition 3.3. *Given a principal T-bundle $p\colon X \to Z$ and an element $\delta \in H^3(X,\mathbb{Z})$ such that $\iota^*\delta = 0$ in $H^3(T,\mathbb{Z})$ and such that $p_!(\delta) = 0$ in $H^1(Z, H^2(T,\mathbb{Z}))$, the nonuniqueness of the lift of δ to an element $\alpha \in \mathrm{Br}_G(X)$ with $M(\alpha) = 0$ does not affect the classical T-dual $p^\#\colon X^\# \to Z$ or the T-dual H-flux $\delta^\# \in X^\#$.*

Proof. Let $A = CT(X,\delta)$. This is a continuous-trace algebra with a free action of the torus T of rank n on its spectrum. Let α be a lift of the T-action to an action of the universal cover G of T on A, such that the Mackey obstruction $M(\alpha)$ is identically 0. Chasing the diagram in Theorem 2.3 shows that any other such lift is of the form $\alpha \otimes \omega$, where ω is an action of G on $\mathcal{K}$ defined by a projective unitary representation of G with Mackey obstruction lying in $\ker\left(\mathbb{R}^k \cong H^2_M(G,\mathbb{T}) \to H^2_M(N,\mathbb{T}) \cong \mathbb{T}^k\right)$, $k = \binom{n}{2}$. Here again N is the lattice subgroup $\ker(G \twoheadrightarrow T)$. (Note that tensoring with

ω does not change the condition of vanishing of the Mackey obstruction because of the fact $\omega|_N$ has trivial obstruction.) We want to prove that the following crossed products are isomorphic:

$$A \rtimes_\alpha G \cong A \rtimes_{\alpha\otimes\omega} G. \tag{3.3}$$

We do this by induction on n. To start the induction, the result is obvious when $n = 1$, since then α is unique up to exterior equivalence (see [14, Corollary 6.1] and [33, Section 4.1]). So assume $n > 1$ and suppose the result is true for smaller values of n. Write N as $N_1 \times \mathbb{Z}$, G as $H \times \mathbb{R}$, where $H = N_1 \otimes_{\mathbb{Z}} \mathbb{R} \cong \mathbb{R}^{n-1}$. Note that $H/N_1 \cong \mathbb{T}^{n-1}$ also acts freely on X (even though the quotient space is different from the original Z). So by inductive hypothesis, with G replaced by H in (3.3), $A \rtimes_{\alpha|_H} H \cong A \rtimes_{\alpha\otimes\omega|_H} H$. So their spectra and Dixmier–Douady classes are the same. In fact, since the action of the other factor of $\mathbb{R}$ on the spectra of these crossed products still comes from the original action of T on $\widehat{A}$, these spectra are homeomorphic not just as spaces, but equivariantly as $\mathbb{T}$-spaces, for the free action of the remaining circle in T. Thus, by the case $n = 1$ applied to this situation,

$$\begin{aligned} A \rtimes_{\alpha\otimes\omega} G &= (A \rtimes_{\alpha\otimes\omega|_H} H) \rtimes_{\alpha\otimes\omega|_{\mathbb{R}}} \mathbb{R} \\ &\cong (A \rtimes_{\alpha|_H} H) \rtimes_{\alpha|_{\mathbb{R}}} \mathbb{R} \\ &\cong A \rtimes_\alpha G. \end{aligned}$$

That completes the inductive step. □

4 Examples

We begin with the following explicit construction, where a field of noncommutative tori appears in the explicit T-dual.

Proposition 4.1. *Let $p\colon\ X = Z \times T \to Z$ be the trivial T-bundle, where $T = \mathbb{T}^n$. Let $\beta \in H^1(Z, \mathbb{Z}^k) = [Z, \mathbb{T}^k]$ be a nonzero element, where $k = \binom{n}{2}$, and let $\delta = \langle \beta, \gamma \rangle \in H^3(X, \mathbb{Z})$, where $\gamma \in H^2(T, \mathbb{Z}^k)$ and the brackets denote the duality pairing. Then there is an explicit construction of $CT(X, \delta)$ together with an action α of $G = \mathbb{R}^n$ on $CT(X, \delta)$, compatible with the free T-action on X, for which $h \circ M(\alpha) = \beta$. As a consequence, we get an explicit description of the T-dual to the trivial principal torus bundle with H-flux equal to δ, that is, an explicit description of $CT(X, \delta) \rtimes_\alpha G$.*

Proof. First of all, notice that δ is in the kernel of $\iota^*\colon\ H^3(X, \mathbb{Z}) \to H^3(T, \mathbb{Z})$, where ι is the inclusion of a fiber. Choose a function $f\colon\ Z \to \mathbb{T}^k$ such that $h(f) = \beta$. For each $f(z) \in \mathbb{T}^k \cong H^2(\mathbb{Z}^n, \mathbb{T})$, choose a multiplier $\sigma_{f(z)}$ on $\mathbb{Z}^n$ such that $[\sigma_{f(z)}] = f(z) \in H^2(\mathbb{Z}^n, \mathbb{T})$. Let $\mathcal{H}$ denote the Hilbert space $\ell^2(\mathbb{Z}^n)$. Then there is a natural projective unitary representation $\rho_{f(z)}$:

$\mathbb{Z}^n \to PU(\mathcal{H})$ which is the right $\sigma_{f(z)}$-regular representation, defined by $\rho_{f(z)}(s)f(s') = f(s+s')\sigma_{f(z)}(s,s')$, and whose Mackey obstruction is equal to $f(z) \in H^2(\mathbb{Z}^n, \mathbb{T})$. Let $\mathbb{Z}^n$ act on $C(Z, \mathcal{K}(\mathcal{H}))$ by α, which is given at the point $z \in Z$ by $\mathrm{Ad}(\rho_{f(z)})$, which is a spectrum-fixing automorphism of $\mathbb{Z}^n$ on $C(Z, \mathcal{K}(\mathcal{H}))$. Define the C^*-algebra

$$\begin{aligned} B &= \mathrm{Ind}_{\mathbb{Z}^n}^{\mathbb{R}^n} \left(C(Z, \mathcal{K}(\mathcal{H})), \alpha\right) \\ &= \left\{\lambda : \mathbb{R}^n \longrightarrow C(Z, \mathcal{K}(\mathcal{H})) : \lambda(t+g) = \alpha(g)(\lambda(t)), \;\; t \in \mathbb{R}^n, g \in \mathbb{Z}^n\right\}. \end{aligned}$$

Since α acts trivially on the spectrum Z of the inducing algebra and B is an algebra of sections of a locally trivial bundle of C^*-algebras with fibers isomorphic to $\mathcal{K}$, it follows that B is a continuous-trace algebra having spectrum $Z \times T$. There is a natural action ξ of G on B by translation, and by construction, the Mackey obstruction $M(\xi) = f$. The computation of the Dixmier–Douady invariant of B is done exactly as [33, Proposition 4.11] and is equal to $\delta = \langle \beta, \gamma \rangle \in H^3(X, \mathbb{Z})$ as claimed. That is, $B = CT(X, \delta)$, and so the T-dual is $B \rtimes_\alpha G$. □

The following is a special case of the above, where we can be even more explicit.

Proposition 4.2. *Let $Z = \mathbb{T}^k$, $T = \mathbb{T}^n$ and consider the trivial torus bundle $X = Z \times T \to Z$. Let $\beta \in H^1(Z, \mathbb{Z}^k) = [\mathbb{T}^k, \mathbb{T}^k]$ be an element of degree one and $\tau \in H^2(T, \mathbb{Z}^k) \cong H^2(\mathbb{Z}^n, \mathbb{Z}^k)$ determine the central extension,*

$$1 \longrightarrow \mathbb{Z}^k \longrightarrow H_\tau \longrightarrow \mathbb{Z}^n \to 1,$$

where H_τ is a 2-step nilpotent group over $\mathbb{Z}$. (In fact, by [41], every torus bundle over a torus is a nilmanifold with fundamental group of the form H_τ.) Then the T-dual of the trivial torus bundle $X = Z \times T \to Z$, with H-flux given by $\delta = \langle \beta, \tau \rangle \in H^3(X, \mathbb{Z})$ (the brackets denote the duality pairing), is the stabilized group C^*-algebra $C^*(H_\tau) \otimes \mathcal{K}$, which can be realized as a continuous field of stabilized noncommutative tori of dimension n that are parameterized by $\mathbb{T}^k$.

