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CODAZZI-EQUIVALENT RIEMANNIAN METRICS*

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Dedicated to S.-T. Yau on the occasion of his 60-th birthday

Abstract. On a smooth manifold M we introduce the concept of Codazzi-equivalent Riemannian metrics. The curvature tensors of two Codazzi-equivalent metrics satisfy a simple relation. The results together with known facts about Codazzi tensors give a method of proof for old and new local and global uniqueness results for Riemannian manifolds and Euclidean hypersurfaces.

Key words. Codazzi-equivalent Riemannian metrics, Codazzi tensors, hypersurfaces with parallel normals.

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1. Introduction.

1.1. Notation and basic notions. Let M be a differentiable manifold of dimension $n \geq 2$. We denote tangent fields by $u, v, w, ...$ and assume M to be equipped with a Riemannian metric g. The Levi-Civita connection is denoted by $\nabla := \nabla(g)$, and the Riemannian curvature data are denoted as follows: $R(g)$ is the $(1,3)$ curvature tensor, and $\kappa(e_i, e_j)$, for $1 \leq i \neq j \leq n$, the sectional curvature with respect to a frame $(e_1, ..., e_n)$. We use the invariant and sometimes the standard local calculus; we raise and lower indices with the metric and adopt the Einstein convention. In the following we admit that affine connections have torsion.

DEFINITION 1.1.

(i) *Let ∇ denote an affine connection and C be a totally symmetric $(0,s)-$ or $(1,s)$-tensor field; assume that the covariant derivative ∇C is also totally symmetric; then we call the pair (∇, C) a Codazzi pair and C a Codazzi tensor with respect to ∇. For $s = 1$ we also call C a Codazzi operator.*

(ii) *A triple (∇, g, ∇^*) of a (semi-)Riemannian metric and two affine connections ∇ and ∇^* is called conjugate if it satisfies the relation*
$ug(v,w) = g(\nabla_u v, w) + g(v, \nabla^*_u w) \quad \text{for all} \quad u, v, w.$

The relation in (ii) generalizes the Ricci Lemma from Riemannian geometry.

1.2. Euclidean hypersurfaces. We consider a hypersurface $x : M^n \to \mathbb{R}^{n+1}$ in Euclidean space and denote the *unit normal* by μ. We assume that the rank of the *shape* or *Weingarten operator* S is maximal on M, thus the three *fundamental forms*, denoted by

$$g := I, \qquad II, \qquad g^* := III,$$

are (semi)-Riemannian metrics; we denote their Levi-Civita connections by $\nabla(g) := \nabla(I)$, $\nabla(II)$, $\nabla(III) =: \nabla^*$, resp., and the eigenvalues of S, the *principal curvatures*, by $k_1, ..., k_n$.

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1.3. Examples. The canonical *inner product* on $\mathbb{R}^{n+1}$ is denoted by

$$< \, , \, >: \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \to \mathbb{R}.$$

1.3.1. *Conjugate triples.* As before, assume that the shape operator of the Euclidean hypersurface x is bijective. Then the Levi-Civita connection $\nabla^* := \nabla(III)$ satisfies

$$\nabla^*_u v = S^{-1} \nabla(g)_u (Sv).$$

We simplify the notation and write $\nabla^* = S^{-1} \nabla(g)\, S$. The triple

$$(\nabla(g), II, \nabla(g^*))$$

is *conjugate*. The pair (∇^*, S^{-1}) is again a Codazzi pair; [13], [16].
At a point $p \in M$ consider a frame $(e_1, ..., e_n)$ of eigenvectors of S; the sectional curvature of g^* satisfies $\kappa^*(e_i, e_j) = \kappa(III)(e_i, e_j) = 1$, thus for the plane span(e_i, e_j) we get the following trivial relation that below similarly will appear in a more general context:

$$\kappa^*(e_i, e_j) = (k_i\, k_j)^{-1} \cdot \kappa(e_i, e_j).$$

1.3.2. *Hypersurfaces with parallel normals.* Consider two Euclidean hypersurfaces $x, x^\sharp : M \to \mathbb{R}^{n+1}$ with $\operatorname{rank} S = n = \operatorname{rank} S^\sharp$. We say that x and $x^\sharp$ have *parallel normalization* if, for all $p \in M$, the normals coincide: $\mu(p) = \mu^\sharp(p)$. For fixed M, obviously the notion of parallel normalization induces an equivalence relation on the set of all Euclidean hypersurface immersions of the type $z : M \to \mathbb{R}^{n+1}$.
As usual, we denote by z also the position vector of the hypersurface z with respect to the origin in $\mathbb{R}^{n+1}$. If rank $S = n$, the *support function* $\rho := - < \mu, z >$ satisfies the well known relation

