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004

HYPOELLIPTICITY OF THE $\bar\partial$-NEUMANN PROBLEM AT A POINT OF INFINITE TYPE*

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Abstract. We prove local hypoellipticity of the complex Laplacian $\Box$ in a domain which has superlogarithmic estimates outside a curve transversal to the CR directions and for which the holomorphic tangential derivatives of a defining function are superlogarithmic multipliers in the sense of [10].

Key words. Hypoellipticity, $\bar\partial$-Neumann problem, superlogarithmic estimate, infinite type.

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1. Introduction. For the pseudoconvex domain $\Omega \subset \mathbb{C}^n$ whose boundary is defined in local coordinates $z = x + iy$ in a neighborhood U of $z_o = 0$, by

$$2x_n = \exp\left(-\frac{1}{(\sum_{j=1}^{n-1}|z_j|^2)^{\frac{s}{2}}}\right), \quad s > 0, \tag{1.1}$$

the tangential Kohn Laplacian $\Box_b = \bar\partial_b\bar\partial_b^* + \bar\partial_b^*\bar\partial_b$ as well as the full Laplacian $\Box = \bar\partial\bar\partial^* + \bar\partial^*\bar\partial$ show very interesting features especially in comparison with the "tube domain" whose boundary is defined by

$$2x_n = \exp\left(-\frac{1}{(\sum_{j=1}^{n-1}|x_j|^2)^{\frac{s}{2}}}\right), \quad s > 0. \tag{1.2}$$

(Here z_j have been replaced by x_j at exponent.) Energy estimates are the same for the two domains. For the problem on the boundary $b\Omega$, they come as

$$||(\log\Lambda)^{\frac{1}{s}}u||_{b\Omega} \underset{\sim}{<} ||\bar\partial_b u||^2_{b\Omega} + ||\bar\partial_b^* u||^2_{b\Omega} + ||u||^2_{b\Omega} \quad \text{for any form } u \in C^\infty_c(b\Omega\cap U)^k \text{ of degree } k \in [1, n-2]. \tag{1.3}$$

Here $\log\Lambda$ is the tangential pseudodifferential operator with symbol $\log(1+|\xi|^2)^{\frac{1}{2}}$, $\xi \in \mathbb{R}^{2n-1}$, the dual real tangent space. As for the problem on the domain Ω, one has simply to replace $\bar\partial_b$, $\bar\partial_b^*$ by $\bar\partial$, $\bar\partial^*$ and take norms over Ω for forms u in $D_{\bar\partial^*}$, the domain of $\bar\partial^*$, of degree $1 \le k \le n-1$; this can be seen, for instance, in [13]. In particular, these are superlogarithmic estimates if and only if $s < 1$; otherwise, for any $s > 0$ they are compactness estimates. A related problem is that of the local hypoellipticity of the Kohn Laplacian $\Box_b$ or, with equivalent terminology, the local regularity of the inverse (modulo harmonics) Green operator $N_b = \Box_b^{-1}$. Similar is the notion of hypoellipticity of the Laplacian $\Box$ or the regularity of the inverse Neumann operator $N = \Box^{-1}$. It has been proved by Kohn in [17] and by the two last authors in [14] that superlogarithmic estimates suffice for local hypoellipticity of

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the problem in the boundary and in the domain. (Note that hypoellipticity for the domain, [17] Theorem 8.3, is deduced from microlocal hypoellipticity for the boundary, [17] Theorem 7.1, but a direct proof is also available, [10] Theorem 5.4.) In particular, for (1.1) and (1.3), there is local hypoellipticity when $s<1$.

As for the more delicate hypoellipticity, in the critical range of indices $s\geq 1$, only the tangential problem has been studied and the striking conclusion is that the behavior of (1.1) and (1.2) split. The first stays always hypoelliptic for any s (Kohn [16]) whereas the second is not for $s\geq 1$ (Christ [6]). When one tries to relate $(\bar\partial_b,\bar\partial_b^*)$ on $b\Omega$ to $(\bar\partial,\bar\partial^*)$ on Ω, estimates go well through (Kohn [17] Section 8 and Khanh [10] Chapter 4) but not regularity. In particular, the two conclusions about tangential hypoellipticity of $\Box_b$ for (1.1) and non-hypoellipticity for (1.2) when $s\geq 1$, cannot be automatically transferred from $b\Omega$ to Ω. Now, for the non-hypoellipticity in Ω in case of the tube (1.2) the authors have obtained in [3] a result of propagation which is not equivalent but intimately related. The real planes of the variables $x_1,...,x_{n-1}$ are propagators of holomorphic extendibility from Ω across $b\Omega$. What we prove in the present paper is the converse, that is, hypoellipticity in Ω for (1.1) when $s\geq 1$. Indeed, we prove local regularity not only for (1.1) but also for the case of higher exponential type at 0. The model of our domain is the one with boundary

$$2x_n=\exp\Big(-\frac{1}{\Big(\sum_{j=1}^{n-1}\exp(-\frac{1}{|z_j|^{m_j}}\Big)^s}\Big),\quad m_j<1,\ s>0. \tag{1.4}$$