Proof. Here we choose the identity map $I : \mathbb{T}^k \to \mathbb{T}^k$ as representing the degree one element $\beta \in H^1(Z, \mathbb{Z}^k)$. For each $\theta \in \mathbb{T}^k \cong H^2(\mathbb{Z}^n, \mathbb{T})$, choose a multiplier σ_θ on $\mathbb{Z}^n$ such that $[\sigma_\theta] = \theta \in H^2(\mathbb{Z}^n, \mathbb{T})$. Let $\mathcal{H}$ denote the Hilbert space $\ell^2(\mathbb{Z}^n)$. Then there is a natural projective unitary representation $\rho_\theta : \mathbb{Z}^n \to PU(\mathcal{H})$ which is the right σ_θ-regular representation, defined by $\rho_\theta(\gamma)f(\gamma') = f(\gamma+\gamma')\sigma_\theta(\gamma, \gamma')$, and whose Mackey obstruction is equal to $\theta \in H^2(\mathbb{Z}^n, \mathbb{T})$. Let $\mathbb{Z}^n$ act on $C(\mathbb{T}^k, \mathcal{K}(\mathcal{H}))$ by α, which is given at the point θ by ρ_θ. Define the C^*-algebra

$$B = \mathrm{Ind}_{\mathbb{Z}^n}^{\mathbb{R}^n} \Big(C(\mathbb{T}^k, \mathcal{K}(\mathcal{H})), \alpha \Big)$$
$$= \Big\{ f : \mathbb{R}^n \longrightarrow C(\mathbb{T}^k, \mathcal{K}(\mathcal{H})) : f(t+g) = \alpha(g)(f(t)), \ \ t \in \mathbb{R}^n, g \in \mathbb{Z}^n \Big\}.$$

That is, B (with an implied action of $\mathbb{R}^n$) is the result of inducing a $\mathbb{Z}^n$-action on $C(\mathbb{T}^k, \mathcal{K}(\mathcal{H}))$ from $\mathbb{Z}^n$ up to $\mathbb{R}^n$. Then B is a continuous-trace C^*-algebra having spectrum $\mathbb{T}^{n+k}$ and an action of $\mathbb{R}^n$ whose induced action on the spectrum of B is the trivial bundle $\mathbb{T}^{n+k} \to \mathbb{T}^k$. The computation of the Dixmier–Douady invariant of B is done exactly as in Proposition 4.11, [33] and is equal to $\delta = \langle \beta, \tau \rangle \in H^3(X, \mathbb{Z})$. The crossed product algebra $B \rtimes \mathbb{R}^n \cong C(\mathbb{T}^k, \mathcal{K}(\mathcal{H})) \rtimes \mathbb{Z}^n$ has fiber over $\theta \in \mathbb{T}^k$ given by $\mathcal{K}(\mathcal{H}) \rtimes_{\rho_\theta} \mathbb{Z}^n \cong A_\theta(n) \otimes \mathcal{K}(\mathcal{H})$, where $A_\theta(n)$ is the noncommutative n-torus. In fact, the crossed product $B \rtimes \mathbb{R}^n$ is Morita equivalent to $C(\mathbb{T}^k, \mathcal{K}(\mathcal{H})) \rtimes \mathbb{Z}^n$ and is even isomorphic to the stabilization of this algebra (by [25]). Thus $B \rtimes \mathbb{R}^n$ is isomorphic to $C^*(H_\tau) \otimes \mathcal{K}$, where H_τ is the 2-step nilpotent group over $\mathbb{Z}$ determined by $\tau \in H^2(\mathbb{Z}^n, \mathbb{Z}^k)$. □

5 The classifying space for topological T-duality and action of the T-duality group

In this section, we discuss how our results are related to a construction of Bunke and Schick [8] concerning a "classifying space" for topological T-duality. We will also discuss the action of the *T-duality group*.

5.1 Review of the Bunke–Schick construction for $n = 1$

In the paper [8], Bunke and Schick have pointed out that there is a *classifying space* R for principal $\mathbb{T}$-bundles with H-flux. In other words, the set of all possible principal $\mathbb{T}$-bundles $p\colon X \to Z$ (for fixed Z), each equipped with a cohomology class $\delta \in H^3(X, \mathbb{Z})$, modulo isomorphism, can be identified with the set of homotopy classes of maps $Z \to R$. Bunke and Schick call $(p\colon X \to Z, \delta)$ a *pair*. They show that there is a *universal pair* $(\mathbf{E}, \mathbf{h})$ over R, such that any pair $(p\colon X \to Z, \delta)$ is pulled back from $(\mathbf{E}, \mathbf{h})$ via a map $Z \to R$ (whose homotopy class is uniquely determined).

The Bunke–Schick classifying space R has an interesting structure. It is a two-stage Postnikov system; i.e., it has exactly two nonvanishing homotopy groups and can be constructed as a principal fibration

$$K(\mathbb{Z},3) \longrightarrow R \xrightarrow{q} K(\mathbb{Z},2) \times K(\mathbb{Z},2). \tag{5.1}$$

Here $K(\mathbb{Z},3)$ is the classifying space for H^3, a space with exactly one nonzero homotopy group and with $\pi_3 = \mathbb{Z}$, and $K(\mathbb{Z},2)$ is the classifying space for H^2, a space with exactly one nonzero homotopy group and with $\pi_2 = \mathbb{Z}$. A useful explicit model for $K(\mathbb{Z},2)$ is the infinite projective space $\mathbb{CP}^\infty$. The homotopy type of the fibration (5.1) is determined by the k-invariant in $H^4(K(\mathbb{Z},2) \times K(\mathbb{Z},2), \mathbb{Z})$, which in this case is the cup-product $\mathbf{c} \cup \widehat{\mathbf{c}}$, where $\mathbf{c}$ and $\widehat{\mathbf{c}}$ are the canonical generators in H^2 of the two $K(\mathbb{Z},2)$ factors. The two classes $\mathbf{c}$ and $\widehat{\mathbf{c}}$ play different, but symmetrical, roles. Let pr_1 and pr_2 be the two projections $K(\mathbb{Z},2) \times K(\mathbb{Z},2) \to K(\mathbb{Z},2)$. Then given a map $\varphi\colon Z \to R$, with $\varphi^*(\mathbf{E},\mathbf{h}) = (p\colon X \to Z, \delta)$, $\mathrm{pr}_1 \circ q \circ \varphi\colon Z \to K(\mathbb{Z},2)$ classifies the Chern class of the $\mathbb{T}$-bundle $p\colon X \to Z$, while $\mathrm{pr}_2 \circ q \circ \varphi\colon Z \to K(\mathbb{Z},2)$ classifies $p_!(\delta)$.

Now T-duality replaces a pair $(p\colon X \to Z, \delta)$ with a dual pair $(p^\#\colon X^\# \to Z, \delta^\#)$, where the Chern class of $p^\#\colon X^\# \to Z$ is $p_!(\delta)$, and $(p^\#)_!(\delta^\#)$ is the Chern class of the original bundle $p\colon X \to Z$. Thus we can understand T-duality in terms of a self-map $\#\colon R \circlearrowleft$ which comes from interchanging the two copies of $K(\mathbb{Z},2)$ in the fibration (5.1). (Note that since $\mathbf{c} \cup \widehat{\mathbf{c}} = \widehat{\mathbf{c}} \cup \mathbf{c}$, this preserves the k-invariant and thus extends to a self-map of R, with $(\#)^2$ homotopic to the identity.)

Using this point of view, we can now explain the notion of the *T-duality group* (still in the relatively trivial case of $n = 1$). Good references include [23] (for the state of the theory up to 1994) and [29] for something more current. The physics literature mostly talks in terms of an $O(n,n;\mathbb{Z})$ symmetry, which when $n = 1$ degenerates to $O(1,1;\mathbb{Z}) = \{\pm 1\} \times \{\pm 1\}$. If this group is to act by T-duality symmetries for general circle bundles, then it should operate on R in a way that induces this symmetry, and indeed it does.

The action of $GL(n,\mathbb{Z})$ as discussed in [8] amounts to the following. Given a principal n-torus bundle $p\colon X \to Z$, we can make a new principal n-torus bundle out of the same underlying spaces X and Z by twisting the free action of $T = \mathbb{T}^n$ on X by an element $g \in \mathrm{Aut}(\mathbb{T}^n) = GL(n,\mathbb{Z})$. In other words, for $t \in T$, we define $x \cdot_g t$ to be $x \cdot g(t)$. (Here $\cdot$ is the original free right action of T on X, with quotient space Z, and $\cdot_g$ is the new twisted action.) In the case where $n = 1$ and $g = -1$ is the nontrivial element of $GL(1,\mathbb{Z})$, this twisting changes the sign of the Chern class of the bundle p. It also changes the sign of $p_!(\delta)$, since δ does not change but the orientation of the fibers of p is reversed (and thus the definition of integration along the fibers changes by a sign). In view of what we said about $\mathbf{c}$ and $\widehat{\mathbf{c}}$, that means that the action of the twisting on R reverses the sign of both $\mathbf{c}$ and $\widehat{\mathbf{c}}$. Since $(-\mathbf{c}) \cup (-\widehat{\mathbf{c}}) = \mathbf{c} \cup \widehat{\mathbf{c}}$, once again the k-invariant of (5.1) is preserved and we get a well-defined action of $GL(1,\mathbb{Z})$ on R. This action can be explicitly realized using the complex conjugation map on each copy of $\mathbb{CP}^\infty = K(\mathbb{Z}, 2)$.

Bunke and Schick do not discuss (in this paper — they do go into it in the sequel paper, [9]) the action of $O(n, n; \mathbb{Z})$, but in the case $n = 1$, we can explain this as follows. First of all, $GL(n, \mathbb{Z})$ is to be embedded in $O(n, n; \mathbb{Z})$ via $a \mapsto \begin{pmatrix} a & 0 \\ 0 & (a^t)^{-1} \end{pmatrix}$ (see [23, Section 2.4]). That means that when $n = 1$, $-1 \in GL(1, \mathbb{Z})$ should correspond to $(-1, -1) \in O(n, n; \mathbb{Z})$, and the whole group is generated by this element and by the T-duality element $\# = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. In fact, there is an action on R of the larger group $GO(1, 1; \mathbb{Z})$, generated by $O(1, 1; \mathbb{Z})$ and by the matrix $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ of determinant -1. This latter element acts by sending $\mathbf{c}$ to $-\mathbf{c}$, $\widehat{\mathbf{c}}$ to itself, and reversing the sign of the generator of $\pi_3(R)$. (The latter compensates for the change of sign in the k-invariant $\mathbf{c} \cup \widehat{\mathbf{c}}$.) On the level of the universal bundle space $\mathbf{E}$, this amounts to a change of sign in the cohomology class $\mathbf{h} \in H^3(\mathbf{E})$.