$$Hess^* \rho + \rho \cdot g^* = II,$$

where $Hess^*$ denotes the Hessian in terms of the g^*-metric; see sections 4.13 and 6.1 in [13]. Moreover, as g^* is a metric of constant curvature, the foregoing relation implies that (∇^*, II) is a Codazzi pair; see [10] and also Proposition 10.3 in [9].
Via *conjugation of the Codazzi relation* (see section 4.4.5 in [13]) we prove that the operator L, defined by

$$L(u) := \nabla(g)_u \operatorname{grad}_{II} \rho + \rho \cdot S(u),$$

gives another Codazzi pair $(\nabla(g), L)$. As the condition $\operatorname{rank} S = n$ is equivalent to the condition $\operatorname{rank} II = n$, this implies that the operator L, defined via conjugation, has also maximal rank. Moreover, L satisfies the relation $II(Lu, v) = II(u, Lv)$,; thus, if II is definite, L has an eigenbasis.
The Codazzi property holds true if we take any smooth function f instead of ρ:

$$H(f) := Hess^* f + f \cdot g^*,$$

then $(\nabla^*, H(f))$ is a Codazzi pair; [9]. But in this case one has to examine whether the operator, defined via conjugation, has maximal rank.
As above, consider the hypersurfaces x, $x^\sharp$ with parallel normalization. From the foregoing it follows that the operator $L^\sharp$, defined by

$$L^\sharp(u) := \nabla(g)_u \operatorname{grad}_{II}\rho^\sharp + \rho^\sharp \cdot S(u),$$

leads to another Codazzi pair $(\nabla, L^\sharp)$, and again $II(L^\sharp u, v) = II(u, L^\sharp v)$. If $\operatorname{rank} L^\sharp = n$ we get the three relations

$$dx^\sharp = dx \cdot L^\sharp, \qquad g^\sharp(u,v) = g(L^\sharp u, L^\sharp v), \qquad L^\sharp \cdot S^\sharp = S,$$

with $g^\sharp$ as another metric on M; its Levi-Civita connection satisfies a relation of the type of Example 1.3.1 above:

$$\nabla^\sharp := \nabla(g^\sharp) = L^{\sharp\,-1} \nabla L^\sharp;$$

finally, also $(\nabla^\sharp, L^{\sharp\,-1})$ is a Codazzi pair. In case that L has an eigenbasis $(e_i)_i$, in analogy to section 1.3.1 one verifies the following relation for the sectional curvatures of the metrics $g^\sharp$ and g and the eigenvalues $\lambda_i^\sharp$ of $L^\sharp$:

$$\kappa^\sharp(e_i, e_j) = \frac{1}{\lambda_i^\sharp \cdot \lambda_j^\sharp} \cdot \kappa(e_i, e_j).$$

If $3 \leq \operatorname{rank} S < n$, the construction of the operator L is more complicated, as one has to use two functions to generate it, namely the support function and additionally the distance function; see the interesting papers [1], [3].

1.3.3. *Parallel hypersurfaces* [3]. The following class of *parallel hypersurfaces* is a subclass of the class of hypersurfaces with parallel normals; this subclass gives a particularly instructive example:
Two hypersurfaces x, $x_t : M \to \mathbb{R}^{n+1}$ are called *parallel* if their position vectors satisfy

$$x_t - x = t \cdot \mu \qquad \text{for some fixed} \qquad 0 \neq t \in \mathbb{R}.$$

Obviously both hypersurfaces have parallel normals, $\mu = \mu_t$, and they satisfy

$$dx_t = dx \cdot L_t \quad \text{and} \quad g_t(u,v) = g(L_t u, L_t v),$$

$$L_t := id - tS \quad \text{and} \quad S_t = L_t^{-1} \cdot S.$$

x_t is an immersion if and only if $\operatorname{rank} L_t = n$. The pair $(\nabla(g), L_t)$ is a Codazzi pair, and trivially $SL_t = L_t S$, thus L_t is $g-$selfadjoint. An eigenbasis of S is an eigenbasis of L_t and also of the shape operator S_t of x_t. The principal curvatures $k_1(t), ..., k_n(t)$ of x_t satisfy

$$k_i(t) = (1 - t\, k_i)^{-1} \cdot k_i.$$

This easily gives the relations for the curvature invariants; for $n = 2$ one can find formulas for the mean curvature H_t and the Gauß curvature K_t in textbooks. For $n \geq 2$ the sectional curvatures are related by

$$\kappa_t(e_i, e_j) = \frac{1}{\lambda_i(t) \cdot \lambda_j(t)} \cdot \kappa(e_i, e_j).$$

