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LOCAL GEOMETRY OF PLANAR ANALYTIC MORPHISMS*

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Abstract. The effect of analytic morphisms on plane curve singularities, via direct and inverse images, is studied in terms of certain objects attached to the morphism, its jacobian, trunk and tangent map among them.

Key words. Analytic map, Analytic morphism, Plane curve singularity, Jacobian, Trunk

AMS subject classifications. 14B05, 32S10, 32S15

1. Introduction. The main purpose of this paper is the study of analytic morphisms between smooth complex analytic surfaces, $\varphi : S \longrightarrow T$, locally at points $O \in S$ and $O' = \varphi(O)$. More precisely we are interested in describing the effect of the morphism on the singularities of curve on either surface, via direct or inverse image. An important role in this description is played by the jacobian (or critical) curve of φ, which in turn is also an object of interest. Indeed, describing and interpreting the singularity of a jacobian curve (usually associated to a pair of functions f, g or germs of curve $f = 0$, $g = 0$, rather than to the morphism of local equations f, g) is an old problem about which not very much is already known beyond the particular case of the polar curves. The reader may see [8], [14], [1], [9] and [13] for recent work in the non-polar case, and also Example 10.11 below for the difficulties in determining equisingularity invariants of the jacobian from the pair of germs of curve $f = 0$, $g = 0$, or even from the pencil $\lambda_0 f + \lambda_1 g = 0$.

We introduce a couple of objects associated to φ. The first one is a finite weighted sequence of points in successive infinitesimal neighbourhoods of O' (i.e., an unibranch weighted cluster) which we call the trunk of φ. The second object is a rational map between the first neighbourhoods of O and the last point of the trunk. This map is a direct generalization of the ordinary differential to the case of a non smooth φ, it is called the tangent map to φ. Both the trunk and the tangent map may be computed by a rational algorithm which is explained in Section 10.

The trunk of φ determines the multiplicities of the inverse images (4.1) and the tangent map helps to describe their tangent cones (9.1). In Section 5, the ratio between the multiplicities of a germ of curve and its inverse image is bounded in terms of the trunk (5.4); this has consequences relative to the dynamics and the asymptotics of the complexity of φ in the case $S = T$ and $O = O'$ (5.5 and 5.6). The trunk of φ also determines the multiplicity of the jacobian curve (6.1), while the tangent cone to the jacobian is closely related to the multiple points of the fibres of the tangent map (10.2). This relationship provides insight on the reasons because of which a jacobian curve does split. On the other hand the trunk of φ gives relevant information on the singularities of direct images, most of them being partially modelled after the points and multiplicities of the trunk (11.1, 11.2).

We attach further trunks to φ by considering, for each point p infinitely near to O, the trunk $\mathcal{T}_p$ of the composition φ_p of the sequence of blowing-ups giving rise to p and φ itself. Only a few of these further trunks carry new information, as in

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most cases $\mathcal{T}_p$ is determined in terms of the trunk $\mathcal{T}_{p'}$ corresponding to the point p' preceding p (14.7, 16.2). The trunks $\mathcal{T}_p$ determine the infinitely near multiplicities, and hence, in particular, the equisingularity types, of the inverse images (13.3) and the jacobian curve (14.1).

In Section 15 it is proved that, under certain conditions on p and $\mathcal{T}_p$ which are satisfied for most of the points p (see 15.2 for more precision), the morphisms φ_p are locally isomorphic to compositions of blowing-ups. This may be understood as a sort of partial resolution of the singularity of φ and furthermore it completely describes the singularities of the direct images of the irreducible germs of curve going through one of these points.

The existence and distribution of irreducible germs of curve whose representatives are not injectively mapped onto their direct images (folded germs), are dealt with in Section 16.

Sections 17 and 18 are devoted to the trunks associated to satellite points and points on the jacobian, respectively. In Section 19, we give conditions for an infinitely near point to belong to the jacobian germ of φ.

2. Preliminaries. If O is a point on a smooth surface S, we will denote by $\mathcal{O}_{S,O}$ the local ring of the germs of analytic functions of S at O. Once local coordinates x, y at O are taken we will identify the elements of $\mathcal{O}_{S,O}$ to convergent series in x, y and for any $h \in \mathcal{O}_{S,O}$ we will denote by $\widehat{h} \in \mathbb{C}[x,y]$ its initial form, and by $o(h) = o_O(h) = \deg(\widehat{h})$ its order. If $\mathcal{M} = (x,y)$ is the maximal ideal of $\mathcal{O}_{S,O}$, then o is the $\mathcal{M}$-adic valuation of $\mathcal{O}_{S,O}$.

The reader is referred to [7] for conventions and known facts about germs of plane curves. Germs of analytic curves on a smooth analytic surface S will be called germs of curve, and often just germs if no confusion may arise; they are allowed to be non-reduced, that is, given by an equation with multiple factors. If f is a germ of analytic function, the notation $\xi : f = 0$ will denote that the curve or germ of curve ξ is defined by the equation f. The multiplicity of a germ of curve ξ at a proper or infinitely near point p will be denoted by $e_p(\xi)$. $TC(\xi)$ will denote the tangent cone to a germ ξ, usually interpreted as an effective divisor on the first neighbourhood of the origin of ξ. As usual, irreducible germs and irreducible components of germs are called branches and square brackets denote intersection multiplicity of germs of curve. Families of germs of the form $\mathcal{P} = \{\xi_\alpha : \alpha_1 f_1 + \alpha_2 f_2 = 0\}$, where $f_1, f_2 \in \mathcal{O}_{S,O}$, $\alpha = \alpha_1/\alpha_2 \in \mathbb{C} \cup \{\infty\}$ and no one of the germs ξ_0, ξ_∞ is included in the other, are called pencils. They are intrinsically structured as projective lines by taking α_1, α_2 as homogeneous coordinates for ξ_α. The *fixed part* of $\mathcal{P}$ is the germ $f_0 = \gcd(f_1, f_2) = 0$ while its *variable part* is the pencil described by the germs $\alpha_1 f_1/f_0 + \alpha_2 f_2/f_0 = 0$ which, clearly, has no (i.e., empty) fixed part. If γ is an irreducible germ not contained in the fixed part of $\mathcal{P}$, an easy computation shows that all the intersection multiplicities $[\gamma.\xi_\alpha]$ are equal, but for a single germ $\xi_{\alpha'}$ for which $[\gamma.\xi_{\alpha'}] > [\gamma.\xi_\alpha]$, $\alpha \neq \alpha'$. We will refer to $\xi_{\alpha'}$ as *the germ in* $\mathcal{P}$ *with higher intersection with* γ. The same applies if a multiple $r\gamma$ of γ is used instead.

A *cluster* with origin at O is a finite set K of points equal or infinitely near to O on S so that for any $p \in K$ all points preceding p also belong to K. By assigning integral multiplicities $\nu = \{\nu_p\}$ to the points p of a cluster K we get a *weighted cluster* $\mathcal{K} = (K, \nu)$, the multiplicities ν being usually called the *virtual multiplicities* of $\mathcal{K}$. Weighted clusters whose multiplicities may be realized by a germ of curve (i.e., for some germ of curve ξ at O, $e_p(\xi) = \nu_p$ for all $p \in K$) are called *consistent clusters*. They are easily characterized in terms of virtual multiplicities and proximity. Indeed, the integer

$$\rho_p = \nu_p - \sum_{q \text{ proximate to } p} \nu_q$$

is called the *excess* of $\mathcal{K}$ at p; consistent clusters are those with no negative excesses, see [7], 4.2. Generic germs ξ going through a consistent cluster $\mathcal{K} = (K, \nu)$ have $e_p(\xi) = \nu_p$ for all $p \in K$ and no singular points outside of K ([7], 4.2.7), which is usually abridged by saying that ξ *goes sharply* through $\mathcal{K}$.

If $\mathcal{K} = (K, \nu)$ is a consistent cluster and ξ is a germ of curve, both with origin at O, we will take the *intersection multiplicity* of $\mathcal{K}$ and ξ as being

$$[\mathcal{K}.\xi] = [\xi.\mathcal{K}] = \sum_{p \in K} \nu_p e_p(\xi).$$

As it is clear, $[\mathcal{K}.\xi]$ equals the intersection multiplicity of ξ with any germ going through $\mathcal{K}$ with effective multiplicities equal to the virtual ones and sharing no point with ξ outside of $\mathcal{K}$, which is in particular true for generic germs through $\mathcal{K}$ (see [7], 4.2.8). If $\mathcal{K}' = (K', \nu')$ is a second weighted cluster with the same origin as $\mathcal{K}$, we will take

$$[\mathcal{K}.\mathcal{K}'] = \sum_{p \in K \cap K'} \nu_p \nu'_p.$$

We will in particular write

$$\mathcal{K}^2 = [\mathcal{K}.\mathcal{K}] = \sum_{p \in K} \nu_p^2,$$

and call it the *self-intersection* of $\mathcal{K}$. Obviously $\mathcal{K}^2 = [\xi.\zeta]$ for any couple of germs ξ, ζ going sharply through $\mathcal{K}$ and sharing no points outside of it.

Any pencil of germs of curve $\mathcal{P}$ without fixed part has a weighted cluster of base points $BP(\mathcal{P})$ that consists of the points and multiplicities shared by all but finitely many germs in the pencil. In the sequel, when saying that a pencil has cluster of base points, we shall implicitly assume that it has no fixed part.

Equalities involving intersection multiplicities or orders of series should be read as meaning that one side is finite if and only if so is the other and then they agree. A pencil of smooth germs of curve with variable tangent at a point p on a on a smooth surface will be called a *pencil of lines* at p. Once such a pencil is fixed, its members will be referred to as *lines*.

If S and T are smooth surfaces, $\varphi : S \longrightarrow T$ an analytic morphism, $O \in S$ and $O' = \varphi(O)$, then the *inverse image* (at O) of a germ $\zeta : f = 0$ at O' is $\varphi^*(\zeta) : \varphi^*(f) = f \circ \varphi = 0$. If γ is an irreducible germ at O and σ is a local parameterization of γ, the composite map $\varphi \circ \sigma$ either is constant or defines an irreducible germ γ' at O'. In the first case one says that γ has been *contracted* by φ and the *direct image* of γ, $\varphi_*(\gamma)$, is taken to be the point O'. Otherwise we take $\varphi_*(\gamma) = d\gamma'$, $d = \deg \varphi \circ \sigma$. Directly from these definitions it follows the equality (*projection formula*) $[\varphi_*(\gamma).\zeta] = [\gamma.\varphi^*(\zeta)]$, for which we agree in taking equal to infinity the intersection multiplicity of any germ of curve and its origin. We will often make use of the following well known fact without further reference: if γ is an irreducible germ at O', $\tilde{\gamma}$ denotes its strict transform by the composition φ of a sequence of blowing-ups and O is the origin of $\tilde{\gamma}$, then $\varphi_*(\tilde{\gamma}) = \gamma$.

3. Fundamental points and local degree. In the sequel S and T will denote smooth complex analytic surfaces, $\varphi : S \longrightarrow T$ an analytic morphism, O a point of S and $O' = \varphi(O)$. Throughout all the paper we shall assume that the pull back morphism induced by φ, $\varphi^* : \mathcal{O}_{T,O'} \longrightarrow \mathcal{O}_{S,O}$, is a monomorphism, or, equivalently, that no analytic curve in a neighbourhood of O' contains the image of a neighbourhood of O. In the sequel we will often write $\varphi^*(h) = h^*$ for $h \in \mathcal{O}_{T,O'}$.

We will take x, y and u, v to be local coordinates on S and T with origins at O and O', respectively, and assume that φ is given in a neighbourhood U of O by the equalities

$$\begin{aligned} u &= f(x, y) \\ v &= g(x, y), \end{aligned}$$

where f and g are non-invertible convergent series in x, y. The above equalities, and sometimes also the series f, g, will be called the *equations* of φ relative to the coordinates x, y, u, v.

The injectivity of φ^* being equivalent to the functional independence of f, g, the jacobian determinant

$$J(\varphi) = J(f, g) = \frac{\partial(f, g)}{\partial(x, y)}$$

is not identically zero. It defines thus a germ of curve at O that will be called the *jacobian germ*, or just the *jacobian*, of φ at O and denoted by $\mathbf{J} = \mathbf{J}(\varphi)$. Its direct image $\Delta = \Delta(\varphi) = \varphi_*(\mathbf{J}(\varphi))$ is the *discriminant* of φ at O. Of course neither $J(\varphi)$, nor $\mathbf{J}(\varphi)$, nor $\Delta(\varphi)$ depend on the choice of coordinates.

The germs in the family $\mathcal{P} = \{\xi_\alpha : \alpha_1 f + \alpha_2 g = 0\}$, $\alpha = \alpha_1/\alpha_2 \in \mathbb{C} \cup \{\infty\}$, are the inverse images of the germs in the pencil of lines $\mathcal{N} = \{\ell_\alpha : \alpha_1 u + \alpha_2 v = 0\}$, $\alpha = \alpha_1/\alpha_2 \in \mathbb{C} \cup \{\infty\}$. As it is clear, all germs in $\mathcal{P}$ but at most one have the same multiplicity at O, which is $\min(o(f), o(g))$. We will call this multiplicity the *multiplicity of* φ *at* O, written $e_O(\varphi)$ or just $e(\varphi)$. We will say that φ is *dicritical at* O (or just dicritical) if and only if all germs in $\mathcal{P}$ have multiplicity $e(\varphi)$ at O. This obviously occurs if and only if the forms of degree $e(\varphi)$ of f and g are linearly independent, or, equivalently, if and only if the tangent cones to the germs in $\mathcal{P}$ do vary with α. Otherwise we will say that φ is non-dicritical (at O): then all germs in $\mathcal{P}$ have the same multiplicity and the same tangent cone at O, but for a single germ which has higher multiplicity.

Put $d = \gcd(f, g)$. The family $\mathcal{P}$ has a fixed part $\Phi : d = 0$ and a variable part $\mathcal{P}' : \alpha_1 f/d + \alpha_2 g/d = 0$, $\alpha_1/\alpha_2 \in \mathbb{C} \cup \{\infty\}$. We will call Φ the *contracted germ* (by φ), as all points on a small enough representative of Φ are mapped to O'. An easy computation shows that Φ is contained in $\mathbf{J}(\varphi)$. Of course Φ may be empty: this occurs if and only if O is isolated in its fibre, or, equivalently, the germ of φ at O is finite (see for instance [11], II.E.6). If both f/d and g/d are non-invertible (which is always the case if $\Phi = \emptyset$), $\mathcal{P}'$ is a pencil of germs at O without fixed part: then we define $BP(\varphi)$, the *cluster of base points of* φ, as being the weighted cluster of base points of $\mathcal{P}'$. Otherwise $\mathcal{P}'$ is a family all whose germs but one are empty and we just take $BP(\varphi) = \emptyset$.

For any p infinitely near to O we define the *multiplicity of* φ *at* p, $e_p(\varphi)$ as the sum of the virtual multiplicity of p in $BP(\varphi)$ (just zero if $p \notin BP(\varphi)$) and the (effective) multiplicity of Φ at p. This obviously extends the former definition of $e_O(\varphi)$ and, for any p equal or infinitely near to O, all but finitely many germs in $\mathcal{P}$ have effective multiplicity $e_p(\varphi)$ at p. A point p equal or infinitely near to O will be called a *fundamental point of* φ if and only if $e_p(\varphi) > 0$, that is, p either is a base point of φ or lies on the contracted germ Φ.

Let M_T denote the maximal ideal of $\mathcal{O}_{T,O'}$. Clearly d is the greatest common divisor of the elements of the ideal $\varphi^*(M_T)$ generated in $\mathcal{O}_{S,O}$ by the inverse images of the elements of M_T. Then $BP(\varphi)$ is the cluster of base points of the linear system of germs defined by $d^{-1}\varphi^*(M_T)$ if $d^{-1}\varphi^*(M_T) \neq (1)$, otherwise it is empty. It easily follows that neither of the notions introduced above depends on the choice of coordinates.

Proposition 3.1. *Fix any pencil of lines $\mathcal{N}$ at O'. For all but finitely many lines $\ell \in \mathcal{N}$, the number of points in $\varphi^{-1}(p)$ that approach O when p approaches O' on a representative of ℓ equals $BP(\varphi)^2 + [BP(\varphi).\Phi]$.*

Proof. After a suitable choice of the coordinates we may take $\mathcal{N} = \{\ell_\alpha : \alpha_1 u + \alpha_2 v = 0\}$, $\alpha = \alpha_1/\alpha_2 \in \mathbb{C} \cup \{\infty\}$ and $\ell = \ell_\alpha$, $\alpha \neq \infty$. If $t = u(p) \neq 0$, for a small enough open neighbourhood U of O, $\varphi^{-1}(p) \cap U$ is the intersection of the curves defined in U by $f = t$ and $\alpha f + g = 0$, which is the same as the intersection of $f = t$ and $d^{-1}(\alpha f + g) = 0$, as $d|f$ and hence no point on $f = t$ may be on $d = 0$. If the germ $\xi'_\alpha : d^{-1}(\alpha f + g) = 0$ is reduced, or empty, the number of points on $f = t$ and $d^{-1}(\alpha f + g) = 0$ that approach O when t approaches 0 is just

$$[\xi_\infty.\xi'_\alpha] = [\Phi.\xi'_\alpha] + [\xi'_\infty.\xi'_\alpha].$$

If $BP(\varphi) = \emptyset$, then $\xi'_\alpha = \emptyset$ for all but one α and the claim is obviously satisfied. Otherwise for any $\alpha \neq \infty$, $[\xi'_\infty.\xi'_\alpha] = BP(\varphi)^2$, while for all but finitely many α, $[\Phi.\xi'_\alpha] = [BP(\varphi).\Phi]$ and ξ'_α is reduced, because the ξ'_α describe a pencil without fixed part ([7], 7.2.10). Hence the claim. □

In the sequel we will call $BP(\varphi)^2 + [BP(\varphi).\Phi]$ the *(local) degree of* φ *at* O, and denote it by $\deg_O(\varphi)$. If $\Phi = \emptyset$, then the local degree is just $BP(\varphi)^2 = \dim_{\mathbb{C}} \mathcal{O}_{S,O}/(f,g)$. It holds

Lemma 3.2. $\deg_O(\varphi) = 0$ *if and only if* $BP(\varphi) = \emptyset$, *while* $\deg_O(\varphi) = 1$ *if and only if φ is a local isomorphism at O*

Proof. Since both $BP(\varphi)^2$ and $[BP(\varphi).\Phi]$ are non-negative, $\deg_O(\varphi) = 0$ forces $BP(\varphi)^2 = 0$ and hence $BP(\varphi) = \emptyset$. The converse is clear. Because of the first claim, in case $\deg_O(\varphi) = 1$, $BP(\varphi)^2 > 0$ and so $BP(\varphi)^2 = 1$ and $[BP(\varphi).\Phi] = 0$. The first equality forces $BP(\varphi)$ to be just O taken with multiplicity one, after which the second one gives $\Phi = \emptyset$. Then f and g have linearly independent initial forms of degree one and φ is a local isomorphism at O. Again, the converse is clear. □

Remark 3.3. Clearly $\deg_O(\varphi) \geq 0$ and $\deg_O(\varphi) > 0$ if $\Phi = \emptyset$, as then $BP(\varphi) \neq \emptyset$.

Example 3.4. If φ is the blowing-up of O' and O is any point on the exceptional divisor of φ, then $\deg_O(\varphi) = 0$ and the equality of 3.1 holds true for all lines but the one going through O.

Example 3.5. If $\pi : T \to Z$ is the blowing-up of a point O'' on a smooth surface Z and φ is as above and has O' on the exceptional divisor of π, then all but one lines ℓ of any fixed pencil of lines at O'' have their inverse images (at O) $(\pi \circ \varphi)^*(\ell)$ equal and therefore $\deg_O(\pi \circ \varphi) = 0$.

Remark 3.6. It follows easily from the proof of 3.1 that if $\Phi = \emptyset$, then the equality of 3.1 holds true for all lines ℓ provided the points in $\varphi^{-1}(p)$ are counted according to the multiplicities of the components of the curve $\alpha f + g = 0$ they belong to.

The multiplicity of a direct image is easily determined:

PROPOSITION 3.7. *If ξ is an irreducible germ at O, not a branch of the contracted germ Φ, $\varphi_*(\xi)$ is a germ and $e_{O'}(\varphi_*(\xi)) = [\xi.BP(\varphi)] + [\xi.\Phi]$*

Proof. For all but finitely many ℓ_α in a pencil of lines $\mathcal{N}$ at O', $e_{O'}(\varphi_*(\xi)) = [\varphi_*(\xi).\ell_\alpha] = [\xi.\varphi^*(\ell_\alpha)] < \infty$, hence the claim. □

4. The main trunk of φ. Fix a pencil of lines $\mathcal{L}$ at O. By a suitable choice of the local coordinates x, y, we may assume that $\mathcal{L} = \{\ell_\alpha : \alpha_1 x - \alpha_2 y = 0\}$, $\alpha = \alpha_1/\alpha_2 \in \mathbb{C} \cup \{\infty\}$. Thus, after dropping the germ of the y-axis, the remaining germs $\ell_\alpha \in \mathcal{L}$ are given by the parametric equations $x = \bar{t}$, $y = \alpha\bar{t}$, $\alpha \in \mathbb{C}$. Assume that the equations of φ are written in the form

$$f = f_n + \cdots + f_i + \cdots$$
$$g = g_n + \cdots + g_i + \cdots ,$$

where f_i and g_i are forms of degree i, n is the multiplicity of φ and, therefore, either f_n or g_n is non-zero. Interchanging u and v, if necessary, we will assume in the sequel that $f_n \neq 0$.