Here, the best possible estimate at $z_1=0,...,z_{n-1}=0$ is worse than for the domain with boundary (1.1), that is,

$$||\log^{\frac{1}{m}}(1+\log^{\frac{1}{s}}\Lambda)u||^2\lesssim Q(u,u),\quad \text{for } m:=\max m_j.$$

When $z_j\neq 0$ for any j we have of course a $\frac{1}{2}$-subelliptic estimate but, if $z_j=0$ for some j, then we do not have a subelliptic estimate as it was for (1.1) but just a $\log^{\frac{1}{m}}$-estimate; however it is strong enough for our need because it is superlogarithmic on account of $m_j<1$ for any j. Also, at $z_1=0,...,z_{n-1}=0$, the functions $\partial_{z_j}r$, $j=1,...,n-1$ are no more subelliptic multipliers (as in (1.1)); however, they are superlogarithmic multipliers (again, for $m_j<1,\ j=1,...,n-1$). Thus, (1.4) serves as a model of our main result

Theorem 1.1. *Let Ω be a pseudoconvex, rigid, domain of $\mathbb{C}^n$ in a neighborhood of $z_o=0$ and assume that the $\bar\partial$-Neumann problem satisfies the following properties for forms of degree ≥ 1*

(i) there is a superlogarithmic estimate for $(z_1,...,z_{n-1})\neq 0$,

(ii) $\partial_{z_j}r,\ j=1,...,n-1$, are superlogarithmic multipliers (cf. [15] and [10]).

Then $\Box$ is locally hypoelliptic at z_o for forms of any degreee $k\geq 0$.

The proof follows in Section 2. It consists in relating the system on Ω to the tangential system on $b\Omega$ along the guidelines of [17] Section 8, and then in using the argument of [16] to control the commutators of the energy Q with the derivatives D^s and the cut-off functions ζ.

Remark 1.2. What we prove is in fact, for a pair $\zeta_o\prec\zeta$ of "nested" cut-off in tangential directions having support in a neighborhood U of z_o,

$$||\zeta_o u||_s\lesssim||\zeta\bar\partial u||_s+||\zeta\bar\partial^*u||_s+||u||_0,\quad \text{for any } u\in\mathcal{H}_k^\perp,\ k\geq 0, \tag{1.5}$$

where $\mathcal{H}_k^\perp$ is the orthogonal to the space of harmonic k-forms and s is the index of the norm in the Sobolev spaces H^s. Note here that we have in fact $\mathcal{H}_k = \{0\}$ for any $k \geq 1$.

We now observe that $b\Omega$ is given only locally in a neighborhodd of z_o. We can continue $b\Omega$ leaving it unchanged in a neighborhood of z_o, making it strongly pseudoconvex elsewhere, in such a way that it bounds a relatively compact pseudoconvex domain $\Omega \subset\subset \mathbb{C}^n$ (cf. [19]). Thus, by the L^2-theory of $\bar\partial$, there is well defined the Neumann operator $N = \Box^{-1}$. As an immediate consequence of (1.5) we have that

$$\bar\partial^* N \text{ and } \bar\partial N \text{ are exactly locally } H^s\text{-regular at } z_o \text{ over } \ker\bar\partial \text{ and } \ker\bar\partial^* \text{ respectively.} \tag{1.6}$$

We specify the action of N on q-forms by the notation N_q and denote by $B_q := \mathrm{Id} - \bar\partial^* N_{q+1}\bar\partial$ the Bergman projection and by $K(z,w)$ the Bergman kernel respectively. From the regularity of $\bar\partial^* N$ it follows that the Bergman projection B is also regular. (Notice that exact regularity is perhaps lost by the presence of $\bar\partial$ in B.) To prove the local regularity at z_o of N itself, we follow now the method of Boas-Straube and exploit formula (5.36) in [20] in unweighted norms, that is, for $t = 0$:

$$\begin{aligned} N_q = {} & B_q(N_q\bar\partial)(\mathrm{Id} - B_{q-1})(\bar\partial^* N_q)B_q \\ & + (\mathrm{Id} - B_q)(\bar\partial^* N_{q+1})B_{q+1}(N_{q+1}\bar\partial)(\mathrm{Id} - B_q). \end{aligned}$$

Now, in the right side, the $\bar\partial N$'s and $\bar\partial^* N$'s are evaluated over $\mathrm{Ker}\,\bar\partial^*$ and $\mathrm{Ker}\,\bar\partial$ respectively; thus they are exactly locally regular at z_o. The B's are also locally regular at z_o and therefore such is N. We put in a separate statement our conclusions for B and also give a complement about K.