1.4. Codazzi-equivalent metrics. The foregoing examples suggest the study of the set of all Riemannian metrics on a given manifold M with respect to the following:
For a given metric g and a Codazzi pair $(\nabla(g), L)$ with an operator L of maximal rank we construct a new metric $g^\sharp$ as above. The examples in sections 1.3 show that also (∇^*, S^{-1}) and $(\nabla^\sharp, L^{-1})$ define Codazzi pairs, that means the Codazzi-relation obviously is symmetric. The geometric situation of hypersurfaces with parallel normals suggests to check whether the relation is also transitive. In fact, one can easily see that this is true; thus we introduce the notion of *Codazzi-equivalence* of metrics as an appropriate concept.

Definition 1.2. *On a smooth, connected manifold M consider two Riemannian metrics g, g^*. We call both metrics Codazzi-equivalent if there exists a bijective operator L s.t. the pair $(\nabla(g), L)$ is a Codazzi pair and $g^*(u,v) = g(Lu, Lv)$ for all u, v. Extending a terminology of Hicks [4], we say that we linearly perturb the metric g with a Codazzi operator L.*

A basic result for our method of proof is the following Theorem.

Theorem 1.3. *Codazzi-equivalence is an equivalence relation on the set of Riemannian metrics on M. We call the equivalence class of g its Codazzi class.*

Section 4 will show that one can state and prove an appropriate generalization for affine connections; we investigate this topic in [11].

Remark 1.4. *Let (V, g) be a Euclidean vector space of dimension $n \geq 2$. Then any positive definite and $g-$self-adjoint operator $\mathcal{L}$ defines an inner product g^*:*

$$g^*(u,v) := g(\mathcal{L}u, v),$$

and in this way we can generate all inner products on V; in particular, given two inner products g and g^* on V, there exists a unique, positive definite and $g-$self-adjoint operator $\mathcal{L}$ s.t. $g^*(u,v) = g(\mathcal{L}u, v)$. For our purposes it is convenient to write $\mathcal{L}$ as an appropriate product $\mathcal{L} = L^2$, where L is bijective, but not necessarily positive definite, and $g^*(u,v) = g(Lu, Lv)$. We note that the eigenspaces of $\mathcal{L}$ and L coincide, and that, for any eigenbasis $(e_1, ..., e_n)$, we have

$$Le_i = \lambda_i \, e_i, \qquad \mathcal{L}e_i = \lambda_i^2 \, e_i \quad \text{for} \quad i = 1, ..., n.$$

The (1,3) curvature tensors of Codazzi equivalent metrics satisfy a simple relation (Proposition 3.1.iii below). In case that the Codazzi operator L in Definition 1.2 additionally has an eigenbasis, the sectional curvatures of Codazzi-equivalent metrics satisfy the following simple relation that generalizes the special relations for the sectional curvatures in the examples above. We would like to point out that Theorem 1.5 holds true for semi-Riemannian metrics, in particular for spacetimes.

Theorem 1.5. *Let g and g^* be Codazzi-equivalent Riemannian metrics on M where $g^*(u,v) = g(Lu, Lv)$. Assume that L has an eigenbasis $(e_1, ..., e_n)$ corresponding to the eigenvalues $\lambda_1, ..., \lambda_n$. Then the sectional curvatures satisfy the relation*

$$\kappa^*(e_i, e_j) = (\lambda_i \cdot \lambda_j)^{-1} \cdot \kappa(e_i, e_j).$$

It is an immediate consequence of the foregoing Theorem that, in dimension $n \geq 3$, we can state the following local result.

Corollary 1.6. *Let $dim M = n \geq 3$. Assume that the metrics g and g^* are Codazzi equivalent on M and that the operator L has an eigenbasis. If the sectional curvatures sastisfy $\kappa^*(e_i, e_j) = q \cdot \kappa(e_i, e_j)$, where $0 < q \in C^\infty$, then g and g^* are homothetic.*

Remark 1.7. **(i)** *Let us recall that, on any Riemannian manifold of dimension $n \geq 3$, the sectional curvature determines the Riemannian curvature tensor (see e.g. [17], p. 60). Now, any perturbation of a Riemannian metric can be written in the form $g^*(u,v) = g(Lu, Lv)$. Thus, for $dim M = n \geq 3$, Theorem 1.5 gives sufficient conditions for L under which the sectional curvature determines the metric. In $dim M = n = 2$ we prove a related global result in Theorem 4.3.*
(ii) *If we cancel the assumption on Codazzi-equivalence, then the assertion of Corollary 1.6 is not any more true in general. For example, consider two hypersurfaces with shape operators of maximal rank. Then both third fundamental forms have sectional curvature equal one, but they are not necessarily isometric.*

In affine hypersurface theory, the concept of *conjugate triples* plays an important role. In this context, the following Proposition reflects the duality of the metrics g, g^* and their Levi-Civita connections $\nabla(g)$, $\nabla(g^*)$, resp.