The direct image of ℓ_α, $\gamma_\alpha = \varphi_*(\ell_\alpha)$ is given by the parametric equations

$$u = f_n(1,\alpha)\bar{t}^n + \cdots + f_i(1,\alpha)\bar{t}^i + \cdots$$
$$v = g_n(1,\alpha)\bar{t}^n + \cdots + g_i(1,\alpha)\bar{t}^i + \cdots ,$$

so, if $f_n(1,\alpha) \neq 0$, one may take

$$t = \bar{t} f_n(1,\alpha)^{1/n}\Big(1 + \cdots + \frac{f_i(1,\alpha)}{f_n(1,\alpha)}\bar{t}^{i-n} + \cdots\Big)^{1/n}$$

as a new parameter, to get a Puiseux-like parameterization of γ_α, namely

$$u = t^n$$
$$v = \sum_{i\geq n} a_i t^i,$$

the coefficients $a_i = a_i(\alpha)$ being algebraic functions of α defined in $\mathbb{C} - Z$, Z the set of zeros of $f_n(1,\alpha)$. In fact all a_i are rational functions of α and $f_n(1,\alpha)^{i/n}$, as taking a n-th root of $(1 + \cdots + \frac{f_i(1,\alpha)}{f_n(1,\alpha)}\bar{t}^{i-n} + \cdots)$ introduces no irrationality.

Note that γ_α need not be an irreducible germ: it is n/n' times the irreducible germ at O' with Puiseux series

$$v = \sum_{i\geq n} a_i u^{i/n},$$

n' being the minimal common denominator of the i/n for which $a_i \neq 0$. Enriques' theorem describing the infinitely near points on an irreducible germ from one of its

Puiseux series (see [10], IV.I or [7], 5.5) still applies to γ_α if one takes for it the above Puiseux series and the division algorithms are performed using the characteristic exponents written with denominator n instead of n', as this has the effect of multiplying all multiplicities by n/n', as required.

At least one of the a_i really depends on α, as otherwise the pull-back of an equation of the constant germ γ_α would be identically zero, contradicting the injectivity of φ^*. Write m for the lowest index i for which the coefficient a_i is a non-constant function: a_i is constant if $i < m$ while a_m is not. According to the way in which the coefficients of a Puiseux series determine the positions of the free points on the germ ([7], 5.7), the germs γ_α for which $\alpha \in \mathbb{C} - Z$ and $a_m(\alpha) \neq 0$, share the point O', all their free points depending on the coefficients a_i, $i < m$, all their satellite points that are satellite of one of the former free points (i.e., the satellite points associated to characteristic exponents less or equal than m/n), and no further point. Furthermore, by the Enriques theorem, the multiplicities of these shared points on γ_α do not depend on α.

We take the points shared by the germs γ_α, $\alpha \in \mathbb{C} - Z$, $a_m(\alpha) \neq 0$, together with their multiplicities on any of the γ_α, to make a weighted cluster $\mathcal{T} = \mathcal{T}(\varphi)$. We shall call it the *trunk* or the *main trunk* of φ. Note that $n = e(\varphi)$ is the virtual multiplicity of O' in $\mathcal{T}$. As it will turn out in a while, $\mathcal{T}$ does not depend on the pencil $\mathcal{L}$ used to define it.

Since the points and the multiplicities of $\mathcal{T}$ belong to a multiple of an irreducible germ, $\mathcal{T}$ is consistent and unibranched (i.e., totally ordered), and all of its points but the last one have excess zero. In other words ([7], 8.4), $\mathcal{T} = r\mathcal{T}_{red}$ where $\mathcal{T}_{red}$ is an irreducible cluster and r a positive integer that will be called the *multiplicity* of $\mathcal{T}$. The *top* of $\mathcal{T}$ will be the last point in $\mathcal{T}$, both its virtual multiplicity and excess equal the multiplicity of $\mathcal{T}$.

Clearly, the multiplicity $n = e(\varphi)$ and the fractional power series

$$\mathcal{S} = \mathcal{S}(u, \theta) = \sum_{i<m} a_i u^{i/n} + \theta u^{m/n},$$

where θ is a free variable, contain the same information as $\mathcal{T}$. We shall call $\mathcal{S}$ *the Puiseux series of* $\mathcal{T}$ (relative to the coordinates u, v) and take it, as usual, as determined up to conjugation over $\mathbb{C}\{u\}$. The partial sum $\sum_{i<m} a_i u^{i/n}$ will be called the *constant part of* $\mathcal{S}$, while the monomial $\theta u^{m/n}$ will be referred to as its *variable part.*

The characteristic exponents of $\mathcal{S}$ will be written $m_1/n, \dots, m_k/n$ and called the *characteristic exponents* of $\mathcal{T}$. Together with n, which is the virtual multiplicity of O' in $\mathcal{T}$, they determine (and are in turn determined by) the virtual multiplicities and proximity relations of the points of $\mathcal{T}$, by the rules of the Enriques theorem.

The integer m will be called the *height* of $\mathcal{T}$. Again by the Enriques theorem, m/n is a (necessarily the last) characteristic exponent of $\mathcal{T}$ if and only if the top of $\mathcal{T}$ is a satellite point.

If $\mathcal{K}$ is an irreducible cluster with origin O' and r a positive integer, it is easy to exhibit a morphism φ with $\mathcal{T}(\varphi) = r\mathcal{K}$. Indeed, according to [7], 5.7.1 and 5.7.2, any (necessarily irreducible) germ γ going sharply through $\mathcal{K}$ has a Puiseux series of the form $s = \sum_{1 \leq i < \ell} b_i u^{i/n'} + cu^{\ell/n'} + \cdots$, where the coefficients b_i are the same for all germs γ going sharply through $\mathcal{K}$, and c depends on the first point on γ not in $\mathcal{K}$. Then it is enough to take φ defined by $f = x^{rn'}$, $g = \sum_{1 \leq i < \ell} b_i x^{ri} + y^{r\ell}$.

Multiplicities of inverse images are controlled by $\mathcal{T}$, namely,

THEOREM 4.1. *If ζ is any germ of curve at O', $e_O(\varphi^*(\zeta)) = [\zeta.\mathcal{T}]$.*

Proof. For all but finitely many α, $e_O(\varphi^*(\zeta)) = [\varphi^*(\zeta).\ell_\alpha]$, which in turn, by the projection formula, equals $[\zeta.\gamma_\alpha]$. Exclude the $\alpha \in Z$. Then, since the first point on γ_α not in $\mathcal{T}$ varies with α, the claim follows from the Noether formula for the intersection multiplicity ([7], 3.3.1) and the definition of $\mathcal{T}$. ◻

If the equisingularity type of the germ ζ is fixed, then the possible intersection multiplicities $[\zeta.\mathcal{T}]$ are described by considering all partial superpositions of the Enriques diagrams of $\mathcal{T}$ and ζ allowed by their own proximity structures. ([7], 3.9). Then, by 4.1 above, we get in this way all possible multiplicities of the inverse images $\varphi^*(\zeta)$. For instance, if $\mathcal{T}$ has first characteristic exponent m_1/n, then the multiplicities of $\varphi^*(\zeta)$ for ζ smooth are m_1 and the multiples in of n for $0 < i < m_1/n$.

Dicritical morphisms, already defined on page 6, have very simple trunks:

Corollary 4.2. *The morphism φ is dicritical if and only if $\mathcal{T}$ has O' as its only point. Furthermore, φ is an isomorphism if and only if $\mathcal{T}$ consists of O' with virtual multiplicity one.*

Proof. Obviously, O' always belongs to $\mathcal{T}$. Take a pencil $\mathcal{N}$ of lines with origin at O' as in Section 3. Then $\mathcal{T}$ contains a point in the first neighbourhood of O' if and only if one of the lines in $\mathcal{N}$ has higher intersection with $\mathcal{T}$ and so, by 4.1, if and only if one of the inverse images of the lines in $\mathcal{N}$ has higher multiplicity, hence the first claim. If in addition $n = e(\varphi) = 1$, then the inverse images of the lines in a pencil at O' describe a pencil of lines at O and thus φ is an isomorphism. The converse is obvious. ◻

Remark 4.3. It is clear from definitions that, $m \geq n$; by Corollary 4.2, the equality holds if and only if φ is dicritical. By [7], 5.7.4, m is the sum of the multiplicities of all non-satellite points on $\mathcal{T}$.

One may associate to φ the set of the multiplicities of the inverse images by φ of all germs at O'. This is an additive semigroup that will be denoted by $\Gamma(\varphi)$ and called the *semigroup of* φ. It is described next:

Proposition 4.4. $\Gamma(\varphi) = r\Gamma(\zeta)$ *where* $\Gamma(\zeta)$ *is the semigroup of any (necessarily irreducible) germ going sharply through* $\mathcal{T}_{red}$

Proof. Since $\mathcal{T}_{red}$ is irreducible, all germs ζ going sharply through it are irreducible and have the same equisingularity type ([7], 4.2.6 and 4.2.8). They all have thus the same semigroup $\Gamma(\zeta)$. For any germ ξ at O', we may choose ζ going sharply through $\mathcal{T}_{red}$ and missing all points on ξ in the first neighbourhood of the top of $\mathcal{T}$ ([7], 4.2.6). For such a ζ, $r[\xi.\zeta] = [\xi.\mathcal{T}]$ and therefore $\Gamma(\varphi) \subset r\Gamma(\zeta)$. On the other hand, for each characteristic exponent κ_i, $i = 1 \dots, k$, of ζ one may choose an irreducible germ ξ_i going with effective multiplicity one through the last free point q_i on ζ associated to κ_i and having no satellite point after it. Then n/r and the intersection multiplicities $[\xi_i.\zeta]$ (minimally) generate $\Gamma(\zeta)$ ([7], 5.8, for instance). Since ζ goes sharply through $\mathcal{T}_{red}$, all the points q_i, $i = 1 \dots, k$, belong to $\mathcal{T}$ and therefore, by the Noether formula, $r[\xi_i.\zeta] = [\xi_i.\mathcal{T}]$. Since obviously $n \in \Gamma(\varphi)$, this proves that a set of generators of $r\Gamma(\zeta)$ is contained in $\Gamma(\varphi)$ and so that $r\Gamma(\zeta) \subset \Gamma(\varphi)$, as needed to complete the proof. ◻

In fact the multiplicities of the inverse images of the germs at O' are the values of the pull-backs of their equations by the $\mathcal{M}$-adic valuation of $\mathcal{O}_{S,O}$. We have:

Corollary 4.5. *The composition of φ^* and the $\mathcal{M}$-adic valuation of $\mathcal{O}_{S,O}$ ($\mathcal{M}$ the maximal ideal of $\mathcal{O}_{S,O}$) is a divisorial valuation, has value semigroup* $\Gamma(\varphi)$ *and its centers and multiplicities are the points and multiplicities of* $\mathcal{T}(\varphi)$.

Proof. The composition obviously is a valuation. That its centers and multiplicities are the points and multiplicities of $\mathcal{T}$ is a direct consequence of 4.1 and the Noether formula for valuations ([7], 8.1.7). Since the divisorial valuations are those with finitely many centers, this completes the proof. □

From 4.5 we get, as already announced:

COROLLARY 4.6. *The trunk of φ does not depend on the pencil of lines at O used to define it.*

As an easy but not quite representative example, the reader may consider the case in which φ is the composition of blowing up points $O', q_1, \dots q_j$, each in the first neighbourhood of the preceding one, and O is any point on the exceptional divisor of the last blowing-up. Then $\mathcal{T}(\varphi)$ is the only irreducible cluster with points $O', q_1, \dots, q_j, O$ and $BP(\varphi) = \emptyset$.

The next geometrical interpretation of the self-intersection of $\mathcal{T}$ is also a direct consequence of 4.1 and the definition of $\mathcal{T}$:

COROLLARY 4.7. *If $\mathcal{L}$ is any pencil of lines at O, for all but finitely many $\ell \in \mathcal{L}$, $e_O(\varphi^*\varphi_*(\ell)) = \mathcal{T}^2$.*

COROLLARY 4.8. *If φ is as above and $\psi : T \to Z$ is a second analytic morphism to a smooth surface Z whose image is not contained in a curve, then*

$$e_O(\psi \circ \varphi) = [BP(\psi).\mathcal{T}(\varphi)] + [\Psi.\mathcal{T}(\varphi)],$$

Ψ being the contracted germ by ψ.

Proof. The multiplicity $e_O(\psi \circ \varphi)$ equals the multiplicity of $\varphi^*(\psi^*(\ell))$ for all but at most one of the lines ℓ in a fixed pencil of lines at $\psi(O')$. By 4.6, $e_O(\varphi^*(\psi^*(\ell))) = [(\psi^*(\ell).\mathcal{T}(\varphi)]$. Now, since for all but finitely many ℓ, $\psi^*(\ell)$ is composed of Ψ and a germ going sharply through $BP(\psi)$ and sharing no base points with $\mathcal{T}(\varphi)$ ([7], 7.2.10), the claim follows. □

5. Ratios of multiplicities. As above, we write $n = e(\varphi)$ and $m_1/n, \dots, m_k/n$, $k \geq 0$, the characteristic exponents of $\mathcal{T}(\varphi)$. Put $n_i = \gcd(n, m_1, \dots, m_i)$, so that $n_0 = n$ and $n_k = r$. Let us recall how the Enriques theorem applied to any germ going sharply through $\mathcal{T}_{red}$ relates the points of $\mathcal{T}$ to its characteristic exponents (see [7], 5.5 for more details). To each characteristic exponent m_i/n there is associated a non-empty set of consecutive free points immediately followed by a non-empty set of consecutive satellite points, in such a way that O' (retained as a free point just for this description) is the first point associated to m_1/n and the first point associated to m_i/n lies in the first neighbourhood of the last satellite point associated to m_{i-1}/n for $i > 1$. There is a further, maybe empty, set of consecutive free points in $\mathcal{T}$ following the last satellite point associated to m_k/n, this set being the whole set of points of $\mathcal{T}$ if $\mathcal{T}$ has no characteristic exponents.

As seen in the proof of 4.4, the minimal system of generators of the semigroup $\Gamma(\varphi)$ may be obtained just as for the semigroup of an irreducible germ: we choose irreducible germs $\xi_1, \dots, \xi_k$, each ξ_i going with effective multiplicity one through the last free point $q_i \in \mathcal{T}$ associated to m_i/n and having no satellite points after q_i. One takes $\check{m}_i = [\xi_i.\mathcal{T}]$ and then $\{n, \check{m}_1, \dots, \check{m}_k\}$ is the minimal system of generators of $\Gamma(\varphi)$. Furthermore, a direct computation gives

$$\check{m}_i = \frac{(n - n_1)m_1}{n_{i-1}} + \frac{(n_1 - n_2)m_2}{n_{i-1}} + \cdots + \frac{(n_{i-2} - n_{i-1})m_{i-1}}{n_{i-1}} + m_i \qquad (1)$$

which allow to compute the $\check{m}_i$ from the m_i and conversely. It is also worth noting that, necessarily, each ξ_i has multiplicity n/n_{i-1} and goes through no point in $\mathcal{T}$ after q_i.

Similarly, a direct computation of $[\mathcal{T}.\gamma]$, γ going sharply through $\mathcal{T}_{red}$, gives the formula

$$\frac{\mathcal{T}^2}{r} = \frac{(n-n_1)m_1}{r} + \frac{(n_1-n_2)m_2}{r} + \cdots + \frac{(n_{k-1}-r)m_k}{r} + m, \tag{2}$$

which shows that, once n and the characteristic exponents of $\mathcal{T}$ are known, m and $\mathcal{T}^2$ determine each other.

Remark 5.1. It follows from the above expressions that $\check{m}_i n_{i-1} < \check{m}_{i+1} n_i$ for $i = 1, \dots, k-1$, and $\check{m}_k n_{k-1} \leq \mathcal{T}^2$, the last inequality being strict if and only if $m_k < m$, this is, $\mathcal{T}$ has a free top.

We adapt the definition of contact between two germs, as given in [15], by taking as *contact* of an irreducible germ γ at O' and $\mathcal{T}$ the rational number

$$\langle \gamma.\mathcal{T} \rangle = n \max_s \{\mathrm{ord}_u(\mathcal{S} - s)\},$$

where s runs on the conjugates of the Puiseux series of γ. It is clear from the definition that the maximal contact of irreducible germs with no more than $i-1$ characteristic exponents ($1 \leq i \leq k$) and $\mathcal{T}$ is m_i, this maximal contact being reached if and only if the germ goes through the last free point in $\mathcal{T}$ associated to the i-th characteristic exponent. Also, the height m is the maximal contact of $\mathcal{T}$ with arbitrary irreducible germs. All germs going sharply through $\mathcal{T}_{red}$ have contact m with $\mathcal{T}$.

For a real c, $0 < c \leq m$, define

$$\rho(c) = \frac{\check{m}_i n_{i-1}}{n} + \frac{(c - m_i) n_i}{n}$$

if $m_i < c \leq m_{i+1}$, $m_{k+1} = m$. Clearly $\rho(c)$ is a continuous, strictly increasing and piecewise linear function of c. We have:

Lemma 5.2. *For any irreducible germ γ at O',*

$$\frac{[\gamma.\mathcal{T}]}{e_{O'}(\gamma)} = \rho(\langle \gamma.\mathcal{T} \rangle).$$

Proof. Follows by direct computation as in the proof of [15], 2.4. $\square$

Proposition 5.3. *Take $0 < i \leq k$. For any non-empty germ ξ at O', no branch of which has i or more characteristic exponents,*

$$\frac{e_O(\varphi^*(\xi))}{e_{O'}(\xi)} \leq \frac{\check{m}_i n_{i-1}}{n},$$

and the equality holds if and only if all branches of ξ go through the last free point in $\mathcal{T}$ associated to the i-th characteristic exponent.

For any non-empty germ ξ at O'

$$\frac{e_O(\varphi^*(\xi))}{e_{O'}(\xi)} \leq \frac{\mathcal{T}^2}{n},$$

and the equality holds for all germs going sharply through $\mathcal{T}_{red}$.

Proof. Assume first ξ irreducible. By 4.1 and 5.2, the first inequality may be written

$$\rho(\langle \xi.\mathcal{T}\rangle) \leq \frac{\check{m}_i n_{i-1}}{n} = \rho(m_i),$$

after which both claims regarding it follow from the properties of the function ρ and the contact stated above. The case of ξ non irreducible follows by using the additivity of multiplicities.

Similar arguments prove the second inequality. The last claim is obvious after 4.1 and the definition of $\mathcal{T}^2$. □

Remark 5.4. One may add to the upper bounds of Proposition 5.3 the rather obvious lower one

$$n \leq \frac{[\mathcal{T}.\xi]}{e_{O'}(\xi)} = \frac{e_O(\varphi^*(\xi))}{e_{O'}(\xi)}.$$

In particular, for a dicritical φ one gets $\mathcal{T}^2 = n^2$ and therefore $e_O(\varphi^*(\xi))/e_{O'}(\xi) = n$ for any non-empty ξ, which also results from an easy direct computation.

Next are two consequences of 5.3 and 5.4 regarding the asymptotics of the complexity of φ in case $S = T$ and $O = O'$ (see for instance [2] and [3], 1994-48, 1994-49):

COROLLARY 5.5. *If $S = T$ and $O = O'$, then for any $j > 0$ and any non-empty germ of curve ξ at O,*

$$e(\varphi)^j \leq \frac{e_O((\varphi^j)^*(\xi))}{e_O(\xi)} \leq \left(\frac{\mathcal{T}^2}{e(\varphi)}\right)^j.$$

Since for any j, $e(\varphi^j)$ equals the multiplicity $e_O((\varphi^j)^*(\ell))$ for all but one lines ℓ in a fixed pencil of lines at O', 5.5 applied to any such ℓ gives:

COROLLARY 5.6. *If $S = T$ and $O = O'$, then for any $j > 0$,*

$$e(\varphi)^j \leq e(\varphi^j) \leq \left(\frac{\mathcal{T}^2}{e(\varphi)}\right)^j.$$

Note that both the inequalities of 5.5 and 5.6 become equalities if φ is dicritical.

6. The multiplicity of the jacobian. This section is devoted to proving the following theorem, which relates the multiplicity of the jacobian and the height of the trunk:

THEOREM 6.1. *If, as above, n is the multiplicity of φ at O and m denotes the height of $\mathcal{T}(\varphi)$,*

$$e_O(\mathbf{J}(\varphi)) = n + m - 2.$$

Before proving 6.1 we set a couple of easy facts about jacobians whose proof is left to the reader:

Lemma 6.2. *If P, P' are homogeneous polynomials in x, y, of degrees d and d', then the jacobian determinant*

$$J(P, P') = \frac{\partial(P, P')}{\partial(x, y)}$$

is zero if and only if $P^{d'} = aP'^{d}$ for some $a \in \mathbb{C}$. Otherwise $J(P, P')$ is homogeneous of degree $d + d' - 2$

We call the couple P, P' *homothetical* when $J(P, P') = 0$.