5.2 The higher rank case and the T-duality group

In the higher rank case where T is a torus of dimension n, n arbitrary, we can mimic the construction of [8] as follows. (A very similar, but not identical, construction can be found in [9].) Let Z be a (nice enough) space, say a locally compact Hausdorff space with the homotopy type of a finite CW complex, and fix $n \geq 1$. A *pair* over Z will mean a principal T-bundle $E \to Z$ together with a class $\alpha \in H^3(E, \mathbb{Z})$, whose restriction to each n-torus fiber is 0. (The reason for this extra condition is that, as we showed in Theorem 2.2, this condition is necessary and sufficient for getting a compatible $G = \mathbb{R}^n$ action on $CT(E, \alpha)$. It also makes the set of pairs over S^i trivial for large i, since for i large, all torus bundles over S^i are trivial, but one has nontrivial 3-cohomology classes, coming from the fiber on such bundles.) The Bunke–Schick argument from [8], copied over essentially line by line, proves the following. (See also [9], which we only saw after completing the first draft of this paper.)

Theorem 5.1. *The set of pairs modulo isomorphism is a representable functor, with representing space*

$$R = ET \times_T \mathrm{Maps}_0(T, K(\mathbb{Z}, 3)), \tag{5.2}$$

where $ET \to BT = K(\mathbb{Z}^n, 2) \cong (\mathbb{CP}^\infty)^n$ *is the universal* T*-bundle and where* Maps_0 *denotes the set of null-homotopic maps* (*those giving the trivial class in* $H^3(T, \mathbb{Z})$). *Note that using* Maps_0 *in place of* Maps *makes* R *path-connected.* (*When* $n = 1$, $\mathrm{Maps}(T, K(\mathbb{Z}, 3))$ *is the free loop space of* $K(\mathbb{Z}, 3)$ *and is already connected.*) *There is an obvious map*

$$c\colon\ R = ET \times_T (\cdots) \longrightarrow ET \times_T * = BT$$

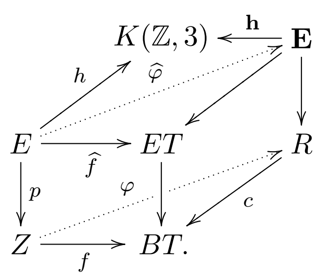
which corresponds to forgetting the second entry of a pair and just taking the underlying bundle. This map is a fibration with fiber $\mathrm{Maps}_0(T, K(\mathbb{Z}, 3))$.

Proof. We follow the outline of the argument in [8]. First we construct a canonical pair $(\mathbf{E}, \mathbf{h})$ over R by letting $\mathbf{E}$ be the pull-back under the map c (defined in the statement of the theorem) of the universal T-bundle $ET \to BT$. Then we define $\mathbf{h}\colon\ \mathbf{E} \to K(\mathbb{Z}, 3)$ to be given by

$$\mathbf{h}(u,\ [v, \gamma]) = \gamma(t),$$

where $\gamma \in \mathrm{Maps}_0(T, K(\mathbb{Z}, 3))$, $u,\ v \in ET$, u and v live over $c([v, \gamma])$, and $tv = u$. One can check that this is independent of the choices of u, v, and γ representing a particular element of $\mathbf{E}$. Clearly any map $Z \to R$ enables us to pull back the canonical pair $(\mathbf{E}, \mathbf{h})$ to a pair over Z.

In the other direction, suppose we have a pair (E, h) over Z. Since $E \xrightarrow{p} Z$ is a principal T-bundle, we know that $E \xrightarrow{p} Z$ is pulled back from the universal bundle $ET \to BT$ via a map $f\colon\ Z \to BT$. We claim we can fill in the diagram



as shown to make it commute and to realize (E, h) as the pull-back of $(\mathbf{E}, \mathbf{h})$. Indeed, we simply define $\varphi(z) = [\widehat{f}(e), \gamma]$, where $e \in p^{-1}(z) \subseteq E$ and where $\gamma \in \mathrm{Maps}(T, K(\mathbb{Z}, 3))$ is defined by $\gamma(t) = h(t \cdot e)$. Since the definition of a pair includes the requirement that h be null-homotopic on each torus fiber, γ indeed lies in $\mathrm{Maps}_0(T, K(\mathbb{Z}, 3))$. Note that $\varphi(z)$ is independent of the choice of e. We can define $\widehat{\varphi}$ by $\widehat{\varphi}(e) = \big[\widehat{f}(e), [\widehat{f}(e), \gamma]\big]$, e as before. The rest of the proof is as in [8]. □

The next step is to make a detailed analysis of the homotopy type of R.

Theorem 5.2. *The space R has only three nonzero homotopy groups, $\pi_1(R) \cong \mathbb{Z}^k$ $(k = \binom{n}{2})$, $\pi_2(R) \cong \mathbb{Z}^{2n}$, and $\pi_3(R) \cong \mathbb{Z}$. Moreover, there is a fibration $\mathbf{E} \to R \to BT$, where $\mathbf{E}$ is a simple space (one with π_1 acting trivially on all the homotopy groups) homotopy equivalent to $K(\mathbb{Z}, 3) \times \mathbf{E}_0$, with $\mathbf{E}_0 \to R$ inducing an isomorphism on π_1 and with the universal cover of $\mathbf{E}_0$ homotopy equivalent to $K(\mathbb{Z}^n, 2)$.*

Proof. The set of pairs over S^i is easy to compute except when $i = 2$ (the only case when the bundle can be nontrivial). Thus one finds that

$$\pi_1(R) \cong \ker\bigl(H^3(\mathbb{T}^n \times S^1, \mathbb{Z}) \to H^3(\mathbb{T}^n, \mathbb{Z})\bigr) \cong H^2(\mathbb{T}^n, \mathbb{Z}) \cong \mathbb{Z}^k, \quad k = \binom{n}{2},$$
$$\pi_3(R) \cong \ker\bigl(H^3(\mathbb{T}^n \times S^3, \mathbb{Z}) \to H^3(\mathbb{T}^n, \mathbb{Z})\bigr) \cong H^3(S^3, \mathbb{Z}) \cong \mathbb{Z},$$
$$\pi_i(R) \cong \ker\bigl(H^3(\mathbb{T}^n \times S^i, \mathbb{Z}) \to H^3(\mathbb{T}^n, \mathbb{Z})\bigr) = 0, \quad i \geq 4.$$

It follows that R has only three nonzero homotopy groups, $\pi_1(R) \cong \mathbb{Z}^k$, $\pi_2(R)$, and $\pi_3(R) \cong \mathbb{Z}$. Furthermore, the proof of Theorem 5.1 showed that we have a fibration $\mathbf{E} \to R \to BT$, with $\mathbf{E}$ homotopy equivalent to $\mathrm{Maps}_0(T, K(\mathbb{Z}, 3))$, with which we replace it (since only the homotopy type of $\mathbf{E}$ matters, anyway).

Next, observe that we can split $\mathbf{E}$ (up to homotopy) as a product $\mathbf{E}_0 \times K(\mathbb{Z}, 3)$, where $\mathbf{E}_0 = \mathrm{Maps}_0^+(T, K(\mathbb{Z}, 3))$, the *based* null-homotopic maps from T to $K(\mathbb{Z}, 3)$. Indeed, we can choose a model K for $K(\mathbb{Z}, 3)$ which is an abelian topological group. (See [35, Chapters V] and [27].) Then $\mathbf{E}$, as a space of maps into K, has a natural topological group structure (coming from pointwise multiplication in K) and thus is simple. Furthermore, we can construct the desired splitting of $\mathbf{E}$ by sending $f\colon T \to K$ (a typical element of $\mathbf{E}$) to the pair $(f(1)^{-1} \cdot f\colon T \to K,\ f(1) \in K)$. (Here we are using the multiplication in K, and $f(1)^{-1} \cdot f$ is based since its value at 1_T is 1_K.)

So to compute $\pi_2(R)$, we just need to compute $\pi_2(\mathbf{E}_0)$. But we have

$$\pi_j(\mathrm{Maps}_0^+(T, K(\mathbb{Z}, 3))) = [\Sigma^j T, K(\mathbb{Z}, 3)], \quad j > 0,$$

by the usual identification of $\mathrm{Maps}^+(S^j, \mathrm{Maps}^+(T, W))$ with $\mathrm{Maps}^+(S^j \wedge T, W)$.

For simplicity, we first consider the case $n = 2$. The suspension of the 2-torus $T^2 = (S^1 \vee S^1) \cup_\psi e^2$ splits as

$$\Sigma T \simeq S^2 \vee S^2 \vee S^3,$$

since the attaching map ψ of the 2-cell is a commutator in $\pi_1(S^1 \vee S^1)$ and is thus stably trivial. So

$$\begin{aligned}[\Sigma^2 T, K(\mathbb{Z},3)] &= [S^3 \vee S^3 \vee S^4, K(\mathbb{Z},3)] \\ &\cong [S^3, K(\mathbb{Z},3)] \oplus [S^3, K(\mathbb{Z},3)] \oplus [S^4, K(\mathbb{Z},3)] \\ &\cong \mathbb{Z} \oplus \mathbb{Z} \oplus 0 = \mathbb{Z}^2.\end{aligned}$$

Thus $\pi_2(\mathrm{Maps}_0(T, K(\mathbb{Z},3))) \cong \pi_2(\mathrm{Maps}_0^+(T, K(\mathbb{Z},3))) \cong \mathbb{Z}^2$, and from the exact sequence

$$0 = \pi_3(BT) \longrightarrow \pi_2(\mathrm{Maps}_0(T, K(\mathbb{Z},3))) \longrightarrow \pi_2(R) \xrightarrow{c_*} \pi_2(BT) = \mathbb{Z}^n, \tag{5.3}$$

we have $\pi_2(R) \cong \mathbb{Z}^2 \oplus \mathbb{Z}^2 = \mathbb{Z}^4$.