Proposition 1.8. *Let $x, x^\sharp : M \to \mathbb{R}^{n+1}$ be Euclidean hypersurfaces with bijective shape operators. Then:*

- *(i) If $x, x^\sharp$ are $I-$isometric then $g^* = III$ and $g^{\sharp *} = III^\sharp$ are Codazzi-equivalent with $L := S^{-1} \cdot S^\sharp$ and $g^{*\sharp}(u,v) = g^*(Lu, Lv)$. Moreover, if one of the shape operators is (positive) definite then the operator L has a basis of eigenvectors.*
- *(ii) If $x, x^\sharp$ are $III-$isometric then $g = I$ and $g^\sharp = I^\sharp$ are Codazzi-equivalent with $L := S \cdot S^{\sharp\,-1}$ and $g^\sharp(u,v) = g(Lu, Lv)$. Moreover, if one of the shape operators is (positive) definite then the operator L has a basis of eigenvectors.*

Remark 1.9. *For $n \geq 3$ it is sufficient to assume that*

$$II(u,u)^2 + II^\sharp(u,u)^2 \neq 0 \quad \textit{for} \quad u \neq 0$$

to find a basis s.t. II and $II^\sharp$ are simultaneously diagonizable (see [2], p.256); from this it follows that L has an eigenbasis.

Parts of the results can be extended to semi-Riemannian manifolds and also to manifolds equiped with affine connections with torsion [11]. In the following we will use tools from the papers [4], [9], [12], [13], [15], [16].

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2. Tools.

2.1. Inner products on vector spaces. We recall some basics. Let (V, g) be a Euclidean vector space of dimension $n \geq 2$. As already stated, any positive definite and $g-$self-adjoint operator $\mathcal{L}$ defines an inner product g^* by

$$g^*(u,v) := g(\mathcal{L}u, v).$$

As above we write $g^*(u,v) = g(Lu, Lv)$. In the same manner we construct another inner product $g^{**}(u,v) := g^*(\mathcal{P}u, v)$, where the operator $\mathcal{P}$ is g^*-self-adjoint and positive definite. Again we write $\mathcal{P} = P^2$ and $g^{**}(u,v) = g^*(Pu, Pv)$, where P is bijective. It is clear that the operator LP is not g-selfadjoint in general. But we have:

REMARK 2.1. *The operator $\mathcal{LP}$ is g-self-adjoint.*

Proof. We have $g^{**}(u,v) = g^*(\mathcal{P}u, v) = g(\mathcal{LP}u, v)$. As g^{**} is symmetric in its arguments, the operator $\mathcal{LP}$ is $g-$self-adjoint. □

2.2. Some known facts on Codazzi tensors. In this section we recall some known results about Codazzi operators and Codazzi (0,2)-tensors, namely how their elementary symmetric functions determine the tensor itself.

LEMMA 2.2. [12], [16]. *On a Riemannian manifold (M, g) consider the Codazzi pair (∇, L) with $\nabla := \nabla(g)$, and assume that the operator L is g-selfadjoint, having a g-orthonormal eigenbasis $(e_1, ..., e_n)$ corresponding to the eigenvalues $\lambda_1, ..., \lambda_n$. Then:*

$$\Delta \|L\|^2 = \sum_{i<j} \kappa(e_i, e_j) \cdot (\lambda_i - \lambda_j)^2 + \|\nabla L\|^2 + L^{ij} \nabla_i \nabla_j (trace\, L);$$

the Laplacian Δ and the norms on tensor spaces are induced by g.

For fixed natural $1 \leq r \leq n$ we denote by $L_{(r)}$ the *normed elementary symmetric function of order r* of the eigenvalues of L. As application of the foregoing Lemma we recall from [12]:

THEOREM 2.3. *Let (M, g) be closed (compact without boundary) with non-negative sectional curvature; assume that $(\nabla(g), L)$ form a Codazzi pair and that the operator L is g-selfadjoint with eigenvalues $\lambda_1, .., \lambda_n$. Then each of the following assumptions (i) - (iv) implies that L is parallel with respect to $\nabla(g)$.*