THEOREM 1.3. *We have, for a neighborhood U of z_o and for any pair of cut-off $\zeta_o \prec \zeta$ with support in U*

$$\begin{cases} ||\zeta_o B_q \alpha||_s \underset{\sim}{<} ||\zeta\alpha||_{s+1} + ||\alpha||_0, \\ K(z,w)|_{U\times U} \in C^\infty((\bar\Omega \times \bar\Omega) \setminus \mathit{Diagonal}). \end{cases} \tag{1.7}$$

Proof. The first of (1.7) has already been discussed. The second follows from the first by the method of Kerzman [9]. Note that it is explicit, in particular according to the note added in the proof at p.158, that only the local regularity of B_q in the form of the first of (1.7) is needed to get the second. □

2. Proofs. We need several preliminary results

PROPOSITION 2.1. *If (i) and (ii) hold for forms u of degree ≥ 1, they also holds for functions $u \in \ker\bar\partial^\perp$.*

Proof. Since $\bar\partial$ has closed range, then, given $u \in \ker\bar\partial^\perp$, we can find a solution v of degree 1 of

$$\begin{cases} \bar\partial^* v = u, \quad \bar\partial v = 0, \\ ||v|| \underset{\sim}{<} ||u||. \end{cases}$$

Let U be the neighborhood of z_o in which (i) and (ii) hold. We have for $\zeta_o \in C_c^\infty(U)$,

(2.1)

$$\begin{aligned} ||\log(\Lambda)\zeta_o u||^2 &= \Big(\log(\Lambda)\zeta_o u, \log(\Lambda)\zeta_o\bar\partial^* v\Big) \\ &= \Big(\bar\partial\zeta_o u, \log^2(\Lambda)\zeta_o v\Big) + \Big(\log(\Lambda)\zeta_o u, \log^{-1}(\Lambda)[\log^2(\Lambda)\zeta_o, \bar\partial^*]v\Big) \\ &\le \underbrace{\epsilon||\bar\partial\zeta_o u||^2}_{\text{good}} + c_\epsilon\underbrace{||\log^2(\Lambda)\zeta_o v||^2}_{\text{(a)}} + \underbrace{\epsilon||\log(\Lambda)\zeta_o u||^2}_{\text{absorbable}} + c_\epsilon\underbrace{||\log^{-1}(\Lambda)[\log^2(\Lambda)\zeta_o, \bar\partial^*]v||^2}_{\text{(b)}}. \end{aligned}$$

Observe that, for $\zeta \succ \zeta_o$

(2.2)

$$\begin{aligned} (a) &\le ||\log(\Lambda)\zeta\log(\Lambda)\zeta_o v||^2 + ||v||^2_{-\infty} \\ &\underset{\text{by (i)}}{\le} \epsilon Q(\zeta\log(\Lambda)\zeta_o v, \zeta\log(\Lambda)\zeta_o v) + c_\epsilon||\zeta\log(\Lambda)\zeta_o v||_0^2 + ||v||^2_{-\infty} \\ &\le \epsilon\Big(\underbrace{||\zeta\log(\Lambda)\zeta_o\bar\partial^* v||^2}_{\text{absorbable}} + \underbrace{||[\bar\partial^{(*)}, \zeta\log(\Lambda)\zeta_o]v||^2}_{(*)}\Big) + c_\epsilon\underbrace{||\zeta\log(\Lambda)\zeta_o v||_0^2}_{(**)} + ||v||^2_{-\infty}, \end{aligned}$$

where $\bar\partial^{(*)}$ denotes either $\bar\partial^*$ or $\bar\partial$. Now,

$$[\bar\partial^{(*)}, \zeta\log(\Lambda)\zeta_o] \sim \dot\zeta\log(\Lambda)\zeta_o + \zeta\log(\Lambda)\zeta_o + \zeta\log(\Lambda)\dot\zeta_o.$$

Hence (*) and (**) are of type $||\zeta''\log(\Lambda)\zeta' v||^2$ and can therefore be estimated by

$$\begin{aligned} ||\zeta''\log(\Lambda)\zeta' v||^2 &\underset{\text{(i)}}{\lesssim} \epsilon Q(\zeta' v, \zeta' v) + c_\epsilon||\zeta'' v||^2 \\ &\le \epsilon(||\zeta' u||^2 + ||[\bar\partial^{(*)}, \zeta']v||^2) + c_\epsilon||\zeta'' v||^2 \\ &\lesssim c_\epsilon||u||_0^2. \end{aligned} \tag{2.3}$$

As for (b), we notice that

$$\log^{-1}(\Lambda)[\log^2(\Lambda)\zeta_o, \bar\partial^*] \sim \log(\Lambda)\dot\zeta_o + \zeta_o,$$

and hence, for $\zeta \succ \zeta_o$, $\dot\zeta_o$

$$\begin{aligned} (b) &\le ||\log(\Lambda)\zeta v||^2 + ||\zeta v||_{\Lambda^{-1}} \\ &\le \epsilon Q(\zeta v, \zeta v) + c_\epsilon||\zeta v||_0^2 \\ &\le \epsilon||\zeta u||_0^2 + ||[\bar\partial^{(*)}, \zeta]v||_0^2 + c_\epsilon||\zeta v||_0^2 \\ &\lesssim c_\epsilon||u||_0^2. \end{aligned}$$