The higher rank case works similarly, once we have the appropriate stable splitting of T^n. In fact, if T is a torus of any rank n, then T has the same integral homology as

$$\bigvee_{k=1}^{n} \Bigl(\overbrace{S^k \vee \cdots \vee S^k}^{\binom{n}{k}} \Bigr).$$

From this, one can show that ΣT splits as a wedge of spheres, where the number of S^{k+1} summands is the binomial coefficient $\binom{n}{k}$. This follows from Whitehead's theorem once one can construct a homology equivalence from ΣT to the wedge of spheres; but such a map can be constructed from the coproduct (over all subsets of $\{1, \cdots, n\}$) of the maps

$$\begin{aligned}\Sigma\Bigl(\overbrace{S^1 \times \cdots \times S^1}^{n}\Bigr) &\xrightarrow{\Sigma(\mathrm{proj})} \Sigma\Bigl(\overbrace{S^1 \times \cdots \times S^1}^{k}\Bigr) \\ &\xrightarrow{\Sigma(\mathrm{quotient})} \Sigma\Bigl(\overbrace{S^1 \wedge \cdots \wedge S^1}^{k}\Bigr) = \Sigma S^k = S^{k+1}.\end{aligned}$$

(Note that we need to suspend here first of all to be able to "add" maps into the wedge of spheres — recall $[\Sigma T, Y]$ has a group structure for any Y, and secondly because Whitehead's theorem only applies to simply connected spaces.) The same argument as before then shows that

$$\pi_2(\mathrm{Maps}_0(T, K(\mathbb{Z},3))) \cong \pi_2(\mathrm{Maps}_0^+(T, K(\mathbb{Z},3))) \cong \mathbb{Z}^n,$$

and from the exact sequence (5.3), $\pi_2(R) \cong \mathbb{Z}^n \oplus \mathbb{Z}^n$. □

The fundamental group $\pi_1(R)$ presents a problem which shows up in Theorem 3.1; if $H^1(Z, \mathbb{Z}) \neq 0$, then not every pair over Z has a classical T-dual. For that reason, it is convenient to work with the universal cover $\widetilde{R}$, which can be viewed as a classifying space for pairs over *simply connected* spaces.[1] The space $\widetilde{R}$ has only two nonzero homotopy groups, π_2 and π_3, and so it is a two-stage Postnikov system just like the Bunke–Schick classifying space for the case $n = 1$. Since every pair over a simply connected space has a (unique) classical T-dual by Theorem 3.1, we expect T-duality to correspond geometrically to an involution on $\widetilde{R}$. Not only this, but we can understand the action of the T-duality group $O(n, n; \mathbb{Z})$ as being the automorphism group of the quadratic form on $H^2(\widetilde{R})$ defined by the k-invariant of $\widetilde{R}$. We formalize this as follows:

Theorem 5.3.

(1) *The universal cover $\widetilde{R}$ of the classifying space R of Theorem* 5.1 *is a two-stage Postnikov system*

$$K(\mathbb{Z}, 3) \longrightarrow \widetilde{R} \longrightarrow K(\mathbb{Z}^n, 2) \times K(\mathbb{Z}^n, 2),$$

with $\pi_3(\widetilde{R}) \cong \mathbb{Z}$ and with $\pi_2(\widetilde{R}) \cong \mathbb{Z}^n \oplus \mathbb{Z}^n$. The k-invariant of $\widetilde{R}$ in $H^4(K(\mathbb{Z}^n, 2) \times K(\mathbb{Z}^n, 2), \mathbb{Z})$ can be identified with $x_1 \cup y_1 + \cdots + x_n \cup y_n$, where $x_1, \ldots, x_n$ is a basis for the H^2 of the first copy of $K(\mathbb{Z}^n, 2)$ and $y_1, \ldots, y_n$ is a basis for the H^2 of the second copy of $K(\mathbb{Z}^n, 2)$. T-duality is implemented by a self-map $\#$ of $\widetilde{R}$, whose square is homotopic to the identity, interchanging the two copies of $K(\mathbb{Z}^n, 2)$. (The involutive automorphism of $K(\mathbb{Z}^n, 2) \times K(\mathbb{Z}^n, 2)$ interchanging the two factors preserves the k-invariant and thus extends to a homotopy involution of $\widetilde{R}$.) The action of the T-duality group $O(n, n; \mathbb{Z})$ is by automorphisms of $\pi_2(\widetilde{R})$ preserving the k-invariant.

(2) *The classifying space R itself is a simple space (i.e., the fundamental group acts trivially on the higher homotopy groups), with $\pi_1(R) \cong H^2(T, \mathbb{Z}) \cong \mathbb{Z}^k$, $k = \binom{n}{2}$.*

(3) *Given a space Z and a map $u\colon Z \to R$ (representing, by Theorem* 5.1*, a pair $\big(p\colon E \to Z, \alpha \in H^3(E, \mathbb{Z})\big)$ with E a principal T-bundle over Z and with α restricting to* 0 *on the fibers of p), u has a lifting to a map $Z \to \widetilde{R}$ if and only if $p_!(\alpha) = 0$ in $H^1(Z, H^2(T))$.*

Proof. We have already computed the homotopy groups of R. We will first check (2) and (3), then go back and finish the last part of (1), which concerns the k-invariant of $\widetilde{R}$. Recall by Theorem 5.2 that we have a fibration $\mathbf{E} \to R \to BT$, where $\mathbf{E} \to R$ induces an isomorphism on π_1 and $\mathbf{E}$ is simple.

[1] Even in the simply connected case, we need to be careful — given Z simply connected and a map $Z \to R$, it has a lift to a map $Z \to \widetilde{R}$, but the lift is not unique, since we can compose with a covering transformation. So $\widetilde{R}$ really classifies pairs together with a choice of lift.

Since BT is simply connected, and the fundamental group of the fiber $\mathbf{E}$ must act trivially on the homotopy groups of the base BT, it follows that R is simple.

For (3), observe that the obstruction to lifting $u\colon Z \to R$ to a map $Z \to \widetilde{R}$ is the induced map $u_*\colon \pi_1(Z) \to \pi_1(R) \cong H^2(T)$, which can be interpreted as an element of

$$\mathrm{Hom}(\pi_1(Z), H^2(T)) \cong \mathrm{Hom}(\pi_1(Z)_{\mathrm{ab}}, H^2(T)) \cong H^1(Z, H^2(T)).$$

Chasing through the various identifications here shows that this map is precisely $p_!(\alpha)$, if u corresponds to the pair $(p\colon E \to Z,\, \alpha \in H^3(E))$.

To finish the proof of the theorem, we need to check that the k-invariant of $\widetilde{R}$ is as described. Clearly $H^2(\widetilde{R}, \mathbb{Z})$ is free abelian with generators $x_1, \dots, x_n$ dual to the generators of $\pi_2(BT)$ and generators $y_1, \dots, y_n$ dual to the generators of $\pi_2(\mathbf{E})$. Now given a principal $\mathbb{T}$-bundle $p\colon X \to Z$ over a space Z and an element $\delta \in H^3(X, \mathbb{Z})$, we can make it into a pair (E_j, α) in the sense of this section, by letting $E_j = X \times T^{n-1} \xrightarrow{p \circ \mathrm{pr}_1} Z$, where the j-th copy of $\mathbb{T}$ in T^n acts by the $\mathbb{T}$-bundle $X \to Z$, and the other $n-1$ copies act by translation in the second factor. This obviously defines a natural transformation from $\mathbb{T}$-pairs to $\mathbb{T}^n$-pairs, and thus a map of classifying spaces $\mu_j\colon R_1 \to R$, where R_1 is the classifying space of [8], which is a fibration

$$K(\mathbb{Z}, 3) \longrightarrow R_1 \longrightarrow K(\mathbb{Z}, 2) \times K(\mathbb{Z}, 2)$$

with k-invariant xy (x and y the two generators of $H^2(R_1)$). It is clear that μ_j induces an isomorphism on π_3 and that it lifts to a map $R_1 \to \widetilde{R}$ that, on $H^2(\widetilde{R})$, kills the canonical generators x_k and y_k for $k \neq j$ and pulls back x_j to x and y_j to y. It also must pull back the k-invariant of $\widetilde{R}$ to the k-invariant of R_1, which is xy. This shows that modulo x_k and y_k for $k \neq j$, the k-invariant of $\widetilde{R}$ is $x_j y_j$. We are almost done — we just need to show there are no cross-terms involving $x_j x_k$, $y_j y_k$, or $x_j y_k$, with $k \neq j$. But there can be no terms of the form $y_j y_k$, by the product splitting of $\mathbf{E}$ in Theorem 5.2, and similarly there can be no terms of the form $x_j x_k$, since these would be T-dual to terms of the form $y_j y_k$. Finally, there can be no terms of the form $x_j y_k$ with $k \neq j$, since these would give rise to terms of the form $y_j y_k$ after T-dualizing in the j-th circle. Thus the k-invariant must be as described.