(i) $nL_{(1)} = trace\, L = const$;
(ii) $L_{(2)} = const > 0$;
(iii) $L_{(1)} \geq 0$ *and* $L_{(2)} \geq 0$, *and there exists a C^1-function $\Phi : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ s.t.*

$$\partial_1 \Phi \cdot \partial_2 \Phi > 0;$$

(iv) $dim\, M = n = 2$ *and, for all $p \in M$ with $\lambda_1(p) \neq \lambda_2(p)$, we have*

$$g(grad\, \lambda_1, grad\, \lambda_2) \leq 0.$$

REMARK 2.4. *In the foregoing Theorem, assume that the sectional curvature is always positive. Then (M, g) is irreducible and thus L is a constant multiple of the identity operator, that means $L = \frac{1}{n} \cdot trace\, L \cdot id$ with $trace\, L = const$.*

The following Theorem is a modest extension of known results, but, as far as we know, nowhere stated in this form. Its proof goes back to a lecture of H.F. Münzner (Oberwolfach 1971, [8]) on compact Euclidean hypersurfaces with constant higher order mean curvature $L_{(k)}$. The essential technical tool, a formula for $\Delta L_{(k)}$, appears again in [14] and also in [7]. This formula can be extended to g-selfadjoint Codazzi operators and their associated elementary symmetric functions $L_{(k)}$ of order k of the eigenvalues of L, where $k = 1, ..., n$.

THEOREM 2.5. *Let (M, g) be a closed (compact without boundary) Riemannian manifold of semi-positive sectional curvature. Let L be a semi-positive definite, g-selfadjoint operator with g-orthonormal eigenbais $(e_1, ..., e_n)$ corresponding to the* *eigenvalues* $(\lambda_1, ..., \lambda_n)$ *s.t.* $(\nabla(g), L)$ *is a Codazzi pair. Then the relation* $L_{(k)} = const > 0$ *implies that* L *is parallel with respect to* $\nabla(g)$*. If the sectional curvature is positive everywhere then* $L = \lambda \cdot id$ *with* $trace\, L = n\lambda = const > 0$.

Proof. We use the foregoing notation and sketch the steps of the proof following closely the proof of Theorem 1 in [5]; there the proof is given for the Schouten tensor which is a Codazzi tensor on conformally flat manifolds.
1. Step. Define the Newton transformations $T_k(L)$, for $k = 0, ..., n$, induced by L, as follows:

$$\det(L + tI) \cdot (L + tI)^{-1} =: \sum_{k=0}^{n-1} T_k(L)\, t^{n-k-1} \tag{2.20}$$

on the subset Ω of $M \times \mathbb{R}$ where $\Omega := \{(p, t) \mid \det(L(p) + tI(p)) \neq 0\}$.
2. Step. For $f \in C^\infty$ denote by $\mathrm{Hess}(f)$ the $\nabla(g)$-covariant Hessian of f, and its associated operator $\mathcal{H}(f)$ implicitly by $g(\mathcal{H}(f)(u), v) := \mathrm{Hess}(f)(u, v)$. Define the differential operator

$$\boxed{k} f := \mathrm{trace}(T_{k-1}(L) \circ \mathcal{H}(f)), \quad 1 \leq k \leq n.$$

3. Step. If L is semi-positive definite and $L_{(k)} > 0$ on M then the operator $\boxed{k}$ is elliptic.
4. Step. At $p \in M$, derive the following formula for $\Delta L_{(k)}$, where $\{e_i\}_i$ is a local orthogonal frame field of eigenvectors of L such that $L_i^j = \lambda_i \cdot \delta_i^j$ at $p \in M$; then we have at p:

$$\Delta L_{(k)} = \boxed{k}\, \mathrm{trace}\, L - \tfrac{1}{2(n-2)} \sum_{i,j} [\partial_i \partial_j L_{(k)}] \cdot (\lambda_i - \lambda_j)^2 \cdot \kappa(e_i, e_j)$$

$$+ \sum_{i,j,l} \partial_j \partial_l L_{(k)} \left(L_{jj,i} L_{ll,i} - (L_{jl,i})^2 \right),$$

where $\partial_j L_{(k)} = \frac{\partial L_{(k)}}{\partial \lambda_j}$ and $\partial_j \partial_l L_{(k)} = \frac{\partial^2 L_{(k)}}{\partial \lambda_j \partial \lambda_l}$.
5. Step. $L_{(k)} = const > 0$ and the formula for $\Delta L_{(k)}$ imply

$$\boxed{k} L_{(1)} \geq 0,$$

and from this $L_{(1)} = const \geq 0$ as M is compact.
6. Step. Apply Lemma 2.2 and Remark 2.4 above; this gives the assertion. ∎

3. Linear perturbations of connections and conjugation. N. Hicks [4] calculated linear perturbations of connections; K. Nomizu and U. Simon extended the results of Hicks in the context of conjugate connections and affine hypersurface theory. We need the following tools from both papers.