Hence (2.1) can be continued by

$$\le \epsilon||\bar\partial\zeta_o u||^2 + c_\epsilon||u||_0^2.$$

Thus (i) also holds for u. The proof that we have the same conclusion for (ii) is the same as above. ∎

We note now that (i) implies a compactness estimate, that is, for any ϵ and for suitable c_ϵ

$$||u||_0^2 \leq \epsilon Q(u,u) + c_\epsilon ||u||^2_{-1} \quad \text{for any } u \in C^\infty_c(\bar\Omega)\cap D^k_{\bar\partial^*},\ k\geq 1 \text{ or } u\in C^\infty_c(\bar\Omega)\cap \ker\bar\partial^\perp_k,\ k=0. \tag{2.4}$$

This follows from a more general fact: a totally real submanifold of $b\Omega$, such as the y_n-line, is a removable set of non-compactness (and we are in this situation since a superlogarithmic estimate is stronger than a compactness estimate).

LEMMA 2.2. *Assume that there are compactness estimates on $b\Omega\cap U$ except from a totally real subset S. Then we have in fact compactness estimates in the whole $b\Omega \cap U$.*

Proof. We first prove (2.4) for $k\geq 1$. For this we introduce the family of weights $\{\varphi_\epsilon\}_\epsilon = \{\frac{d_S^2}{\epsilon^2}\}_\epsilon$ where d_S is the distance to S. These weights are bounded and their Levi form grows by the rate $\frac{1}{\epsilon^2}$ when $d_S<\epsilon$. With these weights in hand and by the compactness outside S, we get (2.4) from the basic estimate for $k\geq 1$ by the same argument as in [12]. To prove the estimate for $k=0$, we make repeated use of (2.4) in degree 1. This first implies that $\bar\partial^*$ has closed range on 1-forms. In particular, $(\ker\bar\partial)^\perp = \operatorname{range}\bar\partial^*$. Thus, if $u\in(\ker\bar\partial)^\perp$, then there exists a solution $v\in(L^2)^1$ to the equation $\bar\partial_b^* v = u$. Moreover, we can choose v belonging to $\ker\bar\partial$. By the basic estimate for v we have

$$||v||^2_0 \underset{\sim}{<} ||\bar\partial^*_b v||^2_0. \tag{2.5}$$

We also have

$$||v||^2_{-1} \leq \epsilon||\bar\partial^*_b v||^2_0 + c_\epsilon ||\bar\partial^*_b v||^2_{-1}. \tag{2.6}$$

This can be proved by contradiction. If (2.6) is violated, then there is a sequence $v_\nu \in (\ker\bar\partial^*_b)^\perp$ such that $||v_\nu||_{-1}\equiv 1$, $||\bar\partial^*_b v_\nu||_{-1}\to 0$ and $||\bar\partial^*_b v_\nu||_0\leq c$. But we also have from (2.5), $||\bar\partial^*_b v_\nu||_0 \underset{\sim}{>} ||v_\nu||_0 \geq ||v_\nu||_{-1} = 1$. Thus any subsequential L^2-weak limit of $\bar\partial^*_b v_\nu$ must be 0 and $\neq 0$. We use the notation lc and sc for a large and small constant respectively. We have for any function u

$$\begin{aligned}
||u||^2 &= (u,\bar\partial^*_b v)\\
&= (\bar\partial_b u, v)\\
&\leq ||\bar\partial_b u|| ||v||\\
&\underset{(2.4)\text{ for } v}{\leq} ||\bar\partial_b u||(\epsilon||\bar\partial^*_b v|| + c_\epsilon||v||_{-1})\\
&\underset{(2.6)}{\underset{\sim}{<}} ||\bar\partial_b u||(\epsilon||u|| + c_\epsilon||u||_{-1})\\
&\leq lc_1\,\epsilon^2||\bar\partial_b u||^2 + sc_1||u||^2 + lc_2\, c^2_\epsilon||u||^2_{-1} + sc_2||\bar\partial_b u||^2\\
&\leq \epsilon'||\bar\partial_b u||^2 + c_{\epsilon'}||u||^2_{-1} + sc_1||u||^2,
\end{aligned} \tag{2.7}$$

for $\epsilon' = lc_1\,\epsilon^2 + sc_2$ and $c_{\epsilon'} = lc_2\, c^2_\epsilon$. By choosing sc_1 so that $sc_1||u||^2$ is absorbed in the left, (2.7) yields (2.4) for u in degree 0. ∎