Finally, it is clear that the automorphism group $O(n, n; \mathbb{Z})$ of the k-invariant acts by homotopy automorphisms on $\widetilde{R}$. The element of order 2 defined by $\begin{pmatrix} 0 & 1_n \\ 1_n & 0 \end{pmatrix}$ pulls back under each μ_j to the T-duality transformation for circle bundles, and so deserves to be called the T-duality transformation in general. One can check that it matches up with the procedure we have described earlier, of lifting the T-action on the total space of a pair $(p\colon X \to Z, \delta)$ to a G-action on $CT(X, \delta)$, then looking at the pair corresponding to the crossed product. This procedure, in turn, replaces the original tori in X by the dual tori in $X^{\#}$. Indeed, we know that our topological description matches the analytic one when $n = 1$ (since the T-dual in this case is uniquely characterized), and thus it must also agree in this higher rank case, since we can (say in the "universal example" of $\widetilde{R}$) dualize one circle at a time. □

Remark 5.4. In fact, a slightly larger T-duality group acts on $\widetilde{R}$, namely $GO(n, n; \mathbb{Z})$. This is the subgroup of $GL(2n, \mathbb{Z})$ that preserves the quadratic form $x_1y_1 + \cdots + x_ny_n$ only up to sign; it is generated by $O(n, n; \mathbb{Z})$ and by the involution fixing the y's and sending $x_j \mapsto -x_j$. Of course, since such an involution reverses the sign of the k-invariant of $\widetilde{R}$, to get it to act on $\widetilde{R}$, we also have to reverse the orientation on $K(\mathbb{Z}, 3)$.

Remark 5.5. We should emphasize that the T-duality element # of Aut $\widetilde{R}$ (the homotopy automorphisms of $\widetilde{R}$) depends on a choice of basis in $H^2(\widetilde{R}, \mathbb{Z})$. If we were to choose a different basis $u_1, \dots, u_n, v_1, \dots, v_n$ (such that the k-invariant is also given by $u_1 \cup v_1 + \cdots + u_n \cup v_n$), then clearly interchanging the u's and v's would give another perfectly valid notion of T-duality. (This was noticed in [8, Section 4] in somewhat different language.) This amounts to saying that for any $g \in GO(n, n; \mathbb{Z})$,

$$g \begin{pmatrix} 0 & 1_n \\ 1_n & 0 \end{pmatrix} g^{-1}$$

gives another choice of T-duality element. Thus only the conjugacy class of # under the T-duality group is really canonical.

Suppose $(\pi : X \to Z, \delta)$ and $(\pi^{\#} : X^{\#} \to Z, \delta^{\#})$ are T-dual pairs. Now we can discuss how, in the case where Z is simply connected or at least has $H^1(Z, \mathbb{Z}) = 0$, the characteristic classes $[\pi]$ and $[\pi^{\#}]$ of π and $\pi^{\#}$ are related to the H-fluxes δ and $\delta^{\#}$. For this purpose, we fix an identification of the n-torus T with $\mathbb{T}^n$, so that we can think of $[\pi]$ and $[\pi^{\#}]$ as living in $H^2(Z, \mathbb{Z}^n) = \big(H^2(Z, \mathbb{Z})\big)^n$. The formula that follows is the higher rank substitute for equation (1) in [33, Section 4.1]. It should replace the formula given in the first version of our preliminary announcement of these results in [34], which was not completely correct.

Theorem 5.6. *Fix an identification of T with $\mathbb{T}^n$, let $H^1(Z, \mathbb{Z}) = 0$, and let $(\pi : X \to Z, \delta)$ and $(\pi^{\#} : X^{\#} \to Z, \delta^{\#})$ be T-dual pairs. Identify the characteristic class $[\pi]$ of π with an n-tuple of classes $[\pi]_j \in H^2(Z, \mathbb{Z})$. For each*

$j = 1, \dots, n$, let $\pi_j : X \to X_j$ be the S^1-bundle defined by the j-th coordinate in T, and let $p_j : X_j \to Z$ be the projection, a principal $\mathbb{T}^{n-1}$-bundle. In other words, p_j is defined by the property that

$$[p_j] = \big([\pi]_1, \cdots, \widehat{[\pi]_j}, \cdots, [\pi]_n\big).$$

Define $X_j^{\#}$, etc., similarly. Then

$$(p_j)^*\big([\pi^{\#}]_j\big) = (\pi_j)_!(\delta), \quad \textit{and} \quad (p_j^{\#})^*\big([\pi]_j\big) = (\pi_j^{\#})_!(\delta^{\#}).$$

Proof. Since we have shown in Theorem 5.1 and in Theorem 5.3 that π is pulled back from the universal bundle $\widetilde{\mathbf{E}} \to \widetilde{R}$, with δ pulled back from the canonical class $\mathbf{h}$ generating $H^3(\widetilde{\mathbf{E}}) \cong \mathbb{Z}$, it is enough to prove the theorem in this case. We begin with some details on the topology of $\widetilde{\mathbf{E}}$ and $\widetilde{R}$ which are perhaps of independent interest. □

Lemma 5.7. *Let $\widetilde{R}$ be the classifying space for pairs over simply connected base spaces, as in Theorem 5.3, and let $(p\colon \widetilde{\mathbf{E}} \to \widetilde{R}, \mathbf{h})$ be the canonical pair over $\widetilde{R}$. Then with notation as in Theorem 5.3, the cohomology ring $H^*(\widetilde{R}, \mathbb{Z})$ is*

$$\frac{\mathbb{Z}[x_1, \dots, x_n, y_1, \dots, y_n]}{(x_1 y_1 + \cdots + x_n y_n)} + \textit{torsion},$$

where the x_j's and y_j's are in degree 2 and all the torsion is in degrees 5 or higher. In particular, $H^1(\widetilde{R}) = H^3(\widetilde{R}) = 0$ and $H^2(\widetilde{R})$ and $H^4(\widetilde{R})$ are torsion-free. The characteristic class of $p\colon \widetilde{\mathbf{E}} \to \widetilde{R}$ is $[p] = (x_1, \dots, x_n)$ and of the T-dual bundle is $[p^{\#}] = (y_1, \dots, y_n)$. The space $\widetilde{\mathbf{E}}$ is homotopy equivalent to $K(\mathbb{Z}, 3) \times K(\mathbb{Z}^n, 2)$, so its cohomology ring is

$$\frac{\mathbb{Z}[y_1, \dots, y_n, \iota_3]}{(\iota_3^2)} + \textit{torsion},$$

where the y_j's are in degree 2 (pulled back from generators of the same name in $H^2(\widetilde{R})$), ι_3 is the canonical generator of $H^3(K(\mathbb{Z}, 3), 3)$ in degree 3, and all the torsion (which can be explicitly described using the Steenrod algebra [36, Theorem 6.19]*, since it comes from the torsion in the cohomology of $K(\mathbb{Z}, 3)$), is in degrees 5 or higher.*

Proof. From the proof of Theorem 5.1, $\mathbf{E}$ is the homotopy fiber of the map $c\colon R \to BT$, and can be identified with $\mathrm{Maps}_0(T, K(\mathbb{Z}, 3))$, which splits as a product of $K(\mathbb{Z}, 3)$ and $\mathrm{Maps}_0^+(T, K(\mathbb{Z}, 3))$. Thus the universal cover $\widetilde{\mathbf{E}}$ of $\mathbf{E}$ has the homotopy type of $K(\mathbb{Z}, 3) \times K(\mathbb{Z}^n, 2)$, since $\pi_2\big(\mathrm{Maps}_0^+(T, K(\mathbb{Z}, 3))\big)$

$\cong \mathbb{Z}^n$. Now consider the Serre spectral sequences for the fibrations

$$K(\mathbb{Z},3) \longrightarrow \widetilde{\mathbf{E}} \longrightarrow K(\mathbb{Z}^n,2) = (\mathbb{CP}^\infty)^n,$$
$$K(\mathbb{Z},3) \longrightarrow \widetilde{R} \longrightarrow K(\mathbb{Z}^{2n},2) = (\mathbb{CP}^\infty)^{2n}.$$

These both have the form of figure 2, with the solid dots representing the nonzero free abelian groups (except that for the first of these, $2n$ should be replaced by n in the label on the horizontal axis). (There is also higher rank torsion in $H^*(K(\mathbb{Z},3))$, starting with $\mathrm{Sq}^2\,\iota_3$ in dimension 5 [36, Theorem 6.19].) The diagonal arrow in the picture is d_4 and sends ι_3 to the k-invariant of the fibration (see [36, Chapter 6]), which in the case of $\widetilde{\mathbf{E}}$ is 0 since the fibration is a product, and in the case of $\widetilde{R}$ is $x_1y_1 + \cdots + x_ny_n$ by Theorem 5.3.