Proposition 3.1. [4]. *Let* (M, g) *be a Riemannian manifold, let* $\nabla = \nabla(g)$*, and let* L *be an operator of maximal rank. Define the metric* g^* *by*

$$g^*(u, v) := g(Lu, Lv).$$

Then

(i) the expression $L^{-1}\nabla L$ defines a connection by $(L^{-1}\nabla L)_u v := L^{-1}\nabla_u(Lv)$;
(ii) $L^{-1}\nabla L$ satisfies the Ricci-Lemma for g^*, that means:

$$wg^*(u,v) = g^*(L^{-1}\nabla_w Lu, v) + g^*(u, L^{-1}\nabla_w Lv);$$

(iii) the curvature tensor $R^* := R(g^*)$ satisfies

$$R^*(u,v)w = L^{-1}(R(u,v)Lw).$$

Remark 3.2. **(i)** *It is essential for the following to check when the connection $L^{-1}\nabla L$ coincides with the Levi-Civita connection $\nabla^* = \nabla(g^*)$; from Proposition 3.1.ii it is necessary and sufficient that the connection $L^{-1}\nabla L$ additionally is torsion free. From affine hypersurface theory it is known that the connection $L^{-1}\nabla L$ is torsion free if and only if (∇, L) is a Codazzi pair, [9]. Thus Proposition 3.3.i below yields.*
(ii) *One can generalize most parts of the foregoing Proposition to affine connections with torsion, see [11]; this gives more insight, as the Codazzi property is related to the torsion tensors of ∇ and ∇^* of conjugate connections. But here we restrict to the metric case.*

Proposition 3.3.
(i) The connection $L^{-1}\nabla L$ coincides with the Levi-Civita connection ∇^* of g^* if and only if (∇, L) is a Codazzi pair.
(ii) Assume that L is $g-$self-adjoint; define the (semi)-Riemannian metric $\tilde{g}$ by $\tilde{g}(u,v) := g(Lu,v)$; then the triple $(\nabla, \tilde{g}, \nabla^*)$ is conjugate.
(iii) Let B be a symmetric (0,r)-form and define B^* by

$$B^*(u_1, ..., u_r) := B(Lu_1, ..., Lu_r);$$

assume that (∇, L) is a Codazzi pair; then

$$(\nabla^*_v B^*)(u_1, ..., u_r) \;=\; (\nabla_v B)(Lu_1, ..., Lu_r);$$

in particular we have

$$(\nabla^*_w g^*)(u,v) \;=\; (\nabla_w g)(Lu, Lv).$$

The proofs are straighforward computations.

Corollary 3.4.

$$\nabla B = 0 \quad \textit{if} \quad \textit{and} \quad \textit{only} \quad \textit{if} \quad \nabla^* B^* = 0.$$

4. Codazzi-equivalent metrics.

4.1. Proof of Theorem 1.3.

Proof.
(i) Symmetry: We show that the pair (∇, L) is a Codazzi pair if and only if the pair (∇^*, L^{-1}) is a Codazzi pair. The relation

$$(\nabla^*_v L^{-1})u = (\nabla^*_v(L^{-1}u)) - L^{-1}(\nabla^*_v u) = L^{-1}[\nabla_v u - L^{-1}\nabla_v(Lu)]$$

implies

$$(\nabla^*_v L^{-1})u - (\nabla^*_u L^{-1})v = L^{-2}\left[(\nabla_v L)u - (\nabla_u L)v\right].$$

(ii) Transitivity: Assume that (∇, L) is a Codazzi pair and that g^* is defined as in Proposition 3.1; moreover, assume that (∇^*, P) with the regular operator P forms another Codazzi pair; define g^{**} by

$$g^{**}(u,v) := g^*(Pu, Pv) = g(LPu, LPv).$$

We have to show that $(\nabla(g), L \cdot P)$ again is a Codazzi pair. The Levi-Civita connection $\nabla^{**} = \nabla(g^{**})$ is torsion free and thus satisfies

$$0 = \nabla^{**}_u v - \nabla^{**}_v u - [u,v] = (LP)^{-1}\left[(\nabla_u LP)v - (\nabla_v LP)u\right],$$

and this is equivalent to the Codazzi equation

$$0 = (\nabla_u LP)v - (\nabla_v LP)u. \quad \square$$

Remark 4.1. *For the proofs of the symmetry and transitivity we do not need the metrics; as stated above, the notion of Codazzi equivalence can be extended to affine connections [11].*