We decompose a k-form into the tangential and normal components $u = u^\tau + u^\nu$ and further decompose microlocally $u^\tau = u^{\tau+} + u^{\tau-} + u^{\tau\,0}$ (cf. [17]). By elliptic estimate for Q over terms which vanish at $b\Omega$, we have, in particular, that (1.5) is fulfilled by u^ν. The same is true for $u^{\tau 0}$ and $u^{\tau -}$ (cf. [17] Lemma 8.5). So we only need to prove (1.5) for $u^{\tau +}$ that we write as u from now on. We further decompose $u = u^{(H)} + u^{(0)}$ where $u^{(H)}$ is the "holomorphic" component in the sense of [11] and $u^{(0)}$ is the complement; note that $u^{(0)}|_{b\Omega} \equiv 0$. Along with $\zeta \prec \zeta'$ with support in U, we consider an additional tangential cut-off σ with $\sigma \prec \zeta$ and denote by R^s the pseudodifferential tangential operator with symbol $(1+|\xi|^2)^{\frac{s\sigma(a)}{2}}$. Here $(a,r) \in \mathbb{R}^{2n-1} \times \mathbb{R}$ is a local system of coordinates and ξ are dual to the a's. We choose all our cut-off as functions of product type $\zeta = \zeta_1(z')\zeta_2(y_n)$, $\zeta' = \zeta_1'(z')\zeta_2'(y_n)$ and $\sigma = \sigma_1(z')\sigma_2(y_n)$. We denote by Q^τ the tangential component of Q; thus $Q(u,u) = Q^\tau(u,u) + ||\bar{L}_n u||^2$ if $L_1, ..., L_n$ is a system of $(1,0)$ vector fields dual to an orthonormal system of forms $\omega_1, ..., \omega_n$ in which $\omega_n = \partial r$. We point out the crucial property of the component $u^{(H)}$, that is, $\bar{L}_n u^{(H)} \sim r\Lambda u^{(H)}$.

Proposition 2.3. *In the hypotheses (i) and (ii) of Theorem 1.1, we have for any ϵ for suitable c_ϵ and for $\zeta' \succ \zeta$*

$$
\begin{aligned}
&||(\zeta R^s \zeta) u^{(H)}||^2 \leq \epsilon Q^\tau_{\zeta R^s \zeta}(u^{(H)}, u^{(H)}) + c_\epsilon ||\zeta' u^{(H)}||_0^2, \\
&\textit{for} \ \ u \in D^k_{\bar\partial^*}, \quad k \geq 1 \ \ \textit{and} \ \ u \in \ker \bar\partial^\perp, \quad k = 0,
\end{aligned} \tag{2.8}
$$

where $Q^\tau_{(\zeta R^s \zeta)}(u^{(H)}, u^{(H)}) = ||(\zeta R^s \zeta)\bar\partial^\tau u^{(H)}||^2 + ||(\zeta R^s \zeta)\bar\partial^{\tau *} u^{(H)}||^2.$

Remark. In our discussion all estimates are obtained from basic estimates and thus they only hold, in principle, for smooth forms u. However, $b\Omega$ being rigid, they are readily converted into genuine estimates. For this, we use an approximation $\chi_\nu(y_n)$ of the identity in the variable y_n, and define $u^+_\nu := u^+ * \chi_\nu$; we have $u^+_\nu \in C^\infty$. By the rigidity of the boundary we have $\bar\partial_b^{(*)}(u^+_\nu) = (\bar\partial_b^{(*)} u)^+_\nu + \widetilde{u^0}$ and $\bar\partial^{(*)}(u^+_\nu) = (\bar\partial^{(*)} u)^+_\nu + \widetilde{u^0}$ where $\widetilde{u^0}$ denotes a microlocal component supported by the elliptic region. Then the a-priori estimate applied to u^+_ν, in addition to the elliptic estimates for $\widetilde{u^0}$ imply the following. If $\bar\partial_b u$, $\bar\partial_b^* u \in H^s$ in a neighborhood of $\mathrm{supp}\,(\zeta)$, then $||(\zeta R^s \zeta)u|| < +\infty$ (in particular $u \in H^s(\{z : \zeta_o(z) \equiv 1\})$ for $\zeta_o \prec \sigma$).

Proof. Proposition 2.1 shows how to transfer (2.8) from forms to functions $u \in \ker \bar\partial_b^\perp$; so we only prove the result for forms. We start by applying the compactness estimate (2.4) for u replaced by $(\zeta R^s \zeta)u^{(H)}$

$$
\begin{aligned}
&||(\zeta R^s \zeta)u^{(H)}||^2 \\
&\leq \epsilon \Big(Q^\tau((\zeta R^s \zeta)u^{(H)}, (\zeta R^s \zeta)u^{(H)})) + ||\bar{L}_n(\zeta R^s \zeta)u^{(H)}||^2 \Big) + c_\epsilon ||\zeta' u^{(H)}||_0^2.
\end{aligned} \tag{2.9}
$$