So modulo torsion starting in dimension 5, $\widetilde{\mathbf{E}}$ has the same cohomology ring as $S^3 \times (\mathbb{CP}^\infty)^n$, i.e., $\mathbb{Z}[y_1,\dots,y_n,\iota_3]/(\iota_3^2)$, with ι_3 identified with the class $\mathbf{h}$ in Theorem 5.1. In the case of $\widetilde{R}$, since d_4 is injective, we see that, again modulo torsion starting in dimension 5, the cohomology ring is the same as the E_5 term in the spectral sequence, which is

$$\frac{\mathbb{Z}[x_1,\dots,x_n,y_1,\dots,y_n]}{(x_1y_1+\cdots+x_ny_n)}.$$

□

Proof of Theorem 5.6. As mentioned above, we just need to compute in the universal example $\pi\colon \widetilde{\mathbf{E}} \to \widetilde{R}$. $H^2(\widetilde{R})$ was computed in the Lemma, and $[\pi] = (x_1,\dots,x_n)$. The H-flux δ is, with respect to the notation of

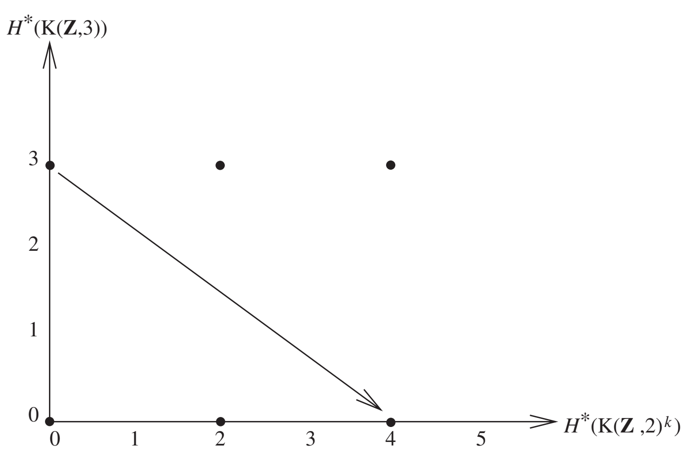


Figure 2: The Serre spectral sequence.

Lemma 5.7, given by ι. The T-dual bundle $\pi^{\#}\colon \widetilde{\mathbf{E}}^{\#} \to \widetilde{R}$ will have $[\pi^{\#}] = (y_1, \dots, y_n)$, as T-duality interchanges the x's and the y's. The intermediate bundles $\widetilde{\mathbf{E}} \xrightarrow{\pi_j} \widetilde{\mathbf{E}}_j \xrightarrow{p_j} \widetilde{R}$ are characterized by

$$[p_j] = (x_1, \cdots, \widehat{x_j}, \dots, x_n).$$

By a calculation identical to that in Lemma 5.7, we find that

$$H^*(\widetilde{\mathbf{E}}_j) \cong \frac{\mathbb{Z}[x_j, y_1, \dots, y_n]}{(x_j y_j)}$$

(each generator pulled back from one of the same name in $H^2(\widetilde{R})$) modulo torsion in degrees 5 and higher, and with respect to this identification, $[\pi_j] = x_j$. Now from the Gysin sequence for the circle bundle $\pi_j\colon \widetilde{\mathbf{E}} \to \widetilde{\mathbf{E}}_j$, we find that

$$(\pi_j)_!(\iota) = y_j = (p_j)^*([\pi^{\#}]_j),$$

as required. The dual formula is now obtained by symmetry. □

5.3 A splitting of the classifying space when $n = 2$

In this subsection, we specialize to the case $n = 2$, which was the subject of our paper [33]. It is possible that similar results hold for general n, but proving them would be more complicated. In any event, our objective here is to show that, at least for $n = 2$, every principal torus bundle with H-flux is, roughly speaking, decomposed into two pieces: one with a classical T-dual and one without. More precisely, we have the following rather surprising theorem:

Theorem 5.8. *When $n = 2$, if R is the classifying space for pairs defined in Theorem 5.1 and if $\widetilde{R}$ is its universal cover, then R is homotopy equivalent to the product $\widetilde{R} \times S^1$.*

Proof. We have seen that $\pi_1(R) \cong \mathbb{Z}$, so there is a map $\mu\colon R \to S^1 = K(\mathbb{Z}, 1)$ which is an isomorphism on π_1. This map is necessarily split, since if γ is a generator of $\pi_1(R)$, then by definition, γ corresponds to a map $S^1 \to R$ splitting μ. It remain to construct a map $\beta\colon R \to \widetilde{R}$ which induces an isomorphism on π_2 and π_3; then $\beta \times \mu$ will be a homotopy equivalence $R \to \widetilde{R} \times S^1$. Since $\widetilde{R}$ is a fibration

$$K(\mathbb{Z}, 3) \longrightarrow \widetilde{R} \xrightarrow{c_2} K(\mathbb{Z}^{2n}, 2), \tag{5.4}$$

where c_2 is a classifying map for $H^2(\widetilde{R}, \mathbb{Z})$, we will begin by constructing a map $R \to K(\mathbb{Z}^{2n}, 2)$, which is an isomorphism on H^2, and lift it to a map to $\widetilde{R}$ by checking that it is compatible with the k-invariant of (5.4). To do this, we need to compute the low-dimensional cohomology of R. We can do this from the spectral sequence

$$E_2^{p,q} = H^p(\pi_1(R), H^q(\widetilde{R}, \mathbb{Z})) \Longrightarrow H^*(R, \mathbb{Z}),$$

together with the simplicity of R (cf. Theorem 5.2), which since the Hurewicz map $\pi_j(R) \to H_j(\widetilde{R}, \mathbb{Z})$ is an isomorphism for $j = 2$ or 3 by Lemma 5.7, guarantees that $\pi_1(R)$ acts trivially on $H^j(\widetilde{R}, \mathbb{Z})$, $j \leq 3$. Furthermore, from the same lemma, the cohomology of $\widetilde{R}$ is generated by elements in degrees 2 and 3, except perhaps for some torsion in degree 5 and up. So at least through dimension 4, $\pi_1(R)$ acts trivially on all the cohomology of $\widetilde{R}$.[2] Since $\pi_1(R) \cong \mathbb{Z}$, $E_2^{p,q} = 0$ for $p > 1$, and thus the spectral sequence collapses.[3] So, at least modulo torsion in dimensions ≥ 5, R has the same cohomology as $\widetilde{R} \times S^1$. That means there is a map $R \to K(\mathbb{Z}^{2n}, 2)$ which induces an isomorphism on H^2 and that this map lifts to a map $\beta\colon\ R \to \widetilde{R}$ which is an isomorphism on π_3 (since $H^4(R)$ is compatible with the k-invariant of $\widetilde{R}$). Then $\beta \times \mu\colon\ R \to \widetilde{R} \times S^1$ induces an isomorphism on all homotopy groups and is equivariant for the action of the fundamental group (which is trivial on both sides). Thus it is a homotopy equivalence by Whitehead's theorem, since R has the homotopy type of a CW-complex. □

Corollary 5.9. *For $n = 2$, the T-duality group $GO(2, 2; \mathbb{Z})$ acts by homotopy automorphisms, not only on $\widetilde{R}$ but also on R. For any pair $(p\colon\ X \to Z, \alpha)$, as in Section 5.2, we can attach two other pairs, $(p_1\colon\ X_1 \to Z, \alpha_1)$ with a classical T-dual, and $(p_2\colon\ X_2 \to Z, \alpha_2)$ which represents the "nonclassical" part.*

Proof. This is immediate from the fact that R splits up to homotopy as $\widetilde{R} \times S^1$, with the T-duality group $GO(2, 2; \mathbb{Z})$ acting on $\widetilde{R}$ by homotopy automorphisms. Given $(p\colon\ X \to Z, \alpha)$, its "classifying map" $c\colon\ Z \to R$ is uniquely defined up to homotopy, by Theorem 5.1. Then $\mu \circ c : Z \to S^1$ is homotopically nontrivial if and only if $p_!(\alpha) \neq 0$ in $H^1(Z, \mathbb{Z})$; thus it measures the "nonclassical" part of the pair. On the other hand, $\beta \circ c : Z \to \widetilde{R}$ does have a classical T-dual, as described in part (1) of Theorem 3.1. □

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[2] As a matter of fact, $\pi_1(R)$ acts trivially on *all* of the cohomology of $\widetilde{R}$, since the torsion in the cohomology of $K(\mathbb{Z}, 3)$ all comes from the canonical element ι via Steenrod operations [36, Theorem 6.19].

[3] It is this step which would not be obvious for $n > 2$. In the higher rank case, it is also not obvious how to construct a map $T^k \to R$ which induces an isomorphism on π_1.

6 The T-duality group in action

6.1 Some classical T-dual examples

6.1.1 Rank one examples

Let $X(p)$ be a circle bundle with H-flux over the 2-dimensional oriented Riemann surface Σ_g that has first Chern class equal to p times the volume form of Σ_g. For example, when $g = 0$, then $\Sigma_g = S^2$ and $X(p) = L(1, p) = S^3/\mathbb{Z}_p$ is the Lens space. The T-duality group in this situation is $GO(1, 1; \mathbb{Z})$, and we will compute the action of the group, as described in Section 5.1. By taking the Cartesian product with a manifold M, and pulling back the H-flux to the product, we see that $(M \times X(p), \delta = q)$ is T-dual to $(M \times X(q), \delta = p)$, and the element $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ of $GO(1, 1; \mathbb{Z})$ interchanges them.

The element $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ of the T-duality group $GO(1, 1; \mathbb{Z})$ lies in the subgroup $GL(1, \mathbb{Z})$, embedded as in Section 5.1, and acts by twisting the S^1 action on $M \times X(p)$. This twisted action makes $M \times X(p)$ into a circle bundle over $M \times \Sigma_g$ having first Chern class equal to $-p$ times the volume form of Σ_g. This bundle is denoted $M \times X(-p)$, and its total space is diffeomorphic to $M \times X(p)$, though by an orientation-reversing diffeomorphism. Therefore the action of $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ on the pair $(M \times X(p), \delta = q)$ and $(M \times X(q), \delta = p)$ gives rise to a new T-dual pair $(M \times X(-p), \delta = -q)$ and $(M \times X(-q), \delta = -p)$.