Lemma 6.3. *If h, h' are germs of analytic functions at O, $o(J(h, h')) \geq o(h) + o(h') - 2$. The equality is true if and only if the initial forms $\widehat{h}, \widehat{h}'$ are not homothetical. In this case $\widehat{J(h, h')} = J(\widehat{h}, \widehat{h}')$.*

Proof. Proof of 6.1: As in the preceding section, choose the coordinates u, v so that the initial form f_n of the first equation f of φ has degree n. For any equation h of any germ of curve $\zeta : h = 0$ at O'

$$J(f, h(f, g)) = \frac{\partial h}{\partial v}(f, g) J(f, g).$$

By taking orders at O and using 6.3 we get

$$\begin{aligned} e_O(\mathbf{J}(\varphi)) = o(J(f, g)) =& o(J(f, h(f, g))) - o\left(\frac{\partial h}{\partial v}(f, g)\right) \\ \geq& o(f) + o(h(f, g)) - 2 - o\left(\frac{\partial h}{\partial v}(f, g)\right) \\ =& n + e_O(\varphi^*(\zeta)) - 2 - e_O(\varphi^*(\zeta')) \\ =& n + [\mathcal{T}.\zeta] - 2 - [\mathcal{T}.\zeta'], \end{aligned}$$

where ζ' denotes the polar germ of ζ, $\zeta' : \partial h / \partial v = 0$. Then,

Lemma 6.4. *If the above inequality is strict, then the difference $[\mathcal{T}.\zeta] - [\mathcal{T}.\zeta']$ is not maximal.*

Proof. Proof of 6.4: By 6.3, if the inequality is strict the initial forms of f and $h(f, g)$ are homothetical, and so there exits $b \in \mathbb{C}$ so that $o(f^{n'} - bh(f, g)^n) > nn'$, where $n' = o(h(f, g)) = e_O(\varphi^*(\zeta)) = [\mathcal{T}.\zeta]$. Take $h_1 = u^{n'} - bh^n$ and $\zeta_1 : h_1 = 0$, so that $\varphi^*(h_1) : f^{n'} - bh(f, g)^n = 0$ and the last inequality may be written $[\mathcal{T}.\zeta_1] > nn'$. On the other hand the polar ζ_1' has equation $h^{n-1} \partial h / \partial v = 0$ and thus

$$[\mathcal{T}.\zeta_1'] = (n - 1)[\mathcal{T}.\zeta] + [\mathcal{T}.\zeta'] = nn' - [\mathcal{T}.\zeta] + [\mathcal{T}.\zeta'],$$

from which

$$[\mathcal{T}.\zeta_1] - [\mathcal{T}.\zeta_1'] > [\mathcal{T}.\zeta] - [\mathcal{T}.\zeta'],$$

as claimed. □

Now, to complete the proof of 6.1, we will show that

$$\max_h \{[\mathcal{T}.\zeta] - [\mathcal{T}.\zeta']\} = m.$$

As in the definition of $\mathcal{T}$, take $\gamma_\alpha = \varphi_*(\ell_\alpha)$: for all but finitely many values of α, γ_α is defined by the parameterization

$$\begin{aligned} u &= t^n \\ v &= s_\alpha(t) = \sum_{i<m,} a_i t^i + a_m(\alpha) t^m + \cdots \end{aligned}$$

and, by the definition of $\mathcal{T}$, for any germ $\xi : z = 0$ at O', $[\mathcal{T}.\xi] = o_t(z(t^n, s_\alpha(t)))$ for all but finitely many α.

We begin by proving the inequality $[\mathcal{T}.\zeta] - [\mathcal{T}.\zeta'] \leq m$. Notice that the polar ζ' depends on the equation h of ζ and not only on ζ itself. If one takes a different equation wh, w invertible, the new polar is $\zeta'' : w\partial h/\partial v + h\partial w/\partial v = 0$. Let us consider two possibilities:

If $[\mathcal{T}.\zeta] - [\mathcal{T}.\zeta'] > 0$ then, for all but finitely many α,

$$o_t(h(t^n, s_\alpha(t))) > o_t(\frac{\partial h}{\partial v}(t^n, s_\alpha(t)))$$

and it is clear from the above equation of ζ'' that $[\mathcal{T}.\zeta''] = [\mathcal{T}.\zeta']$. Thus, the difference $[\mathcal{T}.\zeta] - [\mathcal{T}.\zeta']$ does not depend on the equation of ζ. In particular $[\mathcal{T}.\zeta] - [\mathcal{T}.\zeta']$ is positive no matter which equation of ζ is used to define ζ'.

Otherwise, for one, and therefore for all choices of the equation of ζ, $[\mathcal{T}.\zeta] - [\mathcal{T}.\zeta'] \leq 0$, which makes the inequality $[\mathcal{T}.\zeta] - [\mathcal{T}.\zeta'] \leq m$ obvious.

Thus, it will be enough to prove the inequality for germs for which $[\mathcal{T}.\zeta] - [\mathcal{T}.\zeta']$ is positive: in such a case this difference does not depend on the choice of the equation of ζ and it is enough to consider a single equation for each germ ζ: we take its Weierstrass equation

$$h = u^k \prod_{j=1}^{e} (v - s_j(u)),$$

where $k \geq 0$ and the s_j, $j = 1, \ldots, e$, are the Puiseux series (including all conjugates) of ζ. An equation of ζ' is then

$$\frac{\partial h}{\partial v} = h \left(\sum_j \frac{1}{v - s_j(u)} \right),$$

and for all but finitely many α,

$$\begin{aligned} [\mathcal{T}.\zeta] - [\mathcal{T}.\zeta'] &= - \, o_t \left(\sum_j \frac{1}{s_\alpha(t) - s_j(t^n)} \right) \\ &\leq - \min_j o_t \left(\frac{1}{s_\alpha(t) - s_j(t^n)} \right) \\ &= \max_j o_t (s_\alpha(t) - s_j(t^n)) \leq m \end{aligned} \tag{3}$$

as the term $a_m(\alpha)t^m$ cannot be canceled. The next lemma proves that the bound m is attained and therefore completes the proof. We state it separately for future reference.

Lemma 6.5. *If $\zeta : h = 0$ is any irreducible germ having the last free point $q' \in \mathcal{T}$ as a simple point and no satellite points after it, then $[\mathcal{T}.\zeta] - [\mathcal{T}.\zeta'] = m$ for any u-polar $\zeta' : \partial h/\partial v = 0$ of ζ.*

Proof of 6.5. According to [7], 5.7.1 and 5.7.3, ζ has a Puiseux series of the form $s_1 = \sum_{i<m} a_i u^{i/n} + \cdots$, the dots indicating terms which do not increase the polydromy order. Thus, none of the conjugates s_j of s_1, $j \neq 1$, has partial sum $\sum_{i<m} a_i u^{i/n}$ and so

$$m = o_t(s_\alpha(t) - s_1(t^n)) > o_t(s_\alpha(t) - s_j(t^n)).$$

Then the claim for the u-polar corresponding to the Weierstrass equation follows from the equality 3 above. Then, since $m > 0$, one may argue as in the proof of 6.1 to show that the claim is true for all u-polars. □

The reader may notice that a dicritical morphism φ has $m = n$, because $\mathcal{T}$ has a single point, and 6.1 gives $e_O(\mathbf{J}(\varphi)) = 2n - 2$, as is already well known.

7. Pencils of a particular kind. In this section we will prove some technical results about pencils with irreducible cluster of base points. Recall that irreducible clusters are those weighted clusters that are totally ordered and have all excesses zero but for its last point, which has excess one.

Lemma 7.1. *Let $\mathcal{K}$ be an irreducible cluster and q its last point. A germ ζ goes through $\mathcal{K}$ with effective multiplicities equal to the virtual ones if and only if ζ is irreducible, $e_q(\zeta) = 1$ and the point on ζ in the first neighbourhood of q is free.*

Proof. Assume that ζ goes through $\mathcal{K} = (K, \nu)$ with effective multiplicities equal to the virtual ones: then, clearly, $\rho_q = 1$ gives $e_q(\zeta) = \nu_q = 1$. Furthermore, since $\rho_p = 0$ for $p \neq q$, points on ζ not in K cannot be proximate to a $p \in K$, $p \neq q$. Then all points on ζ not in K are infinitely near to q, which is simple on ζ, and the irreducibility of ζ follows. Furthermore, arguing as above, the point in the first neighbourhood of q on ζ cannot be proximate to a point preceding q, hence it is free.

Conversely, ζ being irreducible and the point on ζ in the first neighbourhood of q being free, all points on ζ proximate to a point preceding q belong to $\mathcal{K}$. Of course $e_q(\zeta) = 1 = \nu_q$. Then, for each p preceding q, $\nu_p = \sum_{p'} \nu_{p'}$, because $\rho_p = 0$, and $e_p(\zeta) = \sum_{p'} e_{p'}(\zeta)$ both summations running on the points p' proximate to p in K. Using induction backwards from q, it follows $e_p(\zeta) = \nu_p$ for any $p \in K$, as wanted. □

By using the same arguments the reader may easily prove:

Lemma 7.2. *If $\mathcal{K} = (K, \nu)$ is an irreducible cluster and ξ is a multiple of an irreducible germ, has effective multiplicity $e > 0$ at the last point q of $\mathcal{K}$ and its point in the first neighbourhood of q is free, then ξ goes through $e\mathcal{K} = (K, e\nu)$ with effective multiplicities equal to the virtual ones.*

Now, let $\mathcal{Q} = \{\zeta_\beta : \beta_1 h_1 + \beta_2 h_2 = 0\}$, $\beta = \beta_1/\beta_2 \in \mathbb{C} \cup \{\infty\}$, be a pencil with irreducible cluster of base points. Denote by q the last point of $BP(\mathcal{Q})$ and by E_q the first neighbourhood of q. The dimension and degree of the linear series cut out by $\mathcal{Q}$ on E_q (see [7], section 7.2) are both equal to one, because q is the last base point and has virtual multiplicity one. Thus, for all β, the virtual transform with origin at q of ζ_β , $\check{\zeta}_\beta$, has a single point q_β in E_q. Hence, the virtual transforms $\check{\zeta}_\beta$ describe a pencil of lines at q. Furthermore, by [7], 7.2.9, the map $\zeta_\beta \mapsto q_\beta$ is a projectivity $\mathcal{Q} \longrightarrow E_q$

and so we may take β as the absolute projective coordinate of q_β. In this situation we will say that the germ ζ_β and the point $q_\beta \in E_q$ *correspond* to each other.

Assume that q_β is a free point. Then, since q_β does not belong to any exceptional divisor other than that of blowing up q, ζ_β effectively goes through q_β, and $e_{q_\beta}(\zeta_\beta) = 1$. Let ζ' be a germ of $\mathcal{Q}$ going sharply through $BP(\mathcal{Q})$ and missing q_β. By 7.1, the (only) branch γ of ζ_β that goes through q_β is going through $BP(\mathcal{Q})$ with effective multiplicities equal to the virtual ones. Using the Noether formula,

$$[\gamma.\zeta'] = BP(\mathcal{Q})^2 = [\zeta'.\zeta_\beta]$$

which forces $\zeta_\beta = \gamma$. Thus ζ_β is irreducible and goes sharply through $BP(\mathcal{Q})$. If ζ_β goes sharply through $BP(\mathcal{Q})$ it is clear that ζ_β is irreducible and effectively goes through q_β. The notations being as above, we have proved

LEMMA 7.3. *The point q_β is free if and only if ζ_β goes sharply through $BP(\mathcal{Q})$. In such a case ζ_β is irreducible and effectively goes through q_β.*

Thus, the satellite points q_β correspond to the germs ζ_β that do not go sharply through $BP(\mathcal{Q})$. These germs are usually called the *special germs* of the pencil. In the trivial case in which $BP(\mathcal{Q})$ is just O counted once, there are no satellite points q_β or special germs. Otherwise there is a unique germ in the pencil having higher multiplicity at O: this germ will be called the *first special germ* in the pencil, and its corresponding point the *first satellite* of q. If q is a free point, there are no other satellite points in its first neighbourhood, and therefore no other special germs in the pencil. If q is satellite, there remains another satellite point in its first neighbourhood and so another special germ in $\mathcal{Q}$. We will call them the *second satellite* of q and the *second special germ* in $\mathcal{Q}$.

In fact the above distinction among the first and second satellite points of q makes no sense unless we check that it does not depend on the pencil $\mathcal{Q}$, which we will do next. Assume that q is any satellite point. There is a unique irreducible cluster $\mathcal{K} = (K, \nu)$ with last point q, as its points are evident and its multiplicities are determined by the irreducibility. Let q' be one of the satellite points in the first neighbourhood of q; add it counted once to $\mathcal{K}$ to make a new weighted cluster $\bar{\mathcal{K}}$, and use unloading to get a consistent cluster $\mathcal{K}' = (K', \nu')$ equivalent to $\bar{\mathcal{K}}$. It is easy to check, using an Enriques diagram of $\mathcal{K}$, that $\nu_O = \nu'_O$ for one choice of q', and $\nu_O < \nu'_O$ for the other. Take q' according to the second choice: for any pencil $\mathcal{Q}$ with $BP(\mathcal{Q}) = \mathcal{K}$, the special germ ζ' corresponding to q' goes through $\mathcal{K}'$ and hence has $e_O(\zeta') \geq \nu'_O > \nu_O$: ζ' is thus the first special germ of $\mathcal{Q}$, and so q' is the first satellite of q, regardless of the choice of $\mathcal{Q}$.

Keep the notations as above for the next lemma.

LEMMA 7.4. *If $\mathcal{Q}$ is a pencil with irreducible cluster of base points and γ is a multiple of an irreducible germ, then:*

(1) *If $\zeta_\beta \in \mathcal{Q}$, then $[\zeta_\beta.\gamma] \geq [BP(\mathcal{Q}).\gamma]$*

(2) *Assume that ζ_β is non-special. Then the above inequality is an equality if and only if ζ_β and γ share no point in the first neighbourhood of the last point q of $BP(\mathcal{Q})$.*

(3) *If γ has a free point q_β in the first neighbourhood of q, then the germ ζ_β corresponding to q_β is the one with higher intersection with γ.*

(4) *If the germ ζ_β with higher intersection with γ is non-special, then γ effectively goes through q_β.*

Proof. The first claim is a direct consequence of the Noether virtual formula ([7], 4.1.2). The claim 2 follows by computing the first member by the ordinary Noether formula, as ζ_β has effective multiplicities equal to the virtual ones at the base points and no points outside of $BP(\mathcal{Q})$ other than the point in the first neighbourhood of q and points infinitely near to it. Claims 3 and 4 result from claim 2 applied to both ζ_β and any other non-special germ of $\mathcal{Q}$. ☐

8. The tangent map to φ. Back to considering the analytic map $\varphi : S \longrightarrow T$, the next proposition introduces a rational map $\widehat{\varphi}$ which may be viewed as the first non-degenerate algebraic approximation to φ at O. If φ is smooth at O, then $\widehat{\varphi}$ agrees with the ordinary linear tangent map. Denote by q the top of $\mathcal{T}(\varphi)$ and by E_O and E_q the first neighbourhoods of O and q, respectively, both endowed with their natural structures of one-dimensional projective spaces.

PROPOSITION 8.1. *There is a rational map $\widehat{\varphi} : E_O \longrightarrow E_q$ so that for any pencil $\mathcal{L}$ of lines at O and all but finitely many $\ell \in \mathcal{L}$, the image of the point on ℓ in E_O is the point on $\varphi_*(\ell)$ in E_q.*

Obviously, $\widehat{\varphi}$ is determined by the condition in the claim and is non-constant, by the definition of q. We will call $\widehat{\varphi}$ the *tangent map* to φ at O.

Proof of 8.1. Fix local coordinates x, y at O. For a suitable choice of the parameters α_1, α_2, $\alpha = \alpha_1/\alpha_2$, the germs $\ell_\alpha \in \mathcal{L}$ may be written in parametric form

$$x = \alpha_1 t + t^2 w(t, \alpha_1, \alpha_2), \quad y = \alpha_2 t + t^2 w'(t, \alpha_1, \alpha_2),$$

w, w' analytic. Indeed, using suitable local coordinates x', y', the germs of $\mathcal{L}$ have the parametric form $x' = \alpha_1' t$, $y' = \alpha_2' t$. Then just change to coordinates x, y and make a suitable linear substitution of the homogeneous parameters α_1', α_2'.

Take a pencil of germs at O', $\mathcal{Q} = \{\zeta_\beta : \beta_1 h_1 + \beta_2 h_2 = 0\}$, $\beta = \beta_1/\beta_2 \in \mathbb{C} \cup \{\infty\}$, so that $BP(\mathcal{Q}) = \mathcal{T}_{red}$. Fix α: to get the germ ζ_β which has higher intersection with $\gamma_\alpha = \varphi_*(\ell_\alpha)$, one has just to cancel the initial form, as series in t, of

$$\beta_1 h_1^*(\alpha_1 t + t^2 w, \alpha_2 t + t^2 w') + \beta_2 h_2^*(\alpha_1 t + t^2 w, \alpha_2 t + t^2 w') \tag{4}$$

On the other hand, by iterated use of the virtual Noether formula ([7], 4.1.2), one easily sees that for any fixed β and all but finitely many β'

$$[\zeta_\beta . \mathcal{T}_{red}] = [\zeta_\beta . \zeta_{\beta'}] = \mathcal{T}_{red}^2$$

and so, for all $\zeta_\beta \in \mathcal{Q}$,

$$e_O(\varphi^*(\zeta_\beta)) = [\zeta_\beta . \mathcal{T}] = \frac{1}{r}\mathcal{T}^2.$$

It follows that the initial forms of h_1^* and h_2^* are linearly independent and both have degree $\mathcal{T}^2/r$. Thus, but for the finitely many values of α for which $\widehat{h_1^*}(\alpha_1, \alpha_2) = \widehat{h_2^*}(\alpha_1, \alpha_2) = 0$, the initial coefficient of the series 4 above is

$$\beta_1 \widehat{h_1^*}(\alpha_1, \alpha_2) + \beta_2 \widehat{h_2^*}(\alpha_1, \alpha_2).$$

It will thus be cancelled if and only if

$$\beta = \frac{\beta_1}{\beta_2} = -\frac{\widehat{h_2^*}(\alpha_1, \alpha_2)}{\widehat{h_1^*}(\alpha_1, \alpha_2)}.$$

We take this equality as the definition of $\widehat{\varphi}$, which obviously does not depend on $\mathcal{L}$. Then, but for the points p_α belonging to the divisor $C : \gcd(\widehat{h_1^*}, \widehat{h_2^*}) = 0$, $\widehat{\varphi}(p_\alpha)$ is the point q_β whose corresponding germ ζ_β has higher intersection with γ_α. We already know from the definition of $\mathcal{T}$ that, for all but finitely many α, the germ γ_α has a free point in the first neighbourhood of q. If this is the case, then, by 7.4.3, this point is $\widehat{\varphi}(p_\alpha)$, as desired. □

Remark 8.2. Enriques' definition of infinitely near points *per abstrazione* identifies each proper or infinitely near point to the class of all irreducible germs of curve through it (see [10], IV.2, [17] or [7], 3.4). Under this viewpoint the top q of $\mathcal{T}$ plays the role of the image of O by φ, after which it makes sense to have the tangent map defined between the first neighbourhoods of O and q.

The proof of 8.1 has given further information on $\widehat{\varphi}$ which is worth keeping for future reference. We have in fact seen:

Corollary 8.3 (of the proof of 8.1). *Let $\mathcal{Q} = \{\zeta_\beta : \beta_1 h_1 + \beta_2 h_2 = 0, \beta = \beta_1/\beta_2\}$ be a pencil with $BP(\mathcal{Q}) = \mathcal{T}_{red}$. Take β as the absolute coordinate of the point $q_\beta \in E_q$ corresponding to ζ_β and $\alpha = \alpha_1/\alpha_2$ as the absolute coordinate of the point $p_\alpha \in E_O$ lying on $\ell_\alpha : \alpha_1 y - \alpha_2 x = 0$.*

(1) *The tangent cones to the inverse images of the germs in $\mathcal{Q}$ describe a one dimensional linear series $TC(\mathcal{Q}^*)$, of degree $\mathcal{T}^2/r$, whose variable part is the linear series described by the fibres of $\widehat{\varphi}$.*

(2) *If p_α does not belong to the fixed part of $TC(\mathcal{Q}^*)$, then $\widehat{\varphi}(p_\alpha) = q_\beta$ if and only if ζ_β is the germ having higher intersection with $\varphi_*(\ell_\alpha)$.*

(3) *The equation of $\widehat{\varphi}$ is*

$$\beta = -\frac{\widehat{h_2^*}(\alpha, 1)}{\widehat{h_1^*}(\alpha, 1)}.$$

(4) $\deg \widehat{\varphi} \leq \mathcal{T}^2/r$

Also,

Corollary 8.4. *Assume that p_α does not belong to the fixed part of $TC(\mathcal{Q}^*)$. If either $\gamma_\alpha = \varphi_*(l_\alpha)$ has a free point in the first neighbourhood of q, or $\widehat{\varphi}(p_\alpha)$ is free, then $\widehat{\varphi}(p_\alpha)$ is the point on γ_α in the first neighbourhood of q .*

Proof. Under the first hypothesis, the claim has been proved when proving 8.1. If $\widehat{\varphi}(p_\alpha)$ is free, it suffices to apply 8.3.2 and 7.4.4. □

If φ is non-dicritical, q still denotes the top of $\mathcal{T}$ and q_1 is the first satellite point in the first neighbourhood of q, we write $F_1 = \widehat{\varphi}^*(q_1)$ and call it the *first special fibre* of $\widehat{\varphi}$. If q is satellite and q_2 is the second satellite in the in the first neighbourhood of q, then $F_2 = \widehat{\varphi}^*(q_2)$ is the *second special fibre* of $\widehat{\varphi}$. In the sequel, when mentioning F_2 (resp. F_1) we will implicitly assume that q is satellite (resp. φ is non-dicritical).