We wish to estimate the terms with a factor of ϵ on the right. First,

$$
\begin{aligned}
||\bar{L}_n(\zeta R^s \zeta)u^{(H)}|| &= ||\bar{L}_n((\zeta R^s \zeta)u)^{(H)}|| \\
&= ||r\Lambda((\zeta R^s \zeta)u)^{(H)}|| \\
&\lesssim lc||(\zeta R^s \zeta)u^{(H)}|| + sc||\partial_r \Lambda^{-1}(\zeta R^s \zeta)u^{(H)}|| \\
&\lesssim lc||(\zeta R^s \zeta)u^{(H)}|| + \underbrace{sc||\bar{L}_n(\zeta R^s \zeta)u^{(H)}||_{\Lambda^{-1}}}_{\text{absorbable}},
\end{aligned} \tag{2.10}
$$

where lc and sc denote a large and small constant respectively and where in the last inequality we have used that $\zeta R^s \zeta$ commutes with the operation of taking holomorphic extension (H). Next, we claim that

$$
\begin{aligned}
&Q^\tau((\zeta R^s\zeta)u^{(H)},(\zeta R^s\zeta)u^{(H)})\\
&\le Q^\tau_{\zeta R^s\zeta}(u^{(H)},u^{(H)})+c||(\zeta R^s\zeta)u^{(H)}||^2+c||(\zeta' R^s\zeta)u^{(H)}||^2_{\Lambda^{-1}}.
\end{aligned}
\tag{2.11}
$$

To see it, we observe that

$$
\begin{cases}
[\bar\partial^{(*)\,\tau},\zeta R^s\zeta]=\dot\zeta R^s\zeta+\zeta[\bar\partial^{(*)\,\tau},R^s]\zeta+\zeta R^s\dot\zeta,\\
[\bar\partial^{(*)\,\tau},R^s]\le\sum_{j=1}^{n-1}s\Big|\sigma_{1\,z_j}(z)\sigma_2(y_n)+\sigma_1(z')r_{z_j}\dot\sigma_2(y_n)\Big|\log(\Lambda)R^s.
\end{cases}
$$

It follows

$$
\begin{aligned}
&Q^\tau((\zeta R^s\zeta)u^{(H)},(\zeta R^s\zeta)u^{(H)})\\
&\lesssim Q^\tau_{\zeta R^s\zeta}(u^{(H)},u^{(H)})+||s\zeta\alpha\log(\Lambda)R^s\zeta u^{(H)}||^2+||\zeta' u^{(H)}||^2_0,
\end{aligned}
\tag{2.12}
$$

where

$$
\alpha=\sum_{j=1}^{n-1}|\sigma_{1\,z_j}(z')\sigma_2(y_n)|+|\sigma_1(z')r_{z_j}\dot\sigma_2(y_n)|.
$$

We recall the hypotheses (i) and (ii) of Theorem 1.1: there is a superlogarithmic estimate for $z'\neq 0$, in particular on $\mathrm{supp}(\sigma_{z_j}(z'))$ for any j and r_{z_j} are superlogarithmic multipliers. It follows

$$
\begin{aligned}
||s\zeta\alpha\log(\Lambda)R^s\zeta u^{(H)}||^2&\le scQ^\tau((\zeta R^s\zeta)u^{(H)},(\zeta R^s\zeta)u^{(H)})\\
&\quad+lc||(\zeta R^s\zeta)u^{(H)}||^2_0+c||(\zeta' R^s\zeta)u^{(H)}||^2_{\Lambda^{-1}},
\end{aligned}
\tag{2.13}
$$

where sc and lc denote again a small and large constant respectively. Combination of (2.13) with (2.12) yields the claim (2.11). If one plugs (2.11) and (2.10) into (2.9) and uses induction to reduce $||(\zeta' R^s\zeta)u^{(H)}||^2_{\Lambda^{-1}}$ to $||\zeta' u^{(H)}||^2_0$ (for a new ζ'), one gets

$$
||(\zeta R^s\zeta)u^{(H)}||^2\lesssim\epsilon\Big(Q^\tau_{(\zeta R^s\zeta)}(u^{(H)},u^{(H)})+\underbrace{||(\zeta R^s\zeta)u^{(H)}||^2}_{\text{absorbable}}\Big)+c_\epsilon||\zeta' u^{(H)}||^2_0,
$$

which concludes the proof of the proposition. ∎

To carry on our proof we introduce our main technical result

Proposition 2.4. *In the hypotheses (i) and (ii), we have*

$$
\begin{aligned}
&||(\zeta R^s\zeta)u||^2\lesssim Q_{\zeta R^s\zeta}(u,u)+Q_{\partial_r\Lambda^{-1}\zeta R^s\zeta}(u,u)+||u||^2+Q_{\Lambda^{-1}\zeta'}(u,u)\\
&\qquad\qquad\textit{for } u\in D^k_{\bar\partial^*}\cap C^\infty(\bar\Omega),\ k\ge 1\ \textit{or } u\in\ker\bar\partial^\perp,\ k=0.
\end{aligned}
\tag{2.14}
$$