The group $GO(1, 1; \mathbb{Z})$ is generated by the two elements of $O(1, 1; \mathbb{Z})$ just discussed and by $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, which replaces the original T-dual pair by the pair consisting of $(M \times X(p), \delta = -q)$ and $(M \times X(-q), \delta = p)$. This refines earlier understood T-duality. Thus in general there are eight different (bundle, H-flux) pairs with equivalent physics, corresponding to $(\pm p, \pm q)$ and $(\pm q, \pm p)$.

6.1.2 An interesting higher rank example

The following interesting test case for our theory was suggested to us by Professor Edward Witten, who had studied it a few years ago in joint work with Diaconescu et al. [16, end of Section 5]. Namely, let $X = \mathbb{RP}^7 \times \mathbb{RP}^3$, which is the total space of a principal $\mathbb{T}^2$ bundle π over $Z = \mathbb{CP}^3 \times \mathbb{CP}^1$. The cohomology ring of Z is $\mathbb{Z}[c, d]/(c^4, d^2)$, where $c \in H^2(\mathbb{CP}^3)$ and $d \in$

$H^2(\mathbb{CP}^1)$ are the usual generators. Note that the characteristic class $[\pi] \in H^2(Z;\mathbb{Z}^2)$ that classifies π is $(2c, 2d)$. The cohomology ring of X over the field $\mathbb{F}_2$ of two elements is $\mathbb{F}_2[a,b]/(a^8, b^4)$, where a and b are the generators of $H^1(\mathbb{RP}^7;\mathbb{F}_2)$ and $H^1(\mathbb{RP}^3;\mathbb{F}_2)$, respectively. The classes a^2 and b^2 are both reductions of integral classes in H^2. So by the Künneth theorem, $H^3(X;\mathbb{Z}) \cong \mathbb{Z}\gamma \oplus \mathbb{Z}/2$, where the $\mathbb{Z}/2$ summand is generated by the unique nonzero 2-torsion class, the integral Bockstein $\beta(ab)$ of ab, and γ comes from $H^3(\mathbb{RP}^3)$. The class $\beta(ab)$ reduces mod 2 to $\mathrm{Sq}^1(ab) = a^2b + ba^2$. Consider the H-flux $\delta = \beta(ab)$ on X together with the $\mathbb{T}^2$-bundle $\pi : X \to Z$. Since Z is simply connected, this data should have a unique T-dual, and this dual should be classical, i.e., should be given by a $\mathbb{T}^2$-bundle $\pi : X^{\#} \to Z$ with dual H-flux $\delta^{\#}$, and there should be an isomorphism on twisted K-theory $K^*(X,\delta) \cong K^*(X^{\#},\delta^{\#})$.

As we have seen, the T-dual is constructed by taking the crossed product $CT(X,\delta) \rtimes \mathbb{R}^2$ of the stable continuous-trace algebra over X with Dixmier–Douady invariant δ by an $\mathbb{R}^2$-action lifting the transitive action of $\mathbb{R}^2$ on X with quotient Z. If we rewrite the crossed product in two stages as $(CT(X,\delta) \rtimes \mathbb{R}) \rtimes \mathbb{R}$, we see that we can compute $X^{\#}$ in two steps. First we dualize with respect to one one-dimensional torus bundle, getting $CT(X,\delta) \rtimes \mathbb{R} \cong CT(X_1,\delta_1)$, then with respect to the other. We have a choice of how to factor the action, but let us say we deal first with the bundle $\mathbb{RP}^7 \to \mathbb{CP}^3$ and then with the bundle $\mathbb{RP}^3 \to \mathbb{CP}^1$.

Dualizing the first circle bundle gives a diagram

$$\begin{array}{ccccc} & & \mathbb{RP}^7 \times S^1 \times S^3 & & \\ & \swarrow^{\pi_1^*(\pi_2)} & & \searrow^{\pi_2^*(\pi_1)} & \\ X = \mathbb{RP}^7 \times \mathbb{RP}^3 & & & & X_1 = \mathbb{CP}^3 \times S^1 \times S^3 \\ & \searrow_{\pi_1} & & \swarrow_{\pi_2} & \\ & & Z_1 = \mathbb{CP}^3 \times \mathbb{RP}^3 & & \end{array} \tag{6.1}$$

which we can check and use to compute δ_1 as follows. We already know the identity of the spaces X and Z_1 and of the bundle π_1. The T-duality (and Raeburn–Rosenberg) conditions give

$$(\pi_1)_!(\delta) = [\pi_2], \quad (\pi_2)_!(\delta_1) = [\pi_1] = 2c, \quad (\pi_1^*(\pi_2))^*(\delta) = (\pi_2^*(\pi_1))^*(\delta_1). \tag{6.2}$$

In the Gysin sequence for π_1, the map $\pi_1^* : H^3(\mathbb{CP}^3 \times \mathbb{RP}^3) \to H^3(X)$ sends the generator γ of $H^3(\mathbb{RP}^3)$ onto itself (if we think of cohomology of the $\mathbb{RP}^3$ factor as being embedded in the cohomology of either product by the Künneth Theorem), so $(\pi_1)_!$ is injective on the torsion in $H^3(X)$ and sends δ to a nonzero 2-torsion class in $H^2(X)$, which must be $\beta(b)$, the integral

Bockstein of $b \in H^1(\mathbb{RP}^3; \mathbb{F}_2)$. But the unique nontrivial principal $\mathbb{T}$-bundle over $\mathbb{RP}^3$ can be identified with the short exact sequence of compact Lie groups

$$\mathbb{T} \longrightarrow \text{Spin}^c(3) \twoheadrightarrow SO(3),$$

where $SO(3)$ is homeomorphic to $\mathbb{RP}^3$ and $\text{Spin}^c(3)$, the quotient of $\mathbb{T} \times SU(2)$ by the diagonal copy of $\{\pm 1\}$, has torsion-free fundamental group, and is thus homeomorphic to $S^1 \times S^3$. That verifies the diagram (6.1).

Now we use the conditions (6.2) to compute δ_1. The cohomology ring of X_1 is $\mathbb{Z}[c]/(c^4) \otimes \bigwedge_{\mathbb{Z}}(\eta, \zeta)$, where the exterior algebra generators η and ζ have degrees 1 and 3, respectively. So $H^3(X_1) \cong \mathbb{Z}c\eta \oplus \mathbb{Z}\zeta$. In the Gysin sequence for π_2, the map $\pi_2^* : H^3(\mathbb{CP}^3 \times \mathbb{RP}^3) \to H^3(X_1)$ must be injective (since $H^1(\mathbb{CP}^3 \times \mathbb{RP}^3) = 0$), and the image of $(\pi_2)_!$ is the kernel of cup product with $\beta(b)$, or $2\mathbb{Z}c \oplus (\mathbb{Z}/2)\beta(b)$. In fact, one can see that $(\pi_2)_!(c\eta) = 2c = [\pi_1]$. So $\delta_1 \equiv c\eta \mod \ker(\pi_2)_! = 2\zeta\mathbb{Z}$.

But we also have the condition $(\pi_1^*(\pi_2))^*(\delta) = (\pi_2^*(\pi_1))^*(\delta_1)$ from (6.2). We have $H^3\big(\mathbb{RP}^7 \times S^1 \times S^3\big) \cong \mathbb{Z}\zeta \oplus (\mathbb{Z}/2)\beta(a)\eta$. To compute $(\pi_1^*(\pi_2))^*(\delta)$, use cohomology with coefficients in $\mathbb{F}_2$: the map $\text{Spin}^c(3) \twoheadrightarrow SO(3)$ is surjective on π_1, so the map $H^*(\mathbb{RP}^3; \mathbb{F}_2) \to H^*(S^1 \times S^3, \mathbb{F}_2)$ sends the generator b to η, and

$$(\pi_2^*(\pi_1))^*(\delta_1) = (\pi_2^*(\pi_1))^*(a^2 b + ab^2) = a^2\eta = \beta(a)\eta,$$

while $c\eta + 2k\zeta$ pulls back to $\beta(a)\eta + 2k\zeta$. So this forces $k = 0$ and $\delta_1 = c\eta$. This already has one surprising consequence that

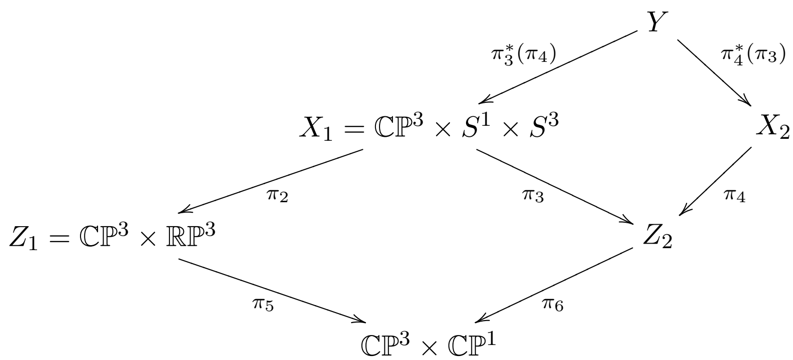
$$K^{*+1}(\mathbb{RP}^7 \times \mathbb{RP}^3,\ \beta(b)) \cong K^*(\mathbb{CP}^3 \times S^1 \times S^3,\ c\eta).$$