4.2. Curvature of Codazzi-equivalent metrics. We are going to prove Theorem 1.5.

Proof. We apply Proposition 3.1.iii:

$$\kappa^*(e_i, e_j) = \frac{g^*(R^*(e_i,e_j)e_j, e_i)}{g^*(e_i,e_i)\, g^*(e_j,e_j) - g^*(e_i,e_j)^2} =$$

$$= (\lambda_i \cdot \lambda_j)^{-2} \cdot \frac{g(R(e_i,e_j)Le_j, Le_i)}{g(e_i,e_i)\, g(e_j,e_j) - g(e_i,e_j)^2} = (\lambda_i \cdot \lambda_j)^{-1} \cdot \kappa(e_i, e_j). \quad \square$$

Corollary 4.2.
(i) We have $\kappa^*(e_i,e_j) \neq 0$ if and only if $\kappa(e_i,e_j) \neq 0$.
(ii) Additionally, if L is positive definite then κ and κ^* have the same sign.

4.3. Isometries. The proof of Corollary 1.6 is immediate: as $n \geq 3$, Theorem 1.5 implies that all eigenvalues of L coincide and $\lambda_i =: \lambda = \frac{1}{\sqrt{q}}$; the Codazzi property gives $\lambda = const$.

In dimension $n = 2$ we have the following global result. A linear perturbation of a metric within its Codazzi class with a g-selfadjoint operator, preserving the Gauß curvature, is trivial, more precisely:

Theorem 4.3. *Let (M, g) be a closed Riemannian manifold of dimension $n = 2$ with positive Gauß curvature. Let the metrics $g^*(u,v) = g(Lu, Lv)$ be Codazzi equivalent and assume that L is g-selfadjoint. If the Gauß curvatures coincide, $\kappa = \kappa^*$, then $g = g^*$.*

Proof. It follows from Theorem 1.5 that $\det L = 1$. Then Theorem 2.3 and Remark 2.4 together with Lemma 2.2 and an integration imply that $L = id$. $\square$

Remark 4.4. *In the foregoing Theorem, if we have the weaker assumption of non-negative sectional curvatures $\kappa = \kappa^* \geq 0$ then $\nabla L = 0$ implies $\lambda_i = const$ for $i = 1, 2$. Then either $\lambda_1 = \lambda_2 = 1$ and $g = g^*$, or $\lambda_1 \neq \lambda_2$, and then both metrics must be flat, and M is a torus.*

We apply Theorem 2.3 to prove:

THEOREM 4.5. *Let M be a closed surface and g and g^* be Codazzi-equivalent metrics, say $g^*(u,v) = g(Lu, Lv)$. If the sectional curvature of g is positive, and if L is g-selfadjoint and satisfies one of the assumptions (i) - (iv) in Theorem 2.3, then g and g^* are homothetic.*

5. Euclidean hypersurfaces. In this section we assume that $x, x^\sharp : M \to \mathbb{R}^{n+1}$ are Euclidean hypersurfaces with $\operatorname{rank} S = n = \operatorname{rank} S^\sharp$, thus the Gauß maps are immersions.

5.1. Proof of Proposition 1.8. (i) Apply Theorem 1.3: $(\nabla^*, L := S^{-1} \cdot S^\sharp)$ is a Codazzi pair. $I = I^\sharp$ implies $II^\sharp(u,v) = II(Lu, v)$ for all u, v, thus L is $II-$self adjoint; if one of the forms II, $II^\sharp$ is positive definite then L has an eigenbasis. Note that, in dimension $n \geq 3$, Remark 1.9 applies.
The proof of (ii) is similar.

5.2. Isometries of the first fundamental form.

THEOREM 5.1. BEEZ-KILLING. *For $\operatorname{rank} S \geq 3$, the first fundamental form of a hypersurface determines the shape operator, that means the first fundamental form determines a hypersurface locally modulo congruence.*

To demonstrate our method we need the stronger assumption that $\operatorname{rank} S = n$.

Proof. We define the operator $L := S^{-1} S^\sharp$ as in the foregoing proof. The sectional curvatures of $g^{\sharp *}$ and g^* are equal: $\kappa^{\sharp *} = 1 = \kappa^*$. For $n \geq 3$ the sectional curvature determines the Riemannian curvature tensor, thus Proposition 3.1.iii gives

$$g^*(Lv, Lw)g^*(Lu, Lz) - g^*(Lu, Lw)g^*(Lv, Lz) =$$

$$= g^{\sharp *}(v, w)g^{\sharp *}(u, z) - g^{\sharp *}(u, w)g^{\sharp *}(v, z) =$$

$$= g^{\sharp *}(R^{\sharp *}(u, v)w, z) = g^*(v, Lw)g^*(u, Lz) - g^*(u, Lw)g^*(v, Lz);$$