Proof. Again, u can be a form or a function in $\ker\bar\partial^\perp$. We first focus our attention to (2.8) and wish to remove (H) from the right. We notice that

$$
\begin{aligned}
Q^\tau_{(\zeta R^s\zeta)}(u^{(H)},u^{(H)})&\le Q^{\tau\,b}_{\Lambda^{-\frac12}(\zeta R^s\zeta)}(u_b,u_b)+||(\zeta R^s\zeta)u_b||^2_{-\frac12}\\
&\le Q^\tau_{(\zeta R^s\zeta)}(u,u)+Q^\tau_{\partial_r\Lambda^{-1}(\zeta R^s\zeta)}(u,u)\\
&\quad+||(\zeta R^s\zeta)u||^2_0+||\partial_r\Lambda^{-1}(\zeta R^s\zeta)u||^2_0.
\end{aligned}
\tag{2.15}
$$

Owing to $\partial_r = \bar{L}_n + \text{Tan}$, we have the estimate for the last term above

$$
\begin{aligned}
&||\partial_r\Lambda^{-1}(\zeta R^s\zeta)u||^2 \\
&\le Q_{\Lambda^{-1}(\zeta R^s\zeta)}(u,u) + ||(\zeta R^s\zeta)u||^2 + \underbrace{||[\bar\partial^{(*)}, \Lambda^{-1}((\zeta R^s\zeta))]u||^2}_{\text{lower order term}}.
\end{aligned}
\tag{2.16}
$$

It follows,

$$
\begin{aligned}
||(\zeta R^s\zeta)u^{(H)}||^2 &\underset{(2.8)}{\lesssim} \epsilon Q^\tau_{\zeta R^s\zeta}(u^{(H)},u^{(H)}) + c_\epsilon||\zeta' u^{(H)}||_0^2 \\
&\underset{(2.15)}{\lesssim} Q^\tau_{\zeta R^s\zeta}(u,u) + Q^\tau_{\partial_r\Lambda^{-1}(\zeta R^s\zeta)}(u,u) \\
&\epsilon\Big(||(\zeta R^s\zeta)u||_0^2 + ||\partial_r\Lambda^{-1}(\zeta R^s\zeta)u||_0^2\Big) + c_\epsilon||\zeta' u^{(H)}||_0^2 \\
&\underset{(2.16)}{\lesssim} Q^\tau_{\zeta R^s\zeta}(u,u) + Q^\tau_{\partial_r\Lambda^{-1}(\zeta R^s\zeta)}(u,u) \\
&+ \underbrace{\epsilon||(\zeta R^s\zeta)u||_0^2}_{\text{absorbable}} + \Big(||\zeta' u||_0^2 + ||\zeta'\partial_r\Lambda^{-1}u||_0^2\Big) + c_\epsilon||\zeta' u^{(H)}||_0^2,
\end{aligned}
\tag{2.17}
$$

where in the last inequality the lower order term which occurs in (2.16) has been reduced to $(||\zeta' u||_0^2 + ||\zeta'\partial_r\Lambda^{-1}u||_0^2)$ by iteration. Next we turn our attention to the term (0) and remark that

$$
\begin{aligned}
||(\zeta R^s\zeta)u^{(0)}||^2 &\underset{\text{Garding}}{\le} Q_{\Lambda^{-1}(\zeta R^s\zeta)}(u^{(0)},u^{(0)}) + ||(\zeta R^s\zeta)u^{(0)}||^2_{\Lambda^{-1}\log(\Lambda)} + ||\zeta' u^{(0)}||_0^2 \\
&\lesssim Q_{\Lambda^{-1}(\zeta R^s\zeta)}(u,u) + Q_{\Lambda^{-1}(\zeta R^s\zeta)}(u^{(H)},u^{(H)}) \\
&+ ||(\zeta R^s\zeta)u^{(0)}||^2_{\Lambda^{-1}\log(\Lambda)} + ||\zeta' u^{(0)}||_0^2 \\
&\underset{(2.10)}{\lesssim} Q_{\Lambda^{-1}(\zeta R^s\zeta)}(u,u) + Q^\tau_{\Lambda^{-1}(\zeta R^s\zeta)}(u^{(H)},u^{(H)}) \\
&\quad + ||(\zeta R^s\zeta)u^{(H)}||^2_{\Lambda^{-1}} + ||(\zeta R^s\zeta)u^{(0)}||^2_{\Lambda^{-1}\log(\Lambda)} + ||\zeta' u^{(0)}||_0^2 \\
&\underset{Q^\tau\le\Lambda}{\le} Q_{\Lambda^{-1}(\zeta R^s\zeta)}(u,u) + ||(\zeta R^s\zeta)u^{(H)}||^2 \\
&\quad + \underbrace{||(\zeta R^s\zeta)u^{(H)}||^2_{\Lambda^{-1}}}_{\text{neglectable}} + ||(\zeta R^s\zeta)u^{(0)}||^2_{\Lambda^{-1}\log(\Lambda)} + ||\zeta' u^{(0)}||_0^2,
\end{aligned}
\tag{2.18}
$$

where in the third inequality above we have decomposed $Q(u,u) = Q^\tau(u,u) + ||\bar{L}_n u||^2$ (over a tangential form u) and used (2.10) to estimate the term with $\bar{L}_n$. The second term in the right of the last inequality is estimated by (2.17). Finally, combination of