Now the right-hand size can be computed from the twisted Atiyah–Hirzebruch spectral sequence (see [42]) with E_2 term

$$E_2^{p,q} = H^p(\mathbb{CP}^3 \times S^1 \times S^3; \mathbb{Z}), \quad q \text{ even (and 0 for } q \text{ odd)}$$

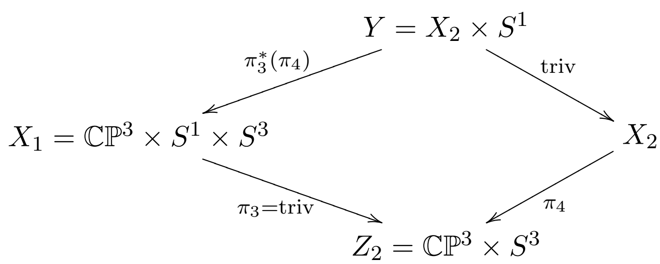
and $d_3 = \_\_ \cup c\eta$ (since the cohomology is torsion-free). Now $H^*(\mathbb{CP}^3 \times S^1 \times S^3) = \mathbb{Z}[c, \eta, \zeta]/(c^4, \eta^2, \zeta^2)$, and $\ker(\_\_ \cup c\eta) = \langle c^3, \eta \rangle$, $\text{im}(\_\_ \cup c\eta) = \langle c\eta \rangle$. So $E_4 = \langle c^3, \eta \rangle / \langle c\eta \rangle$ is torsion-free and is generated (as a free abelian group) by η, $\eta\zeta$, c^3, and $c^3\zeta$. Thus $K^*(X, \delta) \cong \mathbb{Z}^2$ in both even and odd degrees. This seems quite unexpected, since the homology and cohomology of $\mathbb{RP}^7 \times \mathbb{RP}^3$ has a huge amount of torsion in it.

Now we can go ahead and dualize the other circle bundle. We obtain a commutative diagram looking like this:



(6.3)

Here we should have $\pi_2 = \pi_5^*(\pi_6)$ and $\pi_3 = \pi_6^*(\pi_5)$. That means that $[\pi_2] = \beta(b) = \pi_5^*([\pi_6])$, so $[\pi_6]$ must be an odd multiple of $d \in H^2(\mathbb{CP}^1)$. That in turn means Z_2 must be $\mathbb{CP}^3 \times S^3$ or $\mathbb{CP}^3 \times L^3$, where L^3 is a three-dimensional lens space with finite fundamental group, but the latter possibility is ruled out by the requirement that $S^1 \times S^3$ be a principal S^1 bundle over Z_2. So $[\pi_6] = d$ and $Z_2 \cong \mathbb{CP}^3 \times S^3$. That means π_3 is a trivial bundle, so $Y \cong X_2 \times S^1$. So the upper diamond in (6.3) becomes



(6.4)

with $[\pi_4] = (\pi_3)_!(\delta_1) = (\pi_3)_!(c\eta) = c$. Thus $X_2 = S^7 \times S^3$ with π_4 the product of the Hopf bundle $S^7 \to \mathbb{CP}^3$ with the identity on S^3. Since the H-flux δ_2 on $X_2 = S^7 \times S^3$ must pull back to the pull-back of $c\eta \in H^*(\mathbb{CP}^3 \times S^1 \times S^3)$ in $H^*(S^7 \times S^1 \times S^3)$, $\delta_2 = 0$, which is also consistent with our calculation of $K^*(X, \delta)$.

In summary, we have shown that the T-dual of $(\pi : \mathbb{RP}^7 \times \mathbb{RP}^3 \to Z, \delta)$, $Z = \mathbb{CP}^3 \times \mathbb{CP}^1$, and $\delta = \beta(ab)$, is the pair $(\pi^\# : S^7 \times S^3 \to Z, 0)$ with trivial H-flux and with $[\pi] = (2c, 2d) \in H^2(Z; \mathbb{Z}^2)$, $[\pi^\#] = (c, d) \in H^2(Z; \mathbb{Z}^2)$. The element $\begin{pmatrix} 1_2 & 0 \\ 0 & -1_2 \end{pmatrix} \in GO(2, 2; \mathbb{Z})$ replaces the original T-dual by the pair consisting of $(S^7 \times S^3, 0)$, where S^3 and S^7 are given the inverse of the standard circle action, i.e., we change the signs of the Chern classes of the dual bundle. An alternate derivation of the T-dual based on physical arguments was given in [5].

7 Possible generalizations

Instead of just compactifying over torus bundles, one possible generalization is to compactify over the classifying spaces $B\Gamma$ of discrete groups Γ. (Recall that the rank n torus is $B\mathbb{Z}^n$.) Other examples are Riemann surfaces Σ_g of genus $g > 0$, which are classifying spaces of their fundamental groups.

We will give some evidence for this here and begin with a discussion of a special case. Suppose that $\Sigma = B\Gamma$ is a compact, smooth, connected, manifold, where Γ is a discrete group. Also assume for simplicity that Σ has a spin^c structure. (This is not essential but will simplify one step in what follows.) Consider the H-flux $\delta = \tau \times \beta \in H^3(\Sigma \times \mathbb{T}, \mathbb{Z})$; here $\beta \in H^1(\mathbb{T}, \mathbb{Z})$ is the canonical generator, and $\tau \in H^2(\Sigma, \mathbb{Z}) \cong H^2(\Gamma, \mathbb{Z})$ determines the central extension

$$1 \to \mathbb{Z} \to \Gamma_\tau \to \Gamma \to 1.$$

Proposition 7.1. *Suppose that we are in the situation described above. Let n be the dimension of Σ (mod 2), so $n+1$ is the dimension of $B\Gamma_\tau$. Then there is a commutative diagram,*

$$\begin{array}{ccc} K^\bullet(\Sigma \times \mathbb{T}, \delta) & \xrightarrow{T_!} & K_{\bullet+n}(C^*(\Gamma_\tau)) \\ \Big\downarrow \mathrm{Ch}_\delta & & \Big\downarrow \mathrm{Ch} \\ H^\bullet(\Sigma \times \mathbb{T}, \delta) & \xrightarrow{T_*} & HP_{\bullet+n}(\mathcal{S}(\Gamma_\tau)) \end{array} \tag{7.1}$$

where Ch_δ *is the twisted Chern character and* Ch *is the Connes–Chern character.*

If the Baum–Connes conjecture holds for the discrete group Γ_τ, then the horizontal arrows are isomorphisms.

Proof. First observe that if we consider $(\Sigma \times \mathbb{T}, \delta)$ as a circle bundle with H-flux over Σ, then its T-dual is given by the circle bundle $B\Gamma_\tau \to \Sigma$ with first Chern class equal to $\tau \in H^2(\Sigma, \mathbb{Z})$, with vanishing H-flux on $B\Gamma_\tau$. Thus T-duality for circle bundles yields the isomorphism

$$K^\bullet(\Sigma \times \mathbb{T}, \delta) \cong K^{\bullet+1}(B\Gamma_\tau). \tag{7.2}$$

If Σ has a spin^c structure, then so does $B\Gamma_\tau$ (since it is the sphere bundle in a complex line bundle over a spin^c manifold). So by Poincaré duality, $K^{\bullet+1}(B\Gamma_\tau) \cong K_{\bullet+n}(B\Gamma_\tau)$. Thus the assembly map can be viewed as a homomorphism

$$K^{\bullet+1}(B\Gamma_\tau) \to K_{\bullet+n}(C^*(\Gamma_\tau)). \tag{7.3}$$

The composition of the homomorphisms in equations (7.2) and (7.3) yields the upper horizontal homomorphism $T_!$; and the lower homomorphism is similarly defined.

The validity of the Baum–Connes conjecture for Γ_τ is the same as saying that the assembly map given by equation (7.3) is an isomorphism. In this case, it follows that $T_!$ is an isomorphism. The argument to prove that the bottom horizontal arrow is an isomorphism is similar, and so is the commutativity of the diagram. □

This suggests that the T-dual of $(\Sigma \times \mathbb{T}, \delta)$ appears to be the C^*-algebra $C^*(\Gamma_\tau)$, which can be identified with the algebra of sections of a continuous field of C^*-algebras over $\mathbb{T}$, whose fiber at the point $z \in \mathbb{T}$ is the twisted group C^*-algebra $C^*(\Gamma, z)$.

8 Open questions

We conclude with just a few open questions. First of all, the "nonclassical case" still remains a bit of a mystery, because of the lack of symmetry between a torus bundle and its noncommutative dual. The group $GL(n, \mathbb{Z})$ acts as usual by reparameterizations of the action, but it is not clear if there is a meaningful $GO(n, n; \mathbb{Z})$ action in this case.

Secondly, we have been unable to determine whether the splitting result in Theorem 5.8 generalizes to the case of higher rank. An equivalent problem is to determine the Postnikov tower of R itself, not just that of $\widetilde{R}$. This is complicated on two accounts; first of all, one needs to compute the k-invariant for the fibration

$$K(\pi_2(R), 2) = K(\mathbb{Z}^{2n}, 2) \longrightarrow R_1 \longrightarrow K(\pi_1(R), 1) = T^k,$$

with $k = \binom{n}{2}$, in $H^3(T^k, \mathbb{Z}^{2n})$, and even if this vanishes, one then still needs to determine if the k-invariant of the fibration

$$K(\pi_2(R), 3) = K(\mathbb{Z}, 3) \longrightarrow R \longrightarrow R_1,$$

in $H^4(R_1, \mathbb{Z})$ lives entirely on the $K(\mathbb{Z}^{2n}, 2)$ piece.

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