9. Tangent cones to inverse images. We will deal with divisors D on one-dimensional projective spaces: $e_p(D)$ will denote the multiplicity of a point p in D and $\mathrm{supp}(D)$ the set of points with non-zero multiplicity in D. We shall write just $p \in D$ for $p \in \mathrm{supp}(D)$. Without further mention, tangent cones to germs with origin at a point p will be identified to their corresponding divisors on the first neighbourhood of p.

Denote by $\pi : T_q \longrightarrow T$ the composition of the blowing-ups giving rise to the top q of $\mathcal{T}$, so that q belongs to T_q as a proper point. For the whole of this section, a bar denotes pull-back by π. We will make repeated use of the following fact without further mention: if ζ is any germ at O', then

$$[\zeta.\mathcal{T}_{red}] = [\zeta.\gamma] = [\bar{\zeta}.\tilde{\gamma}] = e_q(\bar{\zeta}),$$

where γ is a generic germ through $\mathcal{T}_{red}$ and $\tilde{\gamma}$ denotes its strict transform at q.

Tangent cones to inverse images and inverse images of tangent cones are related by the next theorem:

THEOREM 9.1. *There is a divisor A on E_O so that for any germ of curve ζ at O',*

$$\widehat{\varphi}^*(TC(\bar{\zeta})) = TC(\varphi^*(\zeta)) + [\zeta.\mathcal{T}_{red}]A.$$

Proof. Drop from $\mathcal{T}_{red}$ the last point q and its multiplicity to get a new weighted cluster, still consistent, and choose a germ $\tau : h_0 = 0$ going sharply through it and missing q. Notice that, from its own definition, $[\tau.\mathcal{T}_{red}] = \mathcal{T}_{red}^2 - 1$. As above, let $\mathcal{Q} = \{\zeta_\beta : \beta_1 h_1 + \beta_2 h_2 = 0\}$, $\beta = \beta_1/\beta_2 \in \mathbb{C} \cup \{\infty\}$, be a pencil with $BP(\mathcal{Q}) = \mathcal{T}_{red}$. If $\zeta_\beta \in \mathcal{Q}$ is non-special, both ζ_β and τ have the same mutiplicities at the points preceding q and τ misses q. Because of this

$$\bar{\zeta}_\beta = \bar{\tau} + \tilde{\zeta}_\beta, \tag{5}$$

$\tilde{\zeta}_\beta$ being the strict transform of ζ_β at q. It follows that for any $\zeta_\beta \in \mathcal{Q}$, if $\check{\zeta}_\beta$ is its virtual transform at q (relative to the multiplicities of $BP(\mathcal{Q})$),

$$\bar{\zeta}_\beta = \bar{\tau} + \check{\zeta}_\beta,$$

and so the pencil of lines described by the virtual transforms at q of the germs in $\mathcal{Q}$ is $\{\check{\zeta}_\beta : \beta_1 \bar{h}_1/\bar{h}_0 + \beta_2 \bar{h}_2/\bar{h}_0 = 0\}$. If $\tilde{u}, \tilde{v}$ are local coordinates at q, up to a linear substitution on h_1, h_2, we may assume that the initial forms of $\bar{h}_1/\bar{h}_0$ and $\bar{h}_2/\bar{h}_0$ are $\tilde{u}$ and $\tilde{v}$, respectively.

Let ζ have equation h and denote by $P(\tilde{u}, \tilde{v})$ the initial form of $\bar{h}$. Write

$$\delta = \deg P = e_q(\bar{\zeta}) = [\zeta.\mathcal{T}_{red}].$$

If ξ is the germ $\xi : P(\bar{h}_1/\bar{h}_0, \bar{h}_2/\bar{h}_0) - \bar{h} = 0$, then $e_q(\xi) > \delta$. Take $\xi' : P(\bar{h}_1, \bar{h}_2) - \bar{h}_0^\delta \bar{h} = 0$ to get

$$e_q(\xi') > \delta(1 + [\tau.\mathcal{T}_{red}]) = \delta \mathcal{T}_{red}^2.$$

Clearly $\xi' = \bar{\zeta}'$ where $\zeta' : P(h_1, h_2) - h_0^\delta h = 0$ has thus

$$[\zeta'.\mathcal{T}_{red}] = e_q(\xi') > \delta \mathcal{T}_{red}^2$$

or, equivalently,

$$e_O(\varphi^*(\zeta')) = [\zeta'.\mathcal{T}] > \delta r \mathcal{T}_{red}^2.$$

The germ $\varphi^*(\zeta')$ has equation

$$P(h_1^*, h_2^*) - (h_0^*)^\delta h^* = 0$$

where $(h_0^*)^\delta h^*$ has initial form

$$\widehat{(h_0^*)}^\delta \widehat{h^*}$$

of degree

$$\delta e_O(\varphi^*(\tau)) + e_O(\varphi^*(\zeta)) = \delta r(\mathcal{T}_{red}^2 - 1) + \delta r = \frac{\delta}{r}\mathcal{T}^2.$$

On the other hand, by 8.3.1, the initial forms of h_1^* and h_2^* are linearly independent and both have degree $\mathcal{T}^2/r$. It follows that $P(h_1^*, h_2^*)$ has initial form $P(\widehat{h_1^*}, \widehat{h_2^*})$ of degree $\delta\mathcal{T}^2/r$.

All together, the last inequality forces

$$P(\widehat{h_1^*}, \widehat{h_2^*}) = \widehat{(h_0^*)}^\delta \widehat{h^*}.$$

If $d = \gcd(\widehat{h_1^*}, \widehat{h_2^*})$, the reader may easily check, using 8.3.3, that $P(\widehat{h_1^*}/d, \widehat{h_2^*}/d)$ is an equation of $\widehat{\varphi}^*(TC(\bar{\zeta}))$, after which the claim follows by taking $A = A_1 - C$, $A_1 = TC(\varphi^*(\tau))$ and C the divisor of equation d, that is, the fixed part of the pencil of tangent cones to the germs in $\varphi^*(\mathcal{Q})$. □

Of course, the case in which φ is dicritical is far easier and could have been dealt with directly: then, by 4.2, $q = O'$, $T_q = T$, π is the identical map, $\tau = \emptyset$ and $A_1 = 0$. One may take $\mathcal{Q}$ to be the pencil of the lines $\alpha_1 u + \alpha_2 v = 0$ at O', their inverse images are $\alpha_1 f + \alpha_2 g = 0$. The initial forms $\widehat{f}, \widehat{g}$ provide thus the equation of $\widehat{\varphi}$ and it is enough to note that, $\widehat{f}, \widehat{g}$ being linearly independent by hypothesis, $\widehat{h^*} = \widehat{h}(\widehat{f}, \widehat{g})$.

Remark 9.2. It clearly follows from 9.1 that the divisor A is uniquely determined by φ. Furthermore, the equality 5 in the proof of 9.1 proves that the total transform $\bar{\tau}$ is independent of the choice of τ. Therefore, by 9.1, also $A_1 = TC(\varphi^*(\tau))$ is independent of the choice of τ. This proves that the divisor C is uniquely determined by φ and does not depend on the choice of the pencil $\mathcal{Q}$ with $BP(\mathcal{Q}) = \mathcal{T}_{red}$: we shall call C the *indeterminacy divisor* of $\widehat{\varphi}$ (due to 8.3).

COROLLARY 9.3. *Assume that φ is non-dicritical, let B be the tangent cone shared by all but one inverse images of the lines in a pencil at O' and q' the last free point in $\mathcal{T}$. If ζ is any germ at O' missing q', then $TC(\varphi^*(\zeta)) = ([\zeta.\mathcal{T}]/e(\varphi))B$.*

Proof. Let ξ, ζ be germs at O', both missing q'. Let p be the point preceding q' and E_p the exceptional divisor of blowing up p. Since q' is proximate to no point other than p and both germs are missing q', their total transforms at q' are $\bar{\xi}_{q'} = e_p(\bar{\xi}_p)E_p$ and $\bar{\zeta}_{q'} = e_p(\bar{\zeta}_p)E_p$. Furthermore, for any (necessarily irreducible) germ γ going sharply through $\mathcal{T}_{red}$, if $\tilde{\gamma}_{q'}$ denotes its strict transform at q',

$$[\xi.\mathcal{T}_{red}] = [\xi.\gamma] = [\bar{\xi}_{q'}.\tilde{\gamma}_{q'}] = e_p(\bar{\xi}_p)[E_p.\tilde{\gamma}_{q'}]$$

and similarly for ζ. It follows

$$[\zeta.\mathcal{T}_{red}]\bar{\xi}_{q'} = [\xi.\mathcal{T}_{red}]\bar{\zeta}_{q'}$$

and so

$$[\zeta.\mathcal{T}_{red}]\bar{\xi} = [\xi.\mathcal{T}_{red}]\bar{\zeta}.$$

Then the same relation does hold between the tangent cones to $\bar{\xi}$ and $\bar{\zeta}$, after which the claim follows from 9.1 by taking ξ to be any smooth germ going not through the point of $\mathcal{T}$ in the first neighbourhood of O'. □

As the reader may have noticed, the equality of 9.3 may also be written $TC(\varphi^*(\zeta)) = (e_O(\varphi^*(\zeta))/e(\varphi))B$. The divisor B introduced in 9.3 will play an important role in the sequel, we will call it the *fundamental divisor* of φ. For a dicritical φ we take its *fundamental divisor* B as being the fixed part of the pencil of the tangent cones to the inverse images of the lines of a pencil at O'. As it is clear, in both cases B consists of the fundamental points of φ in E_O and the multiplicity of p in B is $\sum_{p'} e_{p'}(\varphi)$, the summation running on all points p' equal or infinitely near to p and proximate to O.

Remark 9.4. If the top of $\mathcal{T}$ is a free point, then $q = q'$ and 9.3 applies to the germ τ in the proof of 9.1 to give

$$A_1 = \frac{r(\mathcal{T}_{red}^2 - 1)}{e(\varphi)} B.$$

10. Computing $\mathcal{T}(\varphi)$ and $\widehat{\varphi}$. In this section we will describe a rational algorithm that, starting from equations of φ, provides germs $\xi_i : P_i = 0$, $i = 0, \dots, k$, and a pencil of germs $\mathcal{Q}$, both at O', so that $\{\check{m}_i = o(\varphi^*(P_i)) = [\xi_i.\mathcal{T}] | i = 0, \dots, k\}$, is the minimal system of generators of $\Gamma(\varphi)$ and $BP(\mathcal{Q}) = \mathcal{T}(\varphi)_{red}$. The reader may note that the characteristic exponents of $\mathcal{T}(\varphi)$ may be computed from the $\check{m}_i$ using the equalities (1) of Section 5. The multiplicity of $\mathcal{T}(\varphi)$ is $r = \gcd(\check{m}_0, \dots, \check{m}_k)$. The height m of $\mathcal{T}(\varphi)$, which is computable from 6.1, may also be obtained from $\mathcal{Q}$ using the equality 2 of Section 5. The pencil $\mathcal{Q}$ directly gives an equation of $\widehat{\varphi}$, by 8.3.3. Regarding $\mathcal{T}(\varphi)$ itself, the pencil $\mathcal{Q}$ provides general enough germs through it, which often is the easiest way of handling a cluster. Anyway, if needed, one can blow up successive points on a variable germ in the pencil to get the points and multiplicities of $\mathcal{T}(\varphi)$.

Notations being as introduced in Section 5, we will make use of the following lemmas, the first of which is an obvious extension of [7], 6.9.2:

LEMMA 10.1. *Given an infinitely near point p, if ζ and ζ' are multiples of irreducible germs going through p and having a free point in its first neighbourhood, then it holds*

$$\frac{e_q(\zeta)}{e_q(\zeta')} = \frac{e_p(\zeta)}{e_p(\zeta')}$$

for all points q preceding p.

LEMMA 10.2. *If a germ ζ is going through a free point in the first neighbourhood of the last satellite point associated to m_i/n, then $e_{O'}(\zeta) \geq n/n_i$. If furthermore $e_{O'}(\zeta) = n/n_i$, then ζ is irreducible and all points on ζ after the last satellite point associated to m_i/n are simple and free.*

Proof. As it is known (see for example [7], 5.7.3), irreducible germs going through a free point q after the satellite points associated to m_i/n need to have characteristic exponents $m_1/n, \dots, m_i/n$, and hence multiplicity non less than n/n_i. If a germ ζ goes through q, so does one of its branches γ and therefore $e_{O'}(\zeta) \geq e_{O'}(\gamma) \geq n/n_i$.

In case of equality, ζ equals γ ad is thus irreducible. Furthermore, ζ has no further characteristic exponent because $\gcd(n/n_i, m_1/n_i, \dots, m_i/n_i) = 1$ and therefore all its points after the satellite points associated to m_i/n are simple and free. $\square$

Use $\langle\ \rangle$ as meaning *semigroup generated by.*

LEMMA 10.3. *If a germ of curve ξ does not go through the last free point $q_i \in \mathcal{T}$ associated to m_i/n, then $[\xi.\mathcal{T}] \in \langle n, \check{m}_1, \dots, \check{m}_{i-1}\rangle$.*

Proof. Write ν_q for the virtual multiplicity of q in $\mathcal{T}$. Let ξ_i be, as in Section 5, germs having multiplicity one at q_i and no satellite points after it. By 10.1, $\nu_q/e_p(\xi_i) = n_{i-1}$ for all q preceding q_i and hence, by the Noether formula, $[\xi.\mathcal{T}] = n_{i-1}[\xi.\xi_i]$ for any germ ξ missing q_i. By taking $\xi = \xi_j$, $j < i$, one sees that the generators of the semigroup of ξ_i are n/n_{i-1} and $[\xi_i.\xi_j] = \check{m}_j/n_{i-1}$, $j < i$, from which the claim follows using again the above equality for $[\xi.\mathcal{T}]$. $\square$

LEMMA 10.4. *Let q ($\neq O'$) be a free point in $\mathcal{T}$, and m_i/n the last characteristic exponent whose associated points precede q, $i = 0$ if there is no such. If $q = q_{i+1}$ is the last free point associated to m_{i+1}/n take $n' = n_i/n_{i+1}$, and $n' = 1$ otherwise. Let $\zeta : h = 0$ be a (necessarily irreducible) germ of multiplicity n/n_i whose last point in $\mathcal{T}$ is q. Assume that $\tau : P = 0$ is a germ going not through q and having $[\mathcal{T}.\tau] = n'[\mathcal{T}.\zeta]$, and consider the pencil $\mathcal{B} = \{\tau_\alpha : \alpha_1 P + \alpha_2 h^{n'} = 0\}$, $\alpha = \alpha_1/\alpha_2 \in \mathbb{C} \cup \{\infty\}$. Then $BP(\mathcal{B})$ is the irreducible cluster whose last point is either the last satellite point associated to m_{i+1}/n if $q = q_{i+1}$ or, otherwise, the point q itself. Furthermore τ is the first special germ of $\mathcal{B}$, and in case $q = q_{i+1}$, $n'\zeta$ is the second special germ.*

The reader may notice that 10.4 claims, in particular, that the virtual multiplicities of the points $p \in BP(\mathcal{B})$ are either ν_p/n_{i+1} if $q = q_{i+1}$, or ν_p/n_i otherwise, ν_p being the virtual multiplicity of p in $\mathcal{T}$.

Proof of 10.4. A bar will denote pull-back through the composition of blowing-ups giving rise to q, while a tilde will denote strict transform with origin at q. Call E the germ at q of the exceptional divisor: q being free, E is smooth. Since τ misses q, for some positive integer μ,

$$\bar{\tau} = \mu E$$

and so there is an equation z of E for which $\bar{P} = z^\mu$. Also

$$\bar{\zeta} = \mu' E + \tilde{\zeta}$$

where $\tilde{\zeta}$ is smooth (by 10.2) and μ' is a positive integer. If γ is a germ going sharply through $\mathcal{T}_{red}$ and sharing with τ or ζ no point besides those in $\mathcal{T}$, γ is irreducible and

$$r[\tau.\gamma] = [\tau.\mathcal{T}] = n'[\zeta.\mathcal{T}] = rn'[\zeta.\gamma],$$

which in turn gives

$$r\mu[E.\tilde{\gamma}] = r[\bar{\tau}.\tilde{\gamma}] = rn'[\bar{\zeta}.\tilde{\gamma}] = rn'(\mu'[E.\tilde{\gamma}] + [\tilde{\zeta}.\tilde{\gamma}]).$$

Now, if p is the point preceding q, its multiplicity on γ is $\nu_p/r = e_p(\gamma) = [E.\tilde{\gamma}]$. On the other hand, ζ and γ share no point infinitely near to q and $\tilde{\zeta}$ is smooth, hence $\nu_q/r = e_q(\tilde{\gamma}) = [\tilde{\zeta}.\tilde{\gamma}]$ and the last displayed equality gives

$$\mu\nu_p = n'\mu'\nu_p + n'\nu_q.$$

Keep this equality in the form

$$\frac{\mu - n'\mu'}{n'} = \frac{\nu_q}{\nu_p}. \tag{6}$$

In particular $\mu - n'\mu' > 0$ and therefore the strict transform of τ_α, $\alpha \neq \infty$, is

$$\tilde{\tau}_\alpha : \alpha_1 z^{\mu - n'\mu'} + \alpha_2 (hz^{-\mu'})^{n'} = 0.$$

Note that $z_1 = z$ and $z_2 = hz^{-\mu'}$ may be taken as local coordinates at q, because, by 10.2, $\tilde{\zeta}$ is smooth and transverse to E. Using these coordinates, the germs $\tilde{\tau}_\alpha$, $\alpha \neq \infty$, have equations

$$\alpha_1 z_1^{\mu - n'\mu'} + \alpha_2 z_2^{n'} = 0.$$

Assume $q = q_{i+1}$. The above equations and the equality 6 assure that, for $\alpha \neq 0, \infty$, the germs $\tilde{\tau}_\alpha$, and hence also the germs τ_α, have a variable free point in the first neighbourhood of the last satellite point associated to m_{i+1}/n. By 10.2, $e_{O'}(\tau_\alpha) \geq n/n_{i+1}$ if $\alpha \neq 0, \infty$. Since $\tau_0 = n'\zeta$ belongs to the pencil $\mathcal{B}$ and has multiplicity n/n_{i+1}, all but at most one of the τ_α, $\alpha \neq 0, \infty$, have $e_{O'}(\tau_\alpha) = n/n_{i+1}$. By 10.2 again, they are irreducible and then 7.1 proves that $BP(\mathcal{B})$ is as claimed. Since $n' > 1$, $\tau_0 = n'\zeta$ is non-reduced and hence special. Clearly, due to its multiplicity, it is the second special germ. Thus τ_∞, which is also special because is missing the point q, needs to be the first special germ.

If $q \neq q_{i+1}$, then $n' = 1$, $\nu_p = \nu_q$ and hence $\mu - n'\mu' = 1$, after which the $\tilde{\tau}_\alpha$, $\alpha \neq \infty$, together with $E : z_1 = 0$, describe a pencil of lines at q. Then it is clear that the germs τ_α, $\alpha \neq \infty$, go through q and share no point infinitely near to it. Arguing as in the former case, it follows that all but at most one of the τ_α, $\alpha \neq \infty$, are irreducible and $BP(\mathcal{B})$ turns out to be the irreducible cluster with last point q. Furthermore, since q is free, $\mathcal{B}$ has a single special germ, which is τ_∞ by the same reason as above. □

Now we are able to describe the algorithm. Interchanging the coordinates at O' if necessary, we assume that the equations of φ are $u = f(x, y), v = g(x, y)$, with $n = e(\varphi) = o(f) \leq o(g)$.