(2.17) and (2.18) yields

$$
\begin{aligned}
||(\zeta R^s\zeta)u||^2 &\le ||(\zeta R^s\zeta)u^{(H)}||^2+||(\zeta R^s\zeta)u^{(0)}||^2\\
&\lesssim \underbrace{Q_{\zeta R^s\zeta}(u,u)+Q_{\partial_r\Lambda^{-1}(\zeta R^s\zeta)}(u,u)}_{\text{(I)}}+\epsilon\underbrace{||(\zeta R^s\zeta)u^{(H)}||_0^2}_{\text{absorbable}}\\
&\quad+\underbrace{||(\zeta R^s\zeta)u^{(H)}||^2_{\Lambda^{-1}}}_{\text{absorbable}}+||(\zeta R^s\zeta)u^{(0)}||^2_{\Lambda^{-1}\log(\Lambda)}\\
&\quad+\underbrace{\Big(||\zeta' u||_0^2+||\zeta'\Lambda^{-1}\partial_r u||_0^2\Big)+||\zeta' u^{(0)}||_0^2+c_\epsilon||\zeta' u^{(H)}||_0^2}_{\text{(II)}}\\
&\le \text{(I)}+\text{(II)},
\end{aligned}
\tag{2.19}
$$

where in the last inequality we have estimated $||(\zeta R^s\zeta)u^{(0)}||^2_{\Lambda^{-1}\log(\Lambda)}\lesssim (I)+(II)$ because of

$$
\begin{aligned}
\Lambda^{-1}(\zeta R^s\zeta)&\le R^{-1}(\zeta R^s\zeta)\\
&=\zeta R^{s-1}\zeta+\text{Order }0,
\end{aligned}
$$

and from induction. Finally (II) is estimated as follows. As for $u^{(H)}$:

$$
\begin{aligned}
||\zeta' u^{(H)}||_0^2 &\lesssim ||\zeta' u_b||^2_{-\frac12}\\
&\lesssim ||\zeta' u||_0^2+||\Lambda^{-1}\partial_r\zeta' u||^2\\
&\le ||\zeta' u||_0^2+||\Lambda^{-1}\bar L_n\zeta' u||^2+||\Lambda^{-1}\text{Tan }\zeta' u||^2\\
&\lesssim ||u||_0^2+Q_{\Lambda^{-1}\zeta'}(u,u).
\end{aligned}
$$

The same inequality holds for $u^{(H)}$ replaced by $u^{(0)}$ on account of the identity $u^{(0)}=u+u^{(H)}$. Thus $(II)\lesssim c_\epsilon||u||_0^2+Q_{\Lambda^{-1}\zeta'}(u,u)$ and if we plug this into (2.19), we get (2.14). ◻

We are ready for

Proof of Theorem 1.1. We recall that we are writing u for $u^{\tau\,+}$ (or u^+ in case of a function). We begin by noticing that, for $\zeta_o\prec\sigma\prec\zeta$

$$
\begin{aligned}
||\Lambda^s\zeta_o u|| &\lesssim ||R^s\zeta_o u||+||u||\\
&=||R^s\zeta_o\zeta^2 u||+||u||\\
&\le ||R^s\zeta^2 u||+||[R^s,\zeta_o]\zeta^2 u||+||u||\\
&\lesssim ||R^s\zeta^2 u||+||u||\\
&\lesssim ||\zeta R^s\zeta u||+||[R^s,\zeta]\zeta u||+||u||\\
&\lesssim ||\zeta R^s\zeta u||+||u||.
\end{aligned}
$$

Using (2.14) of Proposition 2.4 we get (1.5) in tangential version, that is,

$$
\begin{aligned}
||\Lambda^s\zeta_o u||^2 &\lesssim Q_{\Lambda^s\zeta}(u,u)+Q_{\partial_r\Lambda^{s-1}\zeta}(u,u)+Q_{\Lambda^{-1}\zeta'}(u,u)+||u||_0^2\\
&\lesssim ||\zeta'\bar\partial u||_s^2+||\zeta'\bar\partial^* u||_s^2+||u||_0^2.
\end{aligned}
\tag{2.20}
$$

Finally, by non-characteristicity (cf. eg the end of Section 8 of [17]), we can replace $||\Lambda^s\zeta_o u||^2$ by $||\zeta_o u||^2_s$ in the left of (2.20); we also replace the notation ζ' by ζ on the right and get (1.5). From (1.5) the local regularity of $\bar\partial^{(*)}N$ redily follows which implies the regularity of B and N by the argument before the statement of Theorem 1.3. This concludes the proof of Theorem 1.1. $\square$

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