this is true for all $\tilde{w} := Lw$; we compare the first and last term in the chain of equalities and get:

$$g^*(Lu, Lz)Lv - g^*(Lv, Lz)Lu = g^*(u, Lz)v - g^*(v, Lz)u.$$

This is true for all v; as $\dim(\operatorname{span}(v, Lv)) \leq 2$ and $n \geq 3$, there exists a vector $\tilde{z}$ s.t. $g^*(\tilde{z}, v) = 0 = g^*(\tilde{z}, Lv)$; from the foregoing equation we get $\dim(\operatorname{span}(v, Lv)) = 1$ for any v, thus $L = \lambda \cdot id$ and finally $\lambda = 1$, as $det\, L = 1$; we arrive at $S = S^\sharp$. □

5.3. Hypersurfaces with parallel normals. Recall that rank $S = n =$ rank $S^\sharp$.

REMARK 5.2. *(i) $x, x^\sharp$ have parallel normals modulo congruence if and only if their third fundamental forms coincide on M.*
(ii) Let $x, x^\sharp : M \to \mathbb{R}^{n+1}$ be hypersurfaces of dimension $n \geq 2$ and with parallel normals. If the shape operators coincide, $S = S^\sharp$, then trivially the second and also the first fundamental forms coincide, resp., and thus $x, x^\sharp$ are congruent.

The following Theorem is a modification of results from [1] and [3]; note that here we assume the shape operators to have maximal rank; this stronger assumption admits a simpler, different proof.

THEOREM 5.3. *Let $x, x^\sharp$ be hypersurfaces with $\operatorname{rank} S = n = \operatorname{rank} S^\sharp$. Recall the notation ρ and $\rho^\sharp$ for the support functions, resp.*

(a) Assume that $x, x^\sharp$ have parallel normals. Then:

(a.i) The operator $L^\sharp$, defined by

$$L^\sharp(u) := \nabla(g)_u grad_{II}\rho^\sharp + \rho^\sharp \cdot S(u),$$

and the connection $\nabla(g)$ form a Codazzi pair $(\nabla(g), L^\sharp)$; moreover, we have $\operatorname{rank} L^\sharp = n$ and $dx^\sharp = dx \cdot L^\sharp$, thus the first fundamental forms are related by $g^\sharp(u,v) = g(L^\sharp u, L^\sharp v)$;

(a.ii) the Weingarten operators satisfy $L^\sharp \cdot S^\sharp = S$.

(a.iii) $L^\sharp$ is $II-$self adjoint.

(b) Assume that $x, x^\sharp$ satisfy the relations

(b.i) $g^\sharp(u,v) = g(L^\sharp u, L^\sharp v)$ with $L^\sharp$ defined as in (a.i), and $(\nabla(g), L^\sharp)$ is a Codazzi pair;

(b.ii) $L^\sharp \cdot S^\sharp = S$.

Then $x, x^\sharp$ have parallel normals modulo congruence.

Proof. (a.i) For the proof of $dx^\sharp = dx \cdot L^\sharp$ see Example 1.3.2. As we assume the Weingarten operators to have maximal rank we also can follow the steps from Example 1.3.2 to check the Codazzi property of $(\nabla(g), L^\sharp)$. (a.ii) follows from $dx^\sharp = dx \cdot L^\sharp$. (a.iii) follows from $II^\sharp(u,v) = II(u, L^\sharp v)$.
(b) Prove that the third fundamental forms coincide. □

REMARK 5.4. *We compare the class of hypersurfaces considered here with the class in Theorem 1.i in [3]; while there the authors assume only $\operatorname{rank} S \geq 3$, we restrict to hypersurfaces with regular Gauß map; the class in the foregoing Theorem 5.3 is the class of hypersurfaces with congruent regular Gauß maps.*

Theorem 5.3.a and Corollary 1.6 give:

COROLLARY 5.5. **(a)** *Assume that the hypersurfaces $x, x^\sharp$ have parallel normals and satisfy $\operatorname{rank} S = \operatorname{rank} S^\sharp = n \geq 3$. If $L := S \cdot S^{\sharp\,-1}$ has an eigenbasis $(e_1, ..., e_n)$ and if the sectional curvatures satisfy $\kappa^\sharp(e_i, e_j) = q \cdot \kappa(e_i, e_j)$ with $0 < q \in C^\infty$ then both metrics and finally both hypersurfaces are homothetic.*
(b) *In (a), assume that one of the hypersurfaces is locally strongly convex; then L has an eigenbasis and the assertions in (a) hold true.*

Proof. **(b)** We have $II^\sharp(u,v) = II(Lu, v) = II(u, Lv)$ with positive definite form II. From this L has an eigenbasis. □

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