If $o(g) = n$ and $\widehat{f}/\widehat{g}$ is not constant (that is, φ is dicritical) we end by just taking $P_0 = u$ and $\mathcal{Q} = \{\alpha_1 u + \alpha_2 v = 0\}$. Otherwise take $a \in \mathbb{C}$ such that $o(af - g) > n$. Assume, using induction on $i \geq 0$, to have determined equations $P_j \in \mathbb{C}[u, v]$ of germs $\xi_0 : P_0 = 0, \ldots, \xi_i : P_i = 0$ giving the first $i + 1$ elements $n = \check{m}_0, \check{m}_1, \ldots, \check{m}_i$ of the minimal system of generators of $\Gamma(\varphi)$ as $\check{m}_i = [\xi_i . \gamma]$. Take $n_i = \gcd(\check{m}_0, \ldots, \check{m}_i)$. We assume also that $\mathcal{T}$ contains a free point in the first neighbourhood of the last satellite point associated to m_i/n (of O' if $i = 0$) and that we have determined a further germ $\zeta : h = 0$, $h \in \mathbb{C}[u, v]$, that goes through such a free point and has $e_{O'}(\zeta) = n/n_i$. For $i = 0$ we just take $P_0 = u$ and $h = au - v$.

Now we will inductively determine from $\zeta = \zeta_0$ a finite set of irreducible germs $\zeta_j : h_j = 0$, $h_j \in \mathbb{C}[u, v]$, of multiplicity n/n_i, each ζ_j going through all points of $\mathcal{T}$ on ζ_{j-1} and at least one further point of $\mathcal{T}$ after the satellite points associated to m_i/n. By 10.2, all these further points ζ_j is going through belong to the set of consecutive free points of $\mathcal{T}$ that immediately follow the last satellite point associated to m_i/n.

Take $\zeta_0 = \zeta$, $h_0 = h$ and, inductively, assume to have determined h_{j-1} defining an irreducible germ ζ_{j-1} of multiplicity n/n_i going through some free point in $\mathcal{T}$ after the satellite points associated to m_i/n. Since there is a free point in $\mathcal{T}$ after the satellite points associated to m_i/n, by 9.3, all P_s^*, $s = 0, \dots, i$, have homothetical initial forms. As far as the initial form of h_{j-1}^* is homothetical to that of $f = P_0^*$ and $o(h_{j-1}^*)$ belongs to the semigroup $\langle n, \check{m}_1, \dots, \check{m}_i \rangle$, say $o(h_{j-1}^*) = c_0 n + c_1 \check{m}_1 + \cdots + c_i \check{m}_i$, adjust $a \in \mathbb{C}$ to cancel the initial forms in the difference

$$h_{j-1}^* - a \prod_0^i (P_s^*)^{c_s},$$

thus getting

$$\zeta_j : h_j = h_{j-1} - a \prod_0^i P_s^{c_s} = 0$$

so that

$$[\zeta_j . \mathcal{T}] = o(h_j^*) > o(h_{j-1}^*) = [\zeta_{j-1} . \mathcal{T}]. \tag{7}$$

If q is the last free point on ζ_{j-1} in $\mathcal{T}$, it is clear that $\prod_0^i P_s^{c_s} = 0$ does not go through q. Furthermore $q \neq q_{i+1}$ if $i < k$, as otherwise ζ_{j-1} would give rise to a new generator of $\Gamma(\varphi)$, against the hypothesis $o(h_{j-1}^*) \in \langle n, \check{m}_1, \dots, \check{m}_i \rangle$. Then, case $n' = 1$ of Lemma 10.4 applies and proves that ζ_j is irreducible, has multiplicity n/n_i, and goes through all points of $\mathcal{T}$ on ζ_{j-1} with the same multiplicities as ζ_{j-1}. The Noether formula and the inequality 7 show then that ζ_j goes through at least a further point in $\mathcal{T}$.

Now, $\mathcal{T}$ being finite, the above procedure comes to an end, which means that, for some j, either $o(h_j^*) \notin \langle n, \check{m}_1, \dots, \check{m}_i \rangle$, or $\widehat{h_j^*}$ is not homothetical to $\widehat{f}$

Assume that $[\zeta_j . \mathcal{T}] = o(h_j^*) \notin \langle n, \check{m}_1, \dots, \check{m}_i \rangle$. Then $i < k$ and, by 10.3, ζ_j goes through the last free point q_{i+1} of $\mathcal{T}$ associated to m_{i+1}/n. By 9.1, we may take $P_{i+1} = h_j$ and $\xi_{i+1} = \zeta_j$ as the germ giving the next generator of $\Gamma(\varphi)$, which is computed as $\check{m}_{i+1} = o(h_j^*)$ and allows in turn to compute $n_{i+1} = \gcd(n, \check{m}_1, \dots, \check{m}_{i+1})$. Write $n' = n_i/n_{i+1}$. Since, as it is well known ([4] or [7], ex. 5.10), $n' \check{m}_{i+1} \in \langle n, \check{m}_1, \dots, \check{m}_i \rangle$, take as above a suitable monomial Q in the $P_0, \dots, P_i$ in order to have $o((P_{i+1}^*)^{n'}) = n' \check{m}_{i+1} = o(Q^*)$. Again $Q = 0$ does not go through q_{i+1} and therefore Lemma 10.4 applies, now with $n' > 1$, to show that the cluster of base points of the pencil $\mathcal{B} = \{\tau_\alpha : \alpha_1 Q + \alpha_2 P_{i+1}^{n'} = 0\}$ consists of the points in $\mathcal{T}$ up to the last satellite associated to m_{i+1}/n, taken with $1/n_{i+1}$ times the virtual multiplicities they have in $\mathcal{T}$, and that all τ_α, $\alpha \neq 0, \infty$, are going sharply through $BP(\mathcal{B})$. They are thus, in particular, all irreducible and of multiplicity n/n_{i+1}

If the initial forms of P_{i+1}^* and f are homothetical, we take for the parameter α the (necessarily finite and non-zero) value a that cancels the initial forms of P_{i+1}^* and Q^*. This gives rise to a non-special germ $\tau_a \in B$ with higher intersection with $\mathcal{T}$ and that, therefore, goes through the first free point in $\mathcal{T}$ after the satellite points associated to m_{i+1}/n: we take τ_a as the new ζ and repeat the procedure.

Otherwise, by 9.3, q_{i+1} is the last free point of $\mathcal{T}$, $k = i + 1$, and the algorithm ends by taking $\mathcal{Q} = \mathcal{B}$.

There remains the case in which $o(h_j^*) \in \langle n, \check{m}_1, \dots, \check{m}_i \rangle$ and the initial forms of h_j^* and f are not homothetical: then, again by 9.3, the last point of $\mathcal{T}$ on ζ_j is the last free point in $\mathcal{T}$, and no satellite points follow it in $\mathcal{T}$, as otherwise $o(h_j^*)$ would be a new generator of $\Gamma(\varphi)$ against the hypothesis. Thus $i = k$ and as before, h_j

and a suitable monomial Q in the $P_0,\dots,P_k$ so that $o(Q^*)=o(h_j^*)$, may be used to generate a pencil which, again by 10.4, has cluster of base points $\mathcal{T}_{red}$, as wanted.

Summarizing, the procedure is as follows:

Interchanging u,v if necessary, assume that $o(u^*)\le o(v^*)$. Use double induction on i,j and start by taking $P_0=u, h_{0,1}=v$.

Assume to have determined $P_0,\dots,P_i,h_{i,1},\dots h_{i,j}\in\mathcal{O}_{T,O'}$, $i\ge 0$, $j\ge 1$, so that the initial forms $\widehat{P_0^*},\dots,\widehat{P_i^*},\widehat{h_{i,1}^*},\dots\widehat{h_{i,j-1}^*}$ are all homothetical. Write $\check{m}_\ell=o(P_\ell^*)$, $\ell=0,\dots,i$ and $\Gamma_i=\langle\check{m}_0,\dots,\check{m}_i\rangle$.

1 If $o(h_{i,j}^*)\in\Gamma_i$, then determine non negative integers $c_0,\dots,c_i$ so that

$$o(h_{i,j}^*)=\sum_{\ell=0}^{i}c_\ell\check{m}_\ell$$

and take

$$Q_{i,j}=\prod_{\ell=0}^{i}P_\ell^{c_\ell}.$$

1.a If $\widehat{h_{i,j}^*}$ is homothetical to $\widehat{P_0^*}$, then take

$$h_{i,j+1}=h_{i,j}-aQ_{i,j},$$

$a\in\mathbb{C}$ being chosen so that $\widehat{h_{i,j}^*}-a\widehat{Q_{i,j}^*}=0$

1.b If, otherwise, $\widehat{h_{i,j}^*}$ is not homothetical to $\widehat{P_0^*}$, then end by taking $k=i$ and $\mathcal{Q}=\{\alpha_1 h_{i,j}+\alpha_2 Q_{i,j}=0\}$

2 If $o(h_{i,j}^*)\notin\Gamma_i$, then take $P_{i+1}=h_{i,j}$, $\check{m}_{i+1}=o(P_{i+1}^*)$, $\Gamma_{i+1}=\langle\check{m}_0,\dots,\check{m}_{i+1}\rangle$ and $n'=\gcd(\check{m}_0,\dots,\check{m}_i)/\gcd(\check{m}_0,\dots,\check{m}_{i+1})$. Determine non negative integers $c_0,\dots,c_i$ so that

$$n'o(P_{i+1}^*)=\sum_{\ell=0}^{i}c_\ell\check{m}_\ell$$

and take

$$Q_{i+1}=\prod_{\ell=0}^{i}P_\ell^{c_\ell}.$$

2.a If $\widehat{P_{i+1}^*}$ is homothetical to $\widehat{P_0^*}$, then take

$$h_{i+1,1}=P_{i+1}^{n'}-aQ_{i+1},$$

$a\in\mathbb{C}$ being chosen so that $\widehat{P_{i+1}^*}^{n'}-a\widehat{Q_{i+1}^*}=0$.

2.b If, otherwise, $\widehat{P_{i+1}^*}$ is not homothetical to $\widehat{P_0^*}$, then end by taking $k=i+1$ and $\mathcal{P}=\{\alpha_1 P_{i+1}^{n'}+\alpha_2 Q_{i+1}=0\}$.

We have thus proved:

Theorem 10.5. *The above procedure reaches its end after finitely many steps and provides analytic germs of function $P_0,\dots P_k$ at O' and a pencil of germs of curve at*

O', $\mathcal{Q}$, such that $\{o(\varphi^*(P_i))\}_{i=0,\dots,k}$, is the minimal system of generators of $\Gamma(\varphi)$ and $BP(\mathcal{Q}) = \mathcal{T}_{red}$.

Remark 10.6. Assume that φ is non-dicritical, call q and q' the top and the last free point of $\mathcal{T}$, respectively, and define the *top twist* $\tilde{n}$ of $\mathcal{T}$ as being $\tilde{n} = n_{k-1}/r$ if q is satellite and $\tilde{n} = 1$ otherwise. We have seen that for any irreducible germ $\zeta : h = 0$ having multiplicity one at q' and no satellite points after it, one may construct $\tau : P = 0$ that misses q' and has $[\tau.\mathcal{T}] = \tilde{n}[\zeta.\mathcal{T}]$. Then 10.4 applies and the pencil $\mathcal{Q} = \{\alpha_1 h^{\tilde{n}} + \alpha_2 P = 0\}$ has $BP(\mathcal{Q}) = \mathcal{T}_{red}$, first special germ τ and, in case of q being satellite, second special germ $\tilde{n}\zeta$.

Remark 10.7. It follows from 10.6 and 8.3.1 that $TC(\varphi^*(\tau)) = C + F_1$ and, if q is satellite, $\tilde{n}TC(\varphi^*(\zeta)) = C + F_2$. In particular the multiplicities of all non-fundamental points in the second special fibre are multiples of $\tilde{n}$. Furthermore, by 9.3, $C + F_1 = (\mathcal{T}^2/nr)B$.

Next we will work out some examples. In all of them we will denote by x, y both the local coordinates at O and their initial forms, which are taken as homogeneous coordinates on the first neighbourhood of O. An absolute coordinate α on the first neighbourhood of the top q of $\mathcal{T}$ will be taken in such a way that the first satellite point has $\alpha = \infty$ and, in case of q satellite, the second satellite point has $\alpha = 0$. The remaining notations are as used already, in particular φ will be given by $u = f(x, y), v = g(x, y)$ and $\mathcal{Q}$ is a pencil with $BP(\mathcal{Q}) = \mathcal{T}(\varphi)_{red}$.

EXAMPLE 10.8. We will first consider the easiest non-dicritical case, namely that in which in which $f = f_n + \cdots$, $g = g_m + \cdots$, where f_n, g_m are non-homothetical forms of degrees n, m, respectively, $m > n$ and the dots indicate terms of higher order. Take $r = \gcd(n, m)$, $n' = n/r$, $m' = m/r$. Then $e(\varphi) = n$ and $\mathcal{Q}$ may be taken $u^{m'} - \alpha v^{n'} = 0$, after which $\mathcal{T} = \mathcal{T}(\varphi)$ has Puiseux series $\mathcal{S} = \theta u^{m/n}$, and m and r are the height and multiplicity of $\mathcal{T}$. Note that in case $n \mid m$, $\mathcal{T}$ consists of m/n n-fold free points. Otherwise it has satellite top. The fundamental divisor is $B : f_n = 0$ and $\widehat{\varphi}$ is $[x, y] \mapsto g_m^{n'}(x, y)/f_n^{m'}(x, y)$. Then we may get in different situations by suitable choices of f_n, g_m, namely for $f_n = x^2$ and $g_m = y^3$, $\deg \varphi = 6 = nm$, $\mathrm{supp}(B) = \mathrm{supp}(F_1)$ and $\mathrm{supp}(B) \cap \mathrm{supp}(F_2) = \emptyset$, while for $f_n = xy$ and $g_m = xy^2$, $\deg \varphi = 1$ and $\mathrm{supp}(B) = \mathrm{supp}(F_1) \cup \mathrm{supp}(F_2)$. The case $f_n = xy$, $g_m = xy^7$ has the divisor A of Theorem 9.1 non-effective.

EXAMPLE 10.9. Take $f = x^2 - y^3$, $g = x^3 - y^5$. The divisor B is $x^2 = 0$, in particular $n = 2$. The fundamental points are those on the germ $x^2 - y^3 = 0$ up to the third free point after the satellite one. $\mathcal{Q}$ may be taken $u^3 - v^2 - \alpha u^2 v = 0$. Thus $\mathcal{T}$ consists of the points and multiplicities of the germ $u^3 - v^2$ up to the first free point after the satellite one, $m = 4$ thus. $\mathbf{J}(\varphi)$ is $xy^2(10y^2 - 9x) = 0$, its multiplicity being $4 = n + m - 2$ according to 6.1. Note that the jacobian fails to go through the simple fundamental points after the satellite one. $\widehat{\varphi}$ is given by $-3y^3/x^3$, F_1 is $x^3 = 0$.

EXAMPLE 10.10. Taking $f = x^9 + 9y^{11}$, $g = x^{11} + 11x^2y^{11}$ give $n = 9$, $B : x^9 = 0$. A pencil $\mathcal{Q}$ is $v^9 - u^{11} - \alpha u^9 v^2 = 0$. $\mathcal{T}$ has thus the points on $v^9 - u^{11} = 0$ up to the fourth free point after the satellite ones, with the same multiplicities. The Puiseux series of $\mathcal{T}$ is $\mathcal{S} = u^{11/9} + \theta u^{15/9}$ and so $m = 15$. $\mathbf{J}(\varphi)$ is $xy^{21} = 0$. $\widehat{\varphi}$ is given by $-99y^{22}/x^{22}$, the special fibre being $F_1 : x^{22} = 0$. The fundamental points are those on $g = x^9 + 9y^{11} = 0$ up to the 22-th free point after the satellite ones. Note that the jacobian is missing all fundamental satellite points (some are double) and of course all free points after them.

EXAMPLE 10.11. If $f = x^9 + 2y^{11}$, $g = x^{11} + x^2y^{11}$, again $n = 9$ and B is $x^9 = 0$. Now $\mathcal{Q}$ may be taken $v^9 - u^{11} - \alpha u^{10}v = 0$. $\mathcal{T}$ has thus the points on $v^9 - u^{11} = 0$ up to the second free point after the satellite ones, with the same multiplicities. The Puiseux series of $\mathcal{T}$ is $\mathcal{S} = u^{11/9} + \theta u^{13/9}$ and so $m = 13$. $\mathbf{J}(\varphi)$ is $xy^{10}(13x^9 + 4y^{11}) = 0$. $\widehat{\varphi}$ is given by $-13y^{11}/x^{11}$, the special fibre being $F_1 : x^{11} = 0$. It is worth comparing with Example 10.10: germs $\lambda_0 f + \lambda_1 g = 0$ in the present and the former example are equisingular for any $(\lambda_0, \lambda_1) \neq (0,0)$. In spite of this, there are evident differences. In particular the jacobians are quite different, which shows that neither the equisingularity types of the pair of germs $f = 0$ and $g = 0$, nor those of all germs in the pencil $\lambda_0 f + \lambda_1 g = 0$ can provide complete information on the equisingularity type of the jacobian germ $J(f,g) = 0$, as for instance they do not determine its multiplicity or its splitting in branches.

11. On the behaviour of direct images. We will show in this section that the irreducible germs at O whose tangent line does not belong to the fundamental divisor, have their direct images partially modelled after $\mathcal{T}$.

THEOREM 11.1. *Let ξ be an irreducible germ at O and p its first neighbouring point. If p does not belong to the fundamental divisor B or to the second special fibre, then $\varphi_*(\xi)$ goes through the points of $\mathcal{T}(\varphi)$ with effective multiplicities equal to $e_O(\xi)$ times the virtual ones and goes also through $\widehat{\varphi}(p)$, which is free.*

Proof. The dicritical case being obvious, we assume φ non-dicritical. Take the pencil $\mathcal{Q}$ and coordinates in the first neighbourhoods of O and q as in 8.3, $p = p_\alpha$. By the hypothesis and 10.7, $p_\alpha \notin C$ and $\widehat{\varphi}(p_\alpha) = q_\beta$ is free: then, by 8.3.2, ζ_β has higher intersection with $\gamma_\alpha = \varphi_*(\ell_\alpha)$, and so $\varphi^*(\zeta_\beta) \in \varphi^*(\mathcal{Q})$ has higher intersection with ℓ_α. The germs in $\varphi^*(\mathcal{Q})$ have variable tangent cone at O (8.3.1) while ℓ_α and ξ have the same tangent line at O: this means that $\varphi^*(\zeta_\beta)$ also has higher intersection with ξ and so ζ_β has higher intersection with $\varphi_*(\xi)$. Now, the point q_β on ζ_β being free, by 7.4.4, it belongs to $\varphi_*(\xi)$, and so, by 7.2, $\varphi_*(\xi)$ goes through $e_q(\varphi_*(\xi))\mathcal{T}_{red}$ with effective multiplicities equal to the virtual ones. To conclude, using 10.1 and 3.7,

$$e_q(\varphi_*(\xi)) = \frac{r}{e(\varphi)} e_{O'}(\varphi_*(\xi)) = \frac{r}{e(\varphi)}([\xi.BP(\varphi)] + [\xi.\Phi]) = r e_O(\xi),$$

the last equality coming from the fact that, because of the hypothesis $p \notin B$, ξ shares no point other than O with $BP(\varphi)$ or Φ. ◻

Also the case in which the first neighbouring point belongs to the second special fibre but not to B can be dealt with:

PROPOSITION 11.2. *If ξ and p are as in 11.1, still $p \notin B$ but $p \in F_2$, and q' is the last free point in $\mathcal{T}(\varphi)$, then $\varphi_*(\xi)$ goes through the points of $\mathcal{T}(\varphi)$ preceding q' with effective multiplicities equal to $e_O(\xi)$ times the virtual ones and $e_{q'}(\varphi_*(\xi)) > e_O(\xi)\nu_{q'}$, $\nu_{q'}$ the virtual multiplicity of q' in $\mathcal{T}(\varphi)$.*

Proof. Take the pencil $\mathcal{Q}$ as in 10.6. Denote by ν the system of virtual multiplicities of $\mathcal{T}$. The same argument used in the proof of 11.1 proves that the second special germ of $\mathcal{Q}$, $n'\zeta$, has higher intersection with $\varphi_*(\xi)$. By 10.1, $e_b(n'\zeta) = \nu_b/r$ for all points $b \in \mathcal{T}$ preceding q'. Then $\varphi_*(\xi)$ needs to go through q', otherwise

$[\varphi_*(\xi).n'\zeta] = [\varphi_*(\xi).\tau_\alpha]$ for all non-special $\tau_\alpha \in \mathcal{Q}$. Once we know that $\varphi_*(\xi)$ goes through q', using 10.1 and that, by 3.7, $e_{O'}(\varphi_*(\xi)) = ne_O(\xi)$,

$$e_b(\varphi_*(\xi)) = \frac{e_{O'}(\varphi_*(\xi))}{e_{O'}(n'\zeta)} e_b(n'\zeta) = re_O(\xi)e_b(n'\zeta) = e_O(\xi)\nu_b, \tag{8}$$

for all $b \in \mathcal{T}$ preceding q'.

It remains to take care of the multiplicity at q'. Let q'' be the point just before q' and use $\tilde{}$ and $\bar{}$ to denote strict and total transforms at q'. The strict transform of the first special germ τ being empty, we have

$$\bar{\tau} = \mu E, \quad n'\bar{\zeta} = \mu' E + n'\tilde{\zeta}$$

where E is the germ of the exceptional divisor at q' and μ and μ' are non-negative integers. If τ_α is any non-special germ in $\mathcal{Q}$, we have on one hand $[\tau_\alpha.\tau] = [\tau_\alpha.n'\zeta]$, because the three germs involved belong to the same pencil, while, on the other,

$$[r\tau_\alpha.\tau] = r\mu[\tilde{\tau}_\alpha.E] = \mu\nu_{q''}$$

and

$$[r\tau_\alpha.n'\zeta] = r\mu'[\tilde{\tau}_\alpha.E] + r[\tilde{\tau}_\alpha.n'\tilde{\zeta}] = \mu'\nu_{q''} + n'\nu_{q'},$$

the last equality due to the fact that $\tilde{\zeta}$ is smooth and shares with τ_α no point infinitely near to q'. All together,

$$\mu\nu_{q''} = \mu'\nu_{q''} + n'\nu_{q'}. \tag{9}$$

Assume that $e_{q'}(\varphi_*(\xi)) \leq e_O(\xi)\nu_{q'}$. Then

$$e_{q'}(\varphi_*(\xi)) \leq e_O(\xi)\nu_{q'} < e_O(\xi)\nu_{q''} = e_{q''}(\varphi_*(\xi))$$

and therefore $\widetilde{\varphi_*(\xi)}$ is tangent to E. Since ζ is transverse to E, using 9 and 8 above,

$$\begin{aligned}
[\varphi_*(\xi).n'\zeta] &= \mu'[\widetilde{\varphi_*(\xi)}.E] + [\widetilde{\varphi_*(\xi)}.n'\tilde{\zeta}] \\
&\leq \mu' e_O(\xi)\nu_{q''} + n' e_O(\xi)\nu_{q'} \\
&= \mu e_O(\xi)\nu_{q''} \\
&= \mu e_{q''}(\varphi_*(\xi)) \\
&= \mu[\widetilde{\varphi_*(\xi)}.E] \\
&= [\varphi_*(\xi).\tau]
\end{aligned}$$

contradicting the fact, already proved, that the second special germ has higher intersection with $\varphi_*(\xi)$. □

12. The tangent cone to the jacobian. In this section we will see how the tangent cone to the jacobian germ $\mathbf{J}(\varphi)$, still viewed as a divisor on E_O, is related to the multiple points of the fibres of $\widehat{\varphi}$. Again denote by q the top of $\mathcal{T}$ and by B the fundamental divisor. As above, F_1, F_2 denote the first and second special fibres of $\widehat{\varphi}$. We also write $F_p = \widehat{\varphi}^*(\widehat{\varphi}(p))$ the fibre of any $p \in E_O$. The next lemma is elementary and well known, see [7], 7.1.2 or also [16], XII,2.2 for a more general statement. As usual, the ramification divisor of a rational map between projective lines consists of the e-fold points of its fibres taken $(e-1)$-fold, $e \geq 1$.

LEMMA 12.1. *If $\alpha_1 G_1(X_0, X_1) + \alpha_2 G_2(X_0, X_1) = 0$, G_1, G_2 homogeneous of the same degree d, defines a one-dimensional linear series on $\mathbb{P}^1$, then $J(G_1, G_2)$ has degree $2d-2$ and the group $J(G_1, G_2) = 0$ (the jacobian group of the series) is composed of twice the fixed group of the series,* $\gcd(G_1, G_2) = 0$, *plus the ramification divisor of G_1/G_2.*

Recall that the top twist of $\mathcal{T}(\varphi)$ was defied in 10.6 as $\tilde{n} = n_{k-1}/n_k$ if the top of $\mathcal{T}(\varphi)$ is satellite and $\tilde{n} = 1$ otherwise.

THEOREM 12.2. *The multiplicities of the principal tangents to the jacobian germ of φ are as follows*

$$e_p(TC(\mathbf{J})) = \frac{n+m}{n} e_p(B) - 1 + \begin{cases} -\frac{1}{\tilde{n}} e_p(F_1) & \text{if } p \in F_1 \\ \frac{1}{\tilde{n}} e_p(F_2) & \text{if } p \in F_2 \\ e_p(F_p) & \text{otherwise,} \end{cases}$$

where $n = e(\varphi)$, m is the height of $\mathcal{T}(\varphi)$ and $\tilde{n}$ is the top twist of $\mathcal{T}(\varphi)$.

Proof. If φ is dicritical, then $TC(\mathbf{J})$ has equation $J(\widehat{f}, \widehat{g}) = 0$ and the claim follows applying 12.1 to the series of the tangent cones to the inverse images of the elements of a pencil of lines at O'. Thus we assume φ non-dicritical from now on.

Let q' be the last free point in $\mathcal{T}$. As allowed by 10.6, we take an irreducible germ $\zeta : h = 0$ with multiplicity one at q' and no satellite points after it, as well as a second germ $\tau : P = 0$ going not through q' and having $[\tau.\mathcal{T}] = \tilde{n}[\zeta.\mathcal{T}]$. The pencil $\mathcal{Q} = \{\tau_\alpha : \alpha_1 P + \alpha_2 h^{\tilde{n}} = 0\}$, $\alpha = \alpha_1/\alpha_2 \in \mathbb{C} \cup \{\infty\}$, has thus $BP(\mathcal{Q}) = \mathcal{T}_{red}$, first special germ $\tau = \tau_\infty$ and, if $q' = q_k$, second special germ $\tilde{n}\zeta = \tau_0$.

Let R be the ramification divisor of $\widehat{\varphi}$ and, as before, C the fixed part of the tangent cones to the inverse images of the germs in $\mathcal{Q}$. By 8.3.1 and 12.1, $J(\widehat{P^*}, (\widehat{h^*})^{\tilde{n}}) = 0$ defines the divisor $R + 2C$. Since $P = 0$ does not go through q', by 9.3, $\widehat{P^*} = \widehat{f}^{\mathcal{T}^2/rn}$ up to a non-zero constant factor. Thus the divisor $R + 2C$ is also defined by

$$(\widehat{h^*})^{\tilde{n}-1} \widehat{f}^{\frac{\mathcal{T}^2}{rn}-1} J(\widehat{f}, (\widehat{h^*}))$$

which, by 6.3, equals

$$(\widehat{h^*})^{\tilde{n}-1} \widehat{f}^{\frac{\mathcal{T}^2}{rn}-1} \widehat{J(f, h^*)} = (\widehat{h^*})^{\tilde{n}-1} \widehat{f}^{\frac{\mathcal{T}^2}{rn}-1} \widehat{\left(\frac{\partial h}{\partial v}\right)^*} \widehat{J(f, g)}.$$

Now let us detail the divisors defined by the factors of this form:

- By 8.3.1, $(\widehat{h^*})^{\tilde{n}-1}$ defines $(1 - 1/\tilde{n})(C + F)$, F a fibre of $\widehat{\varphi}$, $F = F_2$, the second special fibre, if the top q of $\mathcal{T}$ is satellite.
- By definition $\widehat{f}$ defines B.
- Take $\zeta' : \partial h/\partial v{=}0$. By our choice of the coordinates u, v, the first neighbouring point of O' in $\mathcal{T}$ lies on the u-axis. Since ζ goes through it, ζ cannot be tangent to the v-axis. Therefore $e_{O'}(\zeta') = e_{O'}(\zeta) - 1$ which prevents ζ' from going through q' (by 10.2) and 9.3 applies: the initial form of $(\partial h/\partial v)^*$ being of degree $\frac{\mathcal{T}^2}{r\tilde{n}} - m$ by 6.5, it defines $(\frac{\mathcal{T}^2}{rn\tilde{n}} - \frac{m}{n})B$.
- $\widehat{J(f, g)}$ obviously defines $TC(\mathbf{J})$.

All together and using 10.7 we get

$$\begin{aligned} R + 2C &= \frac{\tilde{n}-1}{\tilde{n}}(C+F) + (\frac{\mathcal{T}^2}{rn} - 1)B + \frac{1}{n}(\frac{\mathcal{T}^2}{r\tilde{n}} - m)B + TC(\mathbf{J}) \\ &= \frac{\tilde{n}-1}{\tilde{n}}F + \frac{\tilde{n}+1}{\tilde{n}}F_1 + 2C - \frac{m+n}{n}B + TC(\mathbf{J}), \end{aligned}$$

from which the claim directly follows. ☐

Example 12.3. Back to the case of Example 10.8 and using the same notations, the fundamental divisor is $B : f_n = 0$ and we take $B' : g_m = 0$. Then 10.3 easily gives

$$e_p(\mathbf{J}) = \begin{cases} e_p(B) + e_p(B') - 1 \text{ if } p \text{ has non-zero multiplicity in } mB - nB' \\ e_p(B) + e_p(B') + e_p(F_p) - 1 \text{ otherwise.} \end{cases}$$

The reader may check this case by direct computation using 12.1 and the identity $J(f_n^{m'}, g_m^{n'}) = n'm' f_n^{m'-1} g_m^{n'-1} J(f_n, g_m)$.

Multiple points of either especial fibre need not belong to $TC(\mathbf{J})$, as shown below:

Example 12.4. Take $f = x^{54}y^{44}(9x+y)$, $g = x^{66}y^{54}(11x+y)$, which, incidentally, is the composition of the morphism of Example 10.10 and the blowing-ups giving rise to its last satellite fundamental point. Then $B : x^{54}y^{44}(9x+y) = 0$ and after a single step of the algorithm one gets $\mathcal{Q} : \alpha u^{11} - v^9 = 0$ and hence $n = 99$, $m = 121$, $r = 11$, $\tilde{n} = 9$. The tangent map is given by $\alpha = y^2(11x+y)^9/(9x+y)^{11}$ and, by Theorem 12.2, the multiplicity in $TC(\mathbf{J})$ of the only point of the first special fibre $F_1 : (9x+y)^{11} = 0$ is $20/9 - 1 - 11/9 = 0$. For $p : 11x + y = 0$ in the second special fibre, $e_p TC(\mathbf{J}) = -1 + 9/9 = 0$. Of course, a direct computation of the jacobian confirms these facts, as it gives $\mathbf{J} : x^{121}y^{97} = 0$.

Remark 12.5. (on the splitting of the jacobian) If ζ is any germ at O and p belongs to the first neighbourhood of O, let us write ζ^p the component of ζ composed of all branches of ζ having principal tangent p. The germs ζ^p are often called the *tangential components* of ζ and the germ ζ splits in its tangential components, $\zeta = \sum_{p \in TC(\zeta)} \zeta^p$. For any p, $e_O(\zeta^p) = e_p(TC(\zeta))$ and thus the multiplicities of the tangential components of $\mathbf{J}$, $e_O(\mathbf{J}^p)$, and in particular the first neighbouring points $\mathbf{J}$ is going through, are given by Theorem 12.2. The minus sign that affects the multiplicities of the points of the first special fibre F_1 in the formula of Theorem 12.2 is worth a comment. Unlike the points in other fibres, for $p \in F_1$ the multiplicity $e_p(TC(\mathbf{J}))$ is a decreasing function of $e_p(F_p)$. This explains a number of cases in which a component of $\mathbf{J}$ with higher multiplicity at O goes away from the infinitely near base points of the pencil $f + \alpha g = 0$ and seems to have little relation with the singularities of the (general or special) germs in it. Next are two examples:

Example 12.6. In the case of Example 10.10, B consists of the single 9-fold point $x^9 = 0$. There are two fibres of $\widehat{\varphi}$ having multiple points, each consisting of a 22-fold point. The only point of the fibre $y^{22} = 0$ gives rise to a component of $\mathbf{J}$ with multiplicity 21 at O and principal tangent $y = 0$ and that, therefore, goes not through any further base point (this component being in fact $y^{21} = 0$). By contrast, the point $p : x = 0$ of the first especial fibre $x^{22} = 0$ has multiplicity $24 - 1 - 22 = 1$ in $TC(\mathbf{J})$, and so the component $\mathbf{J}^p$ of $\mathbf{J}$ is smooth (equal to x=0 in fact). Incidentally, note that in this case $e_p(TC(\mathbf{J})) < 2 = e_p(\varphi)$.

EXAMPLE 12.7. Take φ with equations $f = y^2 + 2x^4$, $g = y^3 + 3x^4y$. The fundamental points of φ are the origin O, its first neighbouring point p on the x-axis, both double, and two pairs of consecutive free simple points following p. Direct computation gives $\mathbf{J} : x^7 = 0$ and so $\mathbf{J}$ is missing all infinitely near fundamental points, from which the first neighbouring one is double. The algorithm of Section 9 easily shows that $\mathcal{T}$ consists of the points and multiplicities of the germ $u^3 - v^2 = 0$ up to the sixth neighbourhood, hence $n = 2$ and $m = 7$. The tangent map $\hat{\varphi}$ is $[x, y] \mapsto x^8/y^8$. The first exceptional fibre is $8p$ and Theorem 12.2 gives

$$e_p(TC\mathbf{J}) = \frac{2+7}{2} \cdot 2 - 1 - \frac{8}{1} = 0,$$

thus confirming that $\mathbf{J}$ is missing p. Since there are no other points in B, the only point on $\mathbf{J}$ in the first neighbourhood of O comes from to the only non-special fibre with multiple points, namely $x^8 = 0$.

Remark 12.8. The Example 12.7 shows that the jacobian germ may miss multiple free fundamental points. A jacobian germ missing both multiple satellite and simple free fundamental points already appeared in Example 10.10.

The next three corollaries are direct consequences of Theorem 12.2:

COROLLARY 12.9. *A point p belongs to F_1 if and only if*

$$e_p(TC(\mathbf{J})) < \frac{n+m}{n} e_p(B) - 1.$$

COROLLARY 12.10. *If p belongs to a special fibre, then $\tilde{n}e_p(B)$ is a multiple of $n/\gcd(n, m)$. Otherwise $e_p(B)$ is a multiple of $n/\gcd(n, m)$.*

Also,

COROLLARY 12.11. *If φ is non-dicritical and B has a single point, $B = np$, then the number of tangents to $\mathbf{J}$ going not through p, counted according to multiplicities, is*

$$\frac{1}{\tilde{n}} \deg \widehat{\varphi} - 1.$$

In particular $\deg \widehat{\varphi}$ is, in this case, a multiple of $\tilde{n}$.

Proof. By 10.7, $F_1 = (\deg \widehat{\varphi})p$ and so $e_p(TC(\mathbf{J})) = n + m - 1 - \deg \widehat{\varphi}/\tilde{n}$. Since $n + m - 2 = e_O(\mathbf{J}) = \deg TC(\mathbf{J})$ by 6.1, the claim follows. □

COROLLARY 12.12. *If φ is non-dicritical,*

$$\deg \widehat{\varphi} \leq \tilde{n}(n + m - \#(B)),$$

$\#(B)$ being the cardinal of $\operatorname{supp} B$, and this bound is sharp.

Proof. From 12.2,

$$e_p(F_1) \leq \tilde{n}\left(\frac{n+m}{n} e_p(B) - 1\right)$$

for all $p \in B$. Since $\operatorname{supp} F_1 \subset \operatorname{supp} B$ by 10.7, it is enough to add up for $p \in B$. The equality holds for the morphism of Example 12.7. □

The reader may note that for a dicritical φ the obvious bound $\deg \widehat{\varphi} \leq n$ is better than the bound of 12.12, as for such a φ it holds $\tilde{n}(n + m - \#(B)) = 2n - \#(B) > n$.

13. Further trunks. So far we have just dealt with the point O and its first infinitesimal neighbourhood on the source surface S. Now we will extend our considerations to the points infinitely near to O on S. To this end, if p is equal or infinitely near to O, we will denote by $\pi_p : S_p \longrightarrow S$ the composition of the blowing-ups giving rise to p, S_p being thus the surface p is lying on as a proper point, and by φ_p the composite morphism $\varphi_p = \varphi \circ \pi_p$.

Let $\mathcal{N}$ be any pencil of lines at O' and denote by $\mathcal{P}_p$ the family of the inverse images $\varphi_p^*(\ell)$, $\ell \in \mathcal{N}$, $\mathcal{P}_O = \mathcal{P}$ using the notations of Section 3. Clearly the fixed part of $\mathcal{P}_p$ is composed of the total transform of the contracted germ by φ plus the exceptional part shared by all but finitely many total transforms (by π_p) of the germs in the variable part $\mathcal{P}'$ of $\mathcal{P}$. Thus, for $p \neq O$, the contracted germ by φ_p, Φ_p, always contains the germ of the exceptional divisor of π_p at p, and is in particular non-empty. $BP(\varphi_p)$ consists of the points in $BP(\varphi)$ equal or infinitely near to p, with the same virtual multiplicities.

If φ_p is dicritical, we will say that φ is *dicritical at* p, and also that p is a *dicritical point of* φ. Clearly, φ is dicritical at p if and only if p is a dicritical base point of $\mathcal{P}$, that is, p is a base point of $\mathcal{P}$ and the virtual (or total) transforms at p of the germs in $\mathcal{P}$ have a variable tangent. In particular a dicritical point of φ always belongs to $BP(\varphi)$.

The fundamental divisor of φ_p will be denoted by B_p, and still viewed as a divisor on the first neighbourhood E_p of p: its points are the points in the first neighbourhood of p that are either fundamental points of φ or satellite points.

The multiplicity of φ_p, $e(\varphi_p)$, will be also denoted by n_p if no reference to φ is required. By its own definition, $e(\varphi_p)$ is the minimal value taken by the divisorial valuation associated to p on the germs of $\mathcal{P}$. The multiplicities $e(\varphi_p)$ and $e_p(\varphi)$, already defined in Section 3, are related, namely:

PROPOSITION 13.1. *If p is free, proximate to a single point p', then*

$$e(\varphi_p) = e_p(\varphi) + e(\varphi_{p'}).$$

If p is satellite, proximate to points p', p'', then

$$e(\varphi_p) = e_p(\varphi) + e(\varphi_{p'}) + e(\varphi_{p''}).$$

Proof. For all but finitely many choices of $\xi \in \mathcal{P}$, $e_p(\varphi) = e_p(\xi)$, $e(\varphi_p) = e_p(\pi_p^*(\xi))$, $e(\varphi_{p'}) = e_{p'}(\pi_{p'}^*(\xi))$ and, for p satellite, $e(\varphi_{p''}) = e_{p''}(\pi_{p''}^*(\xi))$, after which the claim follows from the well known relations between multiplicities of total and strict transforms of a germ of curve. □

Let us quote an obvious consequence of 13.1 for future reference:

COROLLARY 13.2. *For all p, $e(\varphi_p) > 0$. If p lies in the first neighbourhood of p', then $e(\varphi_p) \geq e(\varphi_{p'})$ and the equality holds if and only if p is free and non-fundamental.*

The trunk $\mathcal{T}(\varphi_p)$ of φ_p will be called the *p-trunk* of φ and denoted by $\mathcal{T}_p(\varphi)$ or just $\mathcal{T}_p$. If $\mathcal{I}_p$ denotes the irreducible cluster with origin at O whose last point is p and $\mathcal{L}_p$ is any pencil with $BP(\mathcal{L}_p) = \mathcal{I}_p$, then $\mathcal{T}_p$ consists of the points and multiplicities shared by all but finitely many direct images of the germs in $\mathcal{L}_p$. Still n_p is the virtual multiplicity of O' in $\mathcal{T}_p$ and φ is dicritical at p if and only if $\mathcal{T}_p = \{O', n_p\}$.

The trunks of φ are thus its main trunk $\mathcal{T}_O(\varphi) = \mathcal{T}(\varphi)$ and all its p-trunks for p infinitely near to O. We will see in the sequel (14.7, 15.4, 16.2) that many of these trunks can be computed from those corresponding to preceding points, and hence do not contain any new information. Before showing some relationship between numerical characters of different trunks, let us just quote a direct consequence of 4.1 that shows that the multiplicities of the inverse images, and so in particular their topological types, are determined by the trunks of φ:

PROPOSITION 13.3. *For any germ ξ at O' and any p equal or infinitely near to O,*

$$e_p(\varphi^*(\xi)) = [\xi.\mathcal{T}_p(\varphi)] - \sum_{p \text{ prox. to } p'} [\xi.\mathcal{T}_{p'}(\varphi)].$$

For any p infinitely near to O, write $m_1^p/n_p, \dots, m_{k_p}^p/n_p$, the characteristic exponents of $\mathcal{T}_p(\varphi)$ and $\check{m}_1^p, \dots, \check{m}_k^p$ the corresponding generators of the semigroup $\Gamma(\varphi_p) = \Gamma_p(\varphi)$, as described in Section 5. Denote by m_p the height of $\mathcal{T}_p$ and put $n_i^p = \gcd(n_p, m_1^p, \dots, m_i^p)$, so that $n_{k_p}^p = r_p$ is the multiplicity of $\mathcal{T}_p$. Then we have:

PROPOSITION 13.4. *If p is infinitely near to O and belongs to the first neighbourhood of p':*

(1) *For $i \leq \min(k_{p'}, k_p)$,*

$$\frac{\check{m}_i^{p'} n_{i-1}^{p'}}{n_{p'}} \leq \frac{\check{m}_i^p n_{i-1}^p}{n_p},$$

this inequality being strict if $p \in B_{p'}$ and $m_i^{p'} < m_{p'}$.

(2) *If $k_{p'} < k_p$,*

$$\frac{\mathcal{T}_{p'}^2}{n_{p'}} \leq \frac{\check{m}_{k_{p'}+1}^p n_{k_{p'}}^p}{n_p},$$

and the inequality is strict if p belongs to no special fibre of $\widehat{\varphi}_{p'}$.

(3) *In any case,*

$$\frac{\mathcal{T}_{p'}^2}{n_{p'}} \leq \frac{\mathcal{T}_p^2}{n_p},$$

and the inequality is strict if p does not belong to the first special fibre of $\widehat{\varphi}_{p'}$.

Proof. It is of course not restrictive to assume $p' = O$. In order to prove part (1), choose an irreducible germ γ at O' with $i-1$ characteristic exponents and going through the last free point of $\mathcal{T}_{p'}$ associated to its i-th characteristic exponent. By 5.3,

$$\frac{e_O(\varphi^*(\gamma))}{e_{O'}(\gamma)} = \frac{\check{m}_i n_{i-1}}{n}$$

and

$$\frac{e_p(\varphi_p^*(\gamma))}{e_{O'}(\gamma)} \leq \frac{\check{m}_i^p n_{i-1}^p}{n_p}.$$

Then the well known properties of the blowing-up give $e_O(\varphi^*(\gamma)) \leq e_p(\varphi_p^*(\gamma))$ and hence the wanted inequality.

If $p \in B$ and $m_i < m$, then, by 9.3, $p \in TC(\varphi^*(\gamma))$ and the strict transform of $\varphi^*(\gamma)$ goes through p. This gives $e_O(\varphi^*(\gamma)) < e_p(\varphi_p^*(\gamma))$ and the inequality is strict.

For claim (2), fix a pencil $\mathcal{Q}$ with $BP(\mathcal{Q}) = \mathcal{T}_{red}$ and choose γ to be any non-special germ of $\mathcal{Q}$. Then $e_{O'}(\gamma) = n/r$, $e_O(\varphi^*(\gamma)) = \mathcal{T}^2/r$. Furthermore, γ has k characteristic exponents, so that, by 5.3,

$$\frac{e_p(\varphi_p^*(\gamma))}{e_{O'}(\gamma)} \leq \frac{\check{m}_{k+1}^p n_k^p}{n_p}$$

and the inequality follows as in the proof of claim (1). If, in addition, p belongs to no special fibre, γ may be chosen in such a way that p belongs to $TC(\varphi^*(\gamma))$ (by 7.4 and 8.3) and so $\varphi^*(\gamma)$ effectively goes through p. As above, this proves the inequality to be strict.

Lastly, choose any $\zeta \in \mathcal{Q}$ different from the first special germ. Then still $e_{O'}(\zeta) = n/r$ (just because ζ is not the first special germ, see Section 7) and $e_O(\varphi^*(\zeta)) = \mathcal{T}^2/r$. Then

$$\mathcal{T}^2/r = [\zeta.\mathcal{T}] = e_O(\varphi^*(\zeta)) \leq e_p(\varphi_p^*(\zeta)),$$

The inequality being strict if ζ goes through p. Claim (3) follows after dividing by $e_{O'}(\zeta) = n/r$ and using the second half of 5.3. □

The reader may have noticed that both sides of the inequalities of claim (1) are polar invariants (of germs going sharply through $(\mathcal{T}_{p'})_{red}$ or $(\mathcal{T}_p)_{red}$, respectively), see [7], 6.8.3. It is also worth noticing that all these inequalities may be equalities, as it will easily turn out after 14.7 below. The easy example $f = x$, $g = y^2$ gives equality in (3). A further remark about 13.4 is:

Remark 13.5. If $\mathcal{T}_{p'}$ has satellite top and $k_{p'} \leq k_p$, then, by 5.1,

$$\frac{\check{m}_{k_{p'}}^{p'} n_{k_{p'}-1}^{p'}}{n_{p'}} = \frac{\mathcal{T}_{p'}^2}{n_{p'}}.$$

Then, due to a well known property of the $\check{m}_i$ (see [4] or [7], ex. 5.11), (1) for $i = k_{p'}$ is stronger than (2).

Corollary 13.6. *If φ is dicritical at a point p in the first neighbourhood of p', then $\mathcal{T}_{p'}^2 \leq 2n_{p'}^2$ if p is free, and $\mathcal{T}_{p'}^2 < 3n_{p'}^2$ if p is satellite.*

Proof. If φ is dicritical at p, $\mathcal{T}_{p'}^2/n_{p'} \leq \mathcal{T}_p^2/n_p = n_p$. On the other hand $e_p(\varphi) \leq e_{p'}(\varphi)$ and so, by 13.1 and 13.2, $n_p \leq e_{p'}(\varphi) + n_{p'} \leq 2n_{p'}$ if p is free, and similarly, $n_p < 3n_{p'}$ if p is satellite, hence the claim. □

To close this section, this corollary follows from the proof of 13.4:

Corollary 13.7. *For any germ ζ but the first special one in a pencil $\mathcal{Q}$ with $BP(\mathcal{Q}) = (\mathcal{T}_{p'})_{red}$, in particular for any germ ζ going sharply through $(\mathcal{T}_{p'})_{red}$, $[\zeta.\mathcal{T}_{p'}] \leq [\zeta.\mathcal{T}_p]$.*

Proof. Just use the equalities $[\zeta.\mathcal{T}_{p'}] = \mathcal{T}_{p'}^2/r$ (definition of $T_{p'}^2$) and $e_p(\varphi_p^*(\zeta)) = [\zeta.\mathcal{T}_p]$ (Theorem 4.1) together with the inequality in the proof of part (3) of 13.4. □

14. Multiplicities of the jacobian or how the trunks grow up . The multiplicities of the jacobian $\mathbf{J}(\varphi)$ at the infinitely near points are determined by the heights of the trunks of φ, to be precise:

Theorem 14.1. *The multiplicities of the jacobian of φ are:*

$$e_p(\mathbf{J}(\varphi)) = \begin{cases} m+n-2 \textit{ if } p=O, \\ m_p+n_p-m_{p'}-n_{p'}-1 \textit{ if } p \textit{ is free, proximate to } p', \\ m_p+n_p-m_{p'}-n_{p'}-m_{p''}-n_{p''} \textit{ if } p \textit{ is proximate to } p' \textit{ and } p''. \end{cases}$$

Theorem 14.1 is a direct consequence of Theorem 6.1 and Lemma 14.2 below. We use a tilde to denote strict transform with origin at p, while for any q, $q<p$, $\mathbf{F}_q$ denotes the germ at p strict transform of the exceptional divisor of blowing up q, or just the germ of the exceptional divisor if $q=p'$.

Lemma 14.2. *If p lies in the first neighbourhood of p' and is free, then*

$$\mathbf{J}(\varphi_p) = \widetilde{\mathbf{J}(\varphi)} + (m_{p'}+n_{p'}-1)\mathbf{F}_{p'}$$

while if p is satellite, proximate also to p'', $p'>p''$,

$$\mathbf{J}(\varphi_p) = \widetilde{\mathbf{J}(\varphi)} + (m_{p'}+n_{p'}-1)\mathbf{F}_{p'} + (m_{p''}+n_{p''}-1)\mathbf{F}_{p''}.$$

Proof of 14.2. Denote by π the blowing-up of p': $\mathbf{J}(\pi)=\mathbf{F}_{p'}$ and therefore, from $\varphi_p=\varphi_{p'}\circ\pi$ we get

$$\mathbf{J}(\varphi_p) = \pi^*(\mathbf{J}(\varphi_{p'})) + \mathbf{F}_{p'}$$

If $p'=O$, then, by 6.1, $\pi^*(\mathbf{J}(\varphi_{p'})) = \widetilde{\mathbf{J}(\varphi_{p'})} + (m_{p'}+n_{p'}-2)\mathbf{F}_{p'}$, which gives the claim. Otherwise, we use induction on the order of the neighbourhood p' belongs to.

If p is free, $\mathbf{F}_q=\emptyset$ if $q\neq p'$. Therefore, the induction hypothesis applied to $\mathbf{J}(\varphi_{p'})$ and 6.1 still give $\pi^*(\mathbf{J}(\varphi_{p'})) = \widetilde{\mathbf{J}(\varphi_{p'})} + (m_{p'}+n_{p'}-2)\mathbf{F}_{p'}$ and, as above, the claim.

If p is satellite, then $\mathbf{F}_{p''}\neq\emptyset$ while $\mathbf{F}_q=\emptyset$ if $q\neq p', p''$. The point p' being also proximate to p'', in this case we get $\pi^*(\mathbf{J}(\varphi_{p'})) = \widetilde{\mathbf{J}(\varphi_{p'})} + (m_{p'}+n_{p'}-2)\mathbf{F}_{p'} + (m_{p''}+n_{p''}-1)\mathbf{F}_{p''}$ and again the claim. ☐

A direct consequence of 14.1 is:

Corollary 14.3. *The multiplicities of $\mathbf{J}(\varphi)$ at O and all points infinitely near to O, and hence the germ $\mathbf{J}(\varphi)$ itself, are determined by the trunks of φ.*

Using 13.1, one may rewrite the last two equalities of 14.1 as $e_p(\mathbf{J}) = m_p - m_{p'} + e_p(\varphi) - 1$ for p free, and $e_p(\mathbf{J}) = m_p - m_{p'} - m_{p''} + e_p(\varphi)$ for p satellite. In particular:

Corollary 14.4. *If p is not a fundamental point of φ, $m_p = m_{p'} + e_p(\mathbf{J}) + 1$ if p is free, and $m_p = m_{p'} + m_{p''} + e_p(\mathbf{J})$ if p is satellite. In any case, $m_p > m_{p'}$.*

The inequality $m_p > m_{p'}$ need not hold if p is a fundamental point. One may even get $m_p < m_{p'}$, as in the next example:

Example 14.5. Taking $f = x^5 + 5x^2y^7$, $g = x^3 + 3y^7$, $p' = O$ and p its first neighbouring point on $x=0$, gives $n=3$ and jacobian $\mathbf{J} : xy^{13} = 0$. It follows $m=13$. Clearly $e_p(\mathbf{J}) = 1$ and $n_p = 6$. Then 14.1 gives $m_p = 12 < m$.

Proposition 14.6. *Let p be a free non-fundamental point infinitely near to O and lying in the first neighbourhood of a point p'. Denote by q the last free point of $\mathcal{T}_{p'}$. We have:*

(a) *If either the top of $\mathcal{T}_{p'}$ is not a satellite point or p does not belong to the second special fibre of $\widehat{\varphi}_{p'}$, then $\mathcal{T}_p$ contains all points in $\mathcal{T}_{p'}$ with the same virtual multiplicities and also the further free point $\widehat{\varphi_{p'}}(p)$ in the first neighbourhood of the top of $\mathcal{T}_{p'}$.*

(b) *If p belongs to the second special fibre of $\widehat{\varphi}_{p'}$, then $\mathcal{T}_p$ contains all points in $\mathcal{T}_{p'}$ preceding q with the same virtual multiplicities and the point q with higher virtual multiplicity.*

Proof. Fix any pencil of germs $\mathcal{L}$ at p' having p' and p, both with virtual multiplicity one, as cluster of base points. The strict transforms at p of the non-special germs in $\mathcal{L}$ describe all but one lines in a pencil at p. Its direct images by φ_p are the germs $(\varphi_{p'})_*(\ell)$ for ℓ non-special in $\mathcal{L}$. The claim follows then from 11.1 and 11.2 applied to them. ◻

We will deal with points in the second special fibre in the forthcoming Section 16. Let us keep now under the hypothesis of Proposition 14.6 (a) and add the further assumption $p \notin \mathbf{J}(\varphi)$. On one hand, if

$$\mathcal{S}'(u,\theta) = \sum_{i<m'} a_i u^{i/n} + \theta u^{m_{p'}/n_{p'}},$$

is the Puiseux series of $\mathcal{T}_{p'}$, by 14.6 (a), the Puiseux series of $\mathcal{T}_p$ has partial sum $\mathcal{S}'(u,a)$ for a certain $a \in \mathbb{C}$, $a \neq 0$ if the end of $\mathcal{T}_{p'}$ is satellite. On the other, the hypothesis $p \notin \mathbf{J}(\varphi)$ implies, by 14.4, $m_p = m_{p'} + 1$ and so the Puiseux series of $\mathcal{T}_p$ is

$$\mathcal{S} = \mathcal{S}'(u,a) + \theta u^{\frac{m_{p'}+1}{n_p}}.$$

Note that if $\mathcal{T}_{p'}$ has multiplicity $r' > 1$, then $(m_{p'}+1)/n_p$ is a characteristic exponent, while otherwise it is not. In any case $\gcd(r', m_{p'}+1) = 1$ and $\mathcal{S}$ has polydromy order n_p. All together we get:

Proposition 14.7. *Let p be a point infinitely near to O, in the first neighbourhood of p'. Assume that p is free, non-fundamental and does not belong to $\mathbf{J}(\varphi)$. Assume also that either $\mathcal{T}_{p'}$ has free top or p does not belong to the second special fibre of $\widehat{\varphi_{p'}}$. Write q' and r' the top and multiplicity of $\mathcal{T}_{p'}$. Then $\mathcal{T}_p$ consists of:*

(a) *the points in $\mathcal{T}_{p'}$ with the same virtual multiplicities,*

(b) *the free point $q = \widehat{\varphi_{p'}}(p)$ in the first neighbourhood of q', with virtual multiplicity one, and*

(c) *$r' - 1$ points in successive neighbourhoods of q and proximate to q', all with virtual multiplicity one too.*

The reader may notice that $\mathcal{T}_p$ is reduced. Also, if p is as in 14.7 and $p = p_0, p_1, \ldots, p_i, \ldots$ is any sequence of consecutive free infinitely near points with p_1 not in the second special fibre, then Proposition 14.7, with $r' = 1$, applies to all points p_i, $i > 0$: it follows that each trunk $\mathcal{T}_{p_i}$ is obtained by adding a virtually simple and free point to $\mathcal{T}_{p_{i-1}}$. This fact, that could be described as a fairly regular growing of the trunks, will become quite obvious after getting a precise description of φ_p in the next section.

15. When φ_p is locally a composition of blowing-ups. Besides local isomorphisms, compositions of blowing-ups are by far the easiest and better understood examples of analytic morphisms between smooth surfaces, their study being just that of the infinitely near points. In this section our morphism φ will be related to compositions of blowing-ups. More precisely, we will prove that the morphism φ_p is locally isomorphic to a composition of blowing-ups π, provided p is a free non-fundamental infinitely near point that does not lie on $\mathbf{J}(\varphi)$, and $\mathcal{T}_p$ is reduced and has free top. This may be understood as a sort of partial resolution theorem for analytic morphisms, as after suitable sequences of blowing-ups (but not after all long enough sequences, hence the word *partial*), φ is turned into a fairly easy morphism. The reader may notice that 14.7 assures that there is a large set of points p infinitely near to O satisfying the above hypothesis: all but finitely many points p in the first neighbourhood of any point p' are free, non-fundamental and do not belong to $\mathbf{J}(\varphi)$ or to the second special fibre. Then either p itself or all free points in its first neighbourhood but those in the second special fibre have reduced trunk with free top. In particular one gets points p in the above conditions after finitely many blowing-ups provided finitely many choices of the point to blow up next are avoided after each blowing-up.

Furthermore, p being as above, π is the composition of blowing up all points in $\mathcal{T}_p$ but the last one. (It cannot be otherwise, as locally isomorphic morphisms should have the same trunk.) Then, in particular, we get an easy description of the singularity of the direct image $\varphi_*(\gamma)$ of any irreducible germ γ at O going through p, in terms of $\mathcal{T}_p$ and the singularity of γ, see 15.3.

LEMMA 15.1. *Assume that $\mathbf{J}(\varphi)$ is a multiple of a smooth germ, that $BP(\varphi) = \emptyset$ and also that $\mathcal{T}(\varphi)$ is reduced and has free top. If $\pi : T_q \longrightarrow T$ is the composition of blowing up all points in $\mathcal{T}(\varphi)$ but its top q, then there are open neighbourhoods U of O in S, and V of q in T_q and an analytic isomorphism $\psi : U \simeq V$ so that $\varphi_{|U} = \pi \circ \psi$.*

Proof. Write $\mathbf{J}(\varphi) = (m + n - 2)\zeta$ with ζ smooth, $n = e_O(\varphi)$ and m equal to the height of $\mathcal{T} = \mathcal{T}(\varphi)$ (6.1). Since $BP(\varphi) = \emptyset$, the contracted germ Φ is non-empty, and since it is contained in $\mathbf{J}(\varphi)$, $\Phi = n\zeta$.

Since $BP(\varphi) = \emptyset$, up to a linear change of the local coordinates u, v on T, we may assume that the first equation f of φ is an equation of Φ while the second equation g has $\operatorname{ord} g > n$. Since f is an equation of $\Phi = n\zeta$, it has a n-th root that defines the smooth germ ζ and that, therefore, may be taken as the first of a pair of local coordinates x, y at O. After these choices $f = x^n$ and $J(\varphi) = x^{n-1}(\partial g/\partial y)$. Since by hypothesis $\mathbf{J}(\varphi) = (m + n - 2)\zeta$, x^{n+m-2} and $J(\varphi)$ differ by an invertible factor, after which an easy computation proves that g has the form

$$g = x^{m-1}yw(x, y) + h(x),$$

with $w(x, y)$ invertible. We make explicit the monomials of g preceding the first monomial depending on y, by writing the equations of φ in the form

$$\begin{aligned} f &= x^n \\ g &= a_1x^{k_1} + \cdots + a_sx^{k_s} + x^{m-1}yw(x, y) + x^m h'(x), \end{aligned}$$

$n < k_1 < \cdots < k_s < m$, $a_i \neq 0$ for $i = 1, \ldots, s$.

The direct images of the lines $x = t$, $y = \alpha t$ are

$$\begin{aligned} u &= t^n \\ v &= a_1t^{k_1} + \cdots + a_st^{k_s} + (\alpha b + h'(0))t^m + \cdots, \end{aligned}$$

where $b = w(0,0) \neq 0$ and the dots indicate terms of higher order. Since we are assuming $\mathcal{T}(\varphi)$ reduced and with free top, $d = \gcd(n, k_1, \dots, k_s) = 1$. The proof will use induction on s

If $s = 0$, then $n = d = 1$, the equations of φ are

$$\begin{aligned} f &= x \\ g &= x^{m-1}(by + xh'(0) + \cdots), \end{aligned}$$

and the claim follows by direct computation.

Otherwise the direct images of the above lines have the form

$$\begin{aligned} u &= t^n \\ v &= a_1 t^{k_1} + \cdots . \end{aligned}$$

Let q' be the free point on these germs corresponding to the coefficient a_1 ([7], 5.7) and $\pi' : T_{q'} \longrightarrow T$ the composition of the blowing-ups giving rise to q'. As it is known (by easy direct computation if $n|k_1$ or from [7], 5.4.2 otherwise), there are local coordinates $\tilde{u}, \tilde{v}$ at q' so that π' is given by

$$(\tilde{u}, \tilde{v}) \mapsto (\tilde{u}^{n/n'}, \tilde{u}^{k_1/n'}(a_1 + \tilde{v})),$$

$n' = \gcd(n, k_1)$. Then $\tilde{\varphi}$, defined in a suitable neighbourhood of O by the rule

$$\tilde{\varphi}(x,y) = (x^{n'}, x^{-k_1} g - a_1)$$

is a lifting of φ to $T_{q'}$, i.e.,

$$\varphi = \pi \circ \tilde{\varphi} \tag{10}$$

in a neighbourhood of O. Note that $x^{-k_1} g - a_1$ has $s-1$ monomials in x preceding the first monomial involving y. Next we will see that either $\tilde{\varphi}$ is a local isomorphism or the induction hypothesis can be applied to it, thus completing the proof.

By (10) above, $\mathbf{J}(\tilde{\varphi})$ is a component of $\mathbf{J}(\varphi)$ and so it is a multiple of ζ too. Also by (10), the direct images by $\tilde{\varphi}$ of the lines in a pencil at O are the strict transforms of their direct images by φ. Then $\mathcal{T}(\tilde{\varphi})$ is reduced and has free top, as it is obtained from $\mathcal{T}$ by dropping all points preceding q.

It remains to examine the base points of $\tilde{\varphi}$. Assume first that $s > 1$ Then the equations of $\tilde{\varphi}$ take the form

$$\begin{aligned} \tilde{f} &= x^{n'} \\ \tilde{g} &= x^{k_2 - k_1}(a_2 + \cdots), \end{aligned}$$

after which, clearly, $\tilde{\varphi}$ has no base points and the proof ends by induction.

In case $s = 1$, $n' = d = 1$ and the equations of φ are

$$\begin{aligned} \tilde{f} &= x \\ \tilde{g} &= bx^{m-k_1-1}(y + xh'(0) + \cdots) \end{aligned}$$

If $m > k_1 + 1$, again $\tilde{\varphi}$ has no base points and the proof ends by induction. Otherwise $\tilde{\varphi}$ is an isomorphism, which in particular implies that $\mathcal{T}(\tilde{\varphi})$ consists of the point q' only and therefore $q = q'$. The morphism $\tilde{\varphi}$ is thus the wanted isomorphism. $\square$

The reader may note that the converse of 15.1 obviously holds: if φ is a composition of blowing-ups and O is a simple point of its exceptional divisor, then all conditions in the hypothesis of 15.1 are satisfied.

THEOREM 15.2. *Let p be a point infinitely near to O, S_p the surface it lies on, q the top of $\mathcal{T}_p$ and $\pi_q : T_q \longrightarrow T$ the composition of blowing up all points in $\mathcal{T}_p$ but q. Assume that the following conditions are satisfied:*
(1) *p is free and non-fundamental,*
(2) *p does not belong to $\mathbf{J}(\varphi)$, and*
(3) *$\mathcal{T}_p$ is reduced and has free top.*
Then there is an analytic isomorphism ψ, from an open neighbourhood U of p in S_p onto an open neighbourhood of q in T_q, so that $(\varphi_p)_{|U} = \pi_q \circ \psi$.

Proof. Condition (1) assures that $BP(\varphi_p) = \emptyset$. The point p being free, by condition (2) and 14.2, $\mathbf{J}(\varphi_p)$ is a multiple of the only component of the germ of the exceptional divisor at p, which is of course smooth. Then 15.1 applies to φ_p and gives the claim. □

The next two corollaries are direct consequences of 15.2, the proof of the second one being left to the reader.

COROLLARY 15.3. *If γ is any irreducible germ at O going through p and p satisfies the conditions of Theorem 15.2, then $\varphi_*(\gamma)$ is reduced and its sequence of infinitely near points may be obtained by appending to the points of $\mathcal{T}_p$ the images by ψ of the infinitely near points on the strict transform $\tilde{\gamma}_p$ of γ at p.*

Proof. Just compute $\varphi_*(\gamma) = (\varphi_p)_*(\tilde{\gamma}_p) = (\pi_q)_*(\psi(\tilde{\gamma}_p))$, and recall that direct images of irreducible germs by blowing-ups are reduced. □

In particular, the Enriques diagram of $\varphi_*(\gamma)$ results by dropping p and the points preceding it from the Enriques diagram of γ, and then appending the remaining of the diagram to the end of the Enriques diagram of $\mathcal{T}_p$.

COROLLARY 15.4. *If p' is infinitely near to p and p satisfies the conditions of Theorem 15.2, then $\mathcal{T}_{p'}$ is the only irreducible cluster whose points are those of $\mathcal{T}_p$ plus the images by ψ of all points p'', $p < p'' \leq p'$.*

16. Non-reduced trunks, second special fibres and folded germs. Irreducible germs of curve γ at O, whose image $\varphi_*(\gamma)$ is a non-reduced germ, will be called *folded germs* (by φ). More precisely, we will say that γ is r-folded when r is the degree of the restricted germ of map $\varphi_{|\gamma} : \gamma \longrightarrow \varphi_*(\gamma)_{red}$. In this section we will gain some insight into the existence and distribution of folded germs. We will deal with unibranched, possibly non reduced, germs $\zeta = r\gamma$, γ irreducible and $r \geq 1$. The Puiseux series and characteristic exponents of such a ζ will be taken to be, by definition, those of $\gamma = \zeta_{red}$. It is convenient to write the characteristic exponents of ζ with common denominator $e(\zeta)$, in which case r may be recovered as the factor that may be cancelled from the whole of characteristic exponents of ζ. Let us begin by stating a direct consequence of 15.3 that needs no proof:

COROLLARY 16.1. *If a non-fundamental free point p not on $\mathbf{J}(\varphi)$ belongs to a folded germ, then either $\mathcal{T}_p$ is non-reduced or it has satellite top.*

We need to pay some attention to points in second special fibres, already not covered by 14.7, namely:

Proposition 16.2. *Let p be a point infinitely near to O, in the first neighbourhood of p'. Assume that p is free, non-fundamental and does not belong to $\mathbf{J}(\varphi)$. Assume also that $\mathcal{T}_{p'}$ has satellite top and p belongs to the second special fibre of $\widehat{\varphi_{p'}}$. If $\mathcal{T}_{p'}$ has Puiseux series*

$$\mathcal{S}' = \sum_{i<m_{p'}} a_i u^{i/n_{p'}} + \theta u^{m_{p'}/n_{p'}},$$

then the Puiseux series of $\mathcal{T}_p$ is

$$\mathcal{S} = \sum_{i<m_{p'}} a_i u^{i/n_{p'}} + \theta u^{(m_{p'}+1)/n_{p'}}.$$

Proof. By 14.6 the Puiseux series of $\mathcal{T}_p$ has as partial sum the constant part of $\mathcal{S}'$ followed by no monomial of degree less or equal than $m_{p'}/n_{p'}$. Since $m_p = m_{p'}+1$ by 14.4, the claim follows. □

Denote by $\tilde{n}$ the top twist of $\mathcal{T}_p$.

Lemma 16.3. *Assume that p is a free non-fundamental point infinitely near to O that does not belong to $\mathbf{J}(\varphi)$. Then $\widehat{\varphi_p}$ has degree $\tilde{n}$, its first special fibre is $\tilde{n}$ times the satellite point and, in case $\tilde{n} > 1$, its second special fibre also consists of an $\tilde{n}$-fold point.*

Proof. The point p being free and non-fundamental, the only point of B_p is the satellite point, that therefore is also the only point of the first special fibre (by 10.7). Since $p \notin \mathbf{J}(\varphi)$, $\mathbf{J}(\varphi_p)$ is a multiple of the exceptional divisor (14.2) and therefore its only point in the first neighbourhood of p is the satellite one. Then 12.11 gives $\deg(\widehat{\varphi}_p) = \tilde{n}$ and the claim about the second special fibre follows from 12.2. □

In the sequel infinite sequences of free points $p, p_1, p_{2,}, \dots$, each in the first neighbourhood of the preceding one, will be called *paths* with origin at p.

Choose a point p with non-reduced trunk, say of multiplicity $r > 1$. Assume that $\mathcal{T}_p$ has Puiseux series

$$\mathcal{S} = \bar{\mathcal{S}} + \theta u^{m/n_p}$$

and characteristic exponents $m_1/n_p, \dots, m_k/n_p$. Then $r = \gcd(n_p, m_1, \dots, m_k)$ divides m. Next we will define a series of paths with origin at p:

(1) Choose p_1 to be any free non-fundamental point in the first neighbourhood of p not belonging to $\mathbf{J}(\varphi)$ or to the second special fibre of $\widehat{\varphi_p}$ if $\mathcal{T}_p$ has satellite top. As it follows from 14.7, $\mathcal{T}_{p_1}$ has Puiseux series

$$\mathcal{S}_1 = \bar{\mathcal{S}} + b_0 u^{m/n_p} + \theta u^{(m+1)/n_p},$$

$b_0 \neq 0$ if the top of $\mathcal{T}_p$ is satellite, and its characteristics exponents are those of $\mathcal{T}_p$ plus the further one $(m+1)/n_p$.

(2) Since 16.3 applies to p_1, we take p_2 to be the only point in the second special fibre of $\widehat{\varphi_{p_1}}$, which is free. By 16.2 the Puiseux series of $\mathcal{T}_{p_2}$ is

$$\mathcal{S}_2 = \bar{\mathcal{S}} + b_0 u^{m/n_p} + \theta u^{(m+2)/n_p}.$$

If $2 < r$ still $\mathcal{T}_{p_2}$ has satellite end and we continue in this way for $i \leq r$:

(i) Take p_i to be the only point in the second special fibre of $\widehat{\varphi_{p_{i-1}}}$. The Puiseux series of $\mathcal{T}_{p_i}$ is

$$\mathcal{S}_i = \bar{\mathcal{S}} + b_0 u^{m/n_p} + \theta u^{(m+i)/n_p}.$$

Once we get the point p_r the situation changes: the trunk $\mathcal{T}_p$ is non-reduced of multiplicity r and no longer has satellite top. Then we restart by taking any free point p_{r+1} in the first neighbourhood of p_r (the remaining conditions in step (1) being now automatically satisfied), and then $p_{r+2}, \dots, p_{2r}$ the points in the successive second special fibres, and so on. Thus we get paths in which there is a free choice of the points p_{rj+1} among the free points in the first neighbourhood of p_{rj}, after which the points $p_{rj+2}, \dots, p_{r(j+1)}$ are uniquely determined. The Puiseux series of $\mathcal{T}_{p_i}$ is

$$\mathcal{S}_i = \bar{\mathcal{S}} + \sum_{0 \le j < i/r} b_j u^{(m+jr)/n_p} + \theta u^{(m+i)/n_p}.$$

The paths defined above will be called the *r-folding paths* with origin at p. A *folding path* will be a r-folding path for any $r > 1$.

Assume that $1 < r' = \gcd(i, r) < r$. Then the point p_i in any one of the r-folding paths described above has $\mathcal{T}_{p_i}$ r'-fold. Therefore it is the origin of a new series of r'-folding paths that contain all free points in the first neighbourhood of p_i but p_{i+1}, the only point in the first neighbourhood of p_i in the r-folding paths with origin p.

The above description of the r-folding paths with origin at p gives:

PROPOSITION 16.4. *If an irreducible germ γ at O goes through the point p_i, $i > 0$, in an r-folding path with origin at p, and has a free point q in its first neighbourhood, then the Puiseux series of $\varphi_*(\gamma)$ has partial sum*

$$\bar{\mathcal{S}} + \sum_{0 \le j < i/r} b_j u^{(m+jr)/n_p}.$$

If furthermore q does not belong to any r-folding path, then next in the above series comes a characteristic term of degree $(m+i)/n_p$.

Proof. After the above description of the Puiseux series of the trunks $\mathcal{T}_{p_i}$, the claim directly follows from either 11.2, in case q still belongs to a r-folding germ, or 11.1 otherwise. □

Remark 16.5. It is worth noting that the first part of 16.4 says in particular that the characteristic exponents of $\varphi_*(\gamma)$ less than $(m+i)/n_p$, are exactly those of $\mathcal{T}_p$. If furthermore q belongs to no r-folding path, then the next characteristic exponent is $(m+i)/n_p$.

COROLLARY 16.6. *Irreducible germs γ at O going through p, all whose points infinitely near to p belong to an r-folding path, are r-folded by φ.*

Proof. All points on γ infinitely near to p being free, $e_p(\gamma) = 1$ and thus, by 11.1, $e_{O'}(\varphi^*(\gamma)) = n_p$. Since 16.5 applies for all i, the characteristic exponents of $\varphi_*(\gamma)$ are those of $\mathcal{T}_p$, from which the claim. □

A partial converse of 16.6 also holds:

Corollary 16.7. *Let p_i, $i > 0$,be the i-th point in a r-folding path with origin at p. An irreducible germ γ at O having p_i as a non-singular point and its first neighbouring point on no r-folding path, is not r-folded.*

Proof. Again by 11.1, $e_{O'}(\varphi^*(\gamma)) = n_p$ and now, by 16.5, r cannot be cancelled from the whole of characteristic exponents of $\varphi_*(\gamma)$ written with denominator n_p. □

It is well known that any path with origin at p is the set of points on a uniquely determined smooth algebroid germ of curve τ at p, this is, a germ given by a order-one formal, non-necessarily convergent, series at p. The germ τ may be analytic or not, depending on the position of the points of the path, as suitable coordinates of the points are the coefficients of a Puiseux series of τ, see for instance [7], 5.7. Therefore, Corollary 16.6 could have an empty claim, as the points p_i, $i \not\equiv 1 \mod r$, in a r-folding path have fixed positions and could give rise to algebroid non-analytic germs, independently on the choice of the remaining points. Next we will check that this is not the case by a direct computation.

Since we are concerned with convergence of series, we may take p_r instead of p and therefore assume that p itself is free, non fundamental and does not belong to $\mathbf{J}(\varphi)$. Then, arguing as in the proof of 15.1, we may choose the local coordinates at O' and p in such a way that the first equation of φ_p is $f = x^{n_p}$, where x is the first local coordinate at p, and also a local equation of the exceptional divisor. Since $p \notin \mathbf{J}(\varphi)$, $\mathbf{J}(\varphi_p)$ is $(n_p + m_p - 2)$-times the exceptional divisor at p, which forces the second equation g of φ_p to have the form:

$$g = \sum_{0<i<m_p} b_i x^i + x^{m_p-1} g'(x,y), \qquad \frac{\partial g'}{\partial y}(0,0) \neq 0.$$

One may thus take $y' = g'$ as a new second coordinate, after which the equations of φ_p take the form:

$$f = x^{n_p}, \quad g = \sum_{0<i<m_p} b_i x^i + x^{m_p-1} y'.$$

The Puiseux series of $\mathcal{T}_p$ is then

$$\mathcal{S} = \sum_{0<i<m_p} b_i u^{i/n_p} + \theta u^{m_p/n_p}$$

and so, in particular, $b_i = 0$ if $i \not\equiv 0 \mod r$, as $\mathcal{T}_p$ is assumed to have multiplicity r.

Now, from these equations, one may directly compute the direct images of the germs

$$y' - \sum_{1 \le rj+1 < \rho} c_j x^{rj+1} - \beta x^{\rho} = 0,$$

and use induction on ρ to easily see that the r-folding paths with origin at p are exactly the paths of points on the algebroid germs

$$y' - \sum_{j \ge 0} c_j x^{rj+1} = 0,$$

for arbitrary $c_j \in \mathbb{C}$. The convergence of these series depends of course on the c_j, which are coordinates of the points p_{rj+1}. Suitable choices of these points give thus analytic germs and we have proved:

THEOREM 16.8. *For infinitely many choices of each of their $(rj+1)$-th points, $j = 0, 1, \dots,$ the r-folding paths with origin at p are on analytic germs which, therefore, are r-folded. In particular there are infinitely many folded germs going through each point p with $\mathcal{T}_p$ non-reduced.*

COROLLARY 16.9. *Any morphism φ with $\deg_O(\varphi) > 1$ has folded germs.*

Proof. We will prove that any such φ has at least one dicritical base point p with $e(\varphi_p) > 1$. Then $\mathcal{T}_p$ consists of O' with virtual multiplicity $e(\varphi_p)$ (4.2) and the claim follows from 16.8.

Since $\deg_O(\varphi) > 0$, O is a base point of φ, by 3.2. If O is the only base point, then it is dicritical because there are no base points in its first neighbourhood. Furthermore $e(\varphi) > 1$, as otherwise $\deg_O(\varphi) = 1$, and we may take $p = O$. If there are base points other than O, take p to be a maximal one. As before, p is dicritical and $e(\varphi_p) > e(\varphi) \geq 1$, by 13.1, as wanted. □

Remark 16.10. The example 3.5 provides examples of morphisms with degree zero and folded germs. On the other hand, in case $d = \deg_O(\varphi) > 1$, d-folded germs need not to exist, as shown by the next example:

EXAMPLE 16.11. The morphism $(x, y) \mapsto (x^2, y^2)$ has degree 4, evident 2-folded germs and no r-folded germs for $r > 2$. Indeed, any irreducible germ of curve γ at O may be given by a (Puiseux) parameterization

$$x = t^n$$
$$y = \sum_{i \geq m_0} a_i t^i, \quad a_{m_0} \neq 0.$$

$\varphi_*(\gamma)$ is thus given by

$$u = t^{2n}$$
$$v = \left(\sum_{i \geq m_0} a_i t^i \right)^2.$$

Obviously γ cannot be r-folded, $r > 2$, if $n = 1$. If $n > 1$ there are characteristic terms $a_{m_j} t^{m_j}$, $j = 1 \dots, k$, each $a_{m_j} t^{m_j}$ being the first non-zero monomial for which $m_j \notin (n, m_1, \dots, m_{j-1})$ and then $(1) = (n, m_1, \dots, m_k)$. It is easy to check that the monomials

$$2a_{m_0} a_{m_j} t^{m_0 + m_j}, \quad j = 1 \dots, k,$$

if $m_0 < m_1$, or

$$a_{m_1}^2 t^{2m_1} \quad \text{and} \quad 2a_{m_1} a_{m_j} t^{m_1 + m_j}, \quad j = 2, \dots, k$$

if $m_0 = m_1$, are not canceled in the above expression for v. If $m_0 < m_1$,

$$(2n, m_0 + m_1, \dots, m_0 + m_k) \supset (2n, 2m_0 + 2m_1, \dots, 2m_0 + 2m_k) = (2)$$

and similarly, in case $m_0 = m_1$,

$$(2n, 2m_1, m_1 + m_2 \dots, m_1 + m_k) \supset (2n, 2m_1, 2m_1 + 2m_2, \dots, 2m_1 + 2m_k) = (2).$$

This shows that no factor other than 2 may divide all exponents effectively appearing in the parameterization of $\varphi_*(\gamma)$ and therefore completes the argument.

Next we show that any folded germ except possibly the components of $\mathbf{J}(\varphi)$, falls in the situation of 16.6 for a suitable choice of p:

Proposition 16.12. *If γ is a folded irreducible germ at O, not a component of $\mathbf{J}(\varphi)$, there is p on γ such that all points on γ infinitely near to p are in a folding path with origin at p.*

Proof. Since γ is not a component of $\mathbf{J}(\varphi)$, nor is it a component of the contracted germ Φ. Thus we may choose a non-singular and non-fundamental p on γ not belonging to $\mathbf{J}(\varphi)$. By 16.1 either $\mathcal{T}_p$ is non-reduced, or it is reduced and has a satellite top. Call $m_1/n_p, \dots, m_k/n_p$ the characteristic exponents of $\mathcal{T}_p$ and assume that $\mathcal{T}_p$ is reduced, so we are in the latter case: by 14.7, 16.1 and 16.3, the point p_1 in the first neighbourhood of p on γ is the one in the second special fibre. If $r = \gcd(n_p, m_1, \dots, m_{k-1}, m_k + 1) > 1$, by 16.2, $\mathcal{T}_{p_1}$ is non-reduced, of multiplicity r. Otherwise, by 16.2, $\mathcal{T}_{p_1}$ has characteristic exponents $m_1/n_p, \dots, m_{k-1}/n_p, (m_k+1)/n_p$ and therefore satellite top. We repeat the argument till getting i such that $\gcd(n_p, m_1, \dots, m_{k-1}, m_k + i) > 1$ and hence $\mathcal{T}_{p_i}$ non-reduced.

Thus, by taking p_i for p if needed, we need just to deal with the case of $\mathcal{T}_p$ non-reduced, say of multiplicity r. Then either all further points on γ are in a folding path with origin at p, as claimed, or if p' is the last point on γ in such a path, by 14.7, 16.1 and the construction of the folding paths, $\mathcal{T}_{p'}$ needs to be again non-reduced, of multiplicity r', $1 < r' < r$. The claim is thus reached after finitely many steps. ☐

We have so far dealt with points p for which the trunk $\mathcal{T}_p$ is non-reduced. We have proved in particular that any such point has folded germs going through, which is a partial converse of 16.1. A similar claim for points p for which $\mathcal{T}_p$ has satellite top is, however, false. Indeed, by taking O a satellite point infinitely near to O' and φ the composition of blowing up all points preceding O, no germ is folded by φ while the main trunk $\mathcal{T}(\varphi)$ has satellite top. Anyway we have:

Proposition 16.13. *If p is equal or infinitely near to O, $\mathcal{T}_p$ has satellite top and the second special fibre of $\widehat{\varphi}_p$ contains some free non-fundamental point p_1 missed by $\mathbf{J}(\varphi)$, then there are infinitely-many folded germs through p.*

Proof. As in the construction of the r-folding paths, by repeated use of 16.2 and 16.3, define a sequence of points $p_1, \dots, p_i$, each p_j in the second special fibre of $\widehat{\varphi}_{j-1}$, till getting, as in the proof of 16.12, p_i with $\mathcal{T}_{p_i}$ non-reduced. Then 16.8 applies. ☐

17. Trunks on the discriminant. In preceding sections we have been dealing with p-trunks for p free, non-fundamental and not belonging to $\mathbf{J} = \mathbf{J}(\varphi)$. In this section we will study the case $p \in \mathbf{J}$. We continue to assume that p is free and non-fundamental.

Proposition 17.1. *If p is free, non-fundamental and belongs to $\mathbf{J}$, then all free points in $\mathcal{T}_p$ belong to the discriminant germ $\Delta = \varphi_*(\mathbf{J})$. More precisely, if p lies in the first neighborhood of p', then all free points in $\mathcal{T}_p$ belong to the direct image of any branch γ of $\mathbf{J}$ going through p for which the ratio $e_p(\gamma)/e_{p'}(\gamma)$ is maximal.*

Proof. Since p is non-fundamental, we choose the local coordinates at O' so that the first equation f of φ_p defines the germ of n_p times the exceptional divisor at p, and take a root f^{1/n_p} as the first of the local coordinates x, y at p. Then the equations of φ_p may be written

$$f = x^{n_p}$$
$$g = P(x) + h(x,y)$$

where h is a series whose initial form effectively depends on y and P is a polynomial in x of degree strictly less than $o(h)$, say

$$P = \sum_{i=n_p}^{o(h)-1} b_i x^i.$$

The direct images of the lines $x = t$, $y = \alpha t$ are

$$u = t^{n_p}$$
$$v = P(t) + \widehat{h}(1,\alpha)t^{o(h)} + \cdots,$$

after which, clearly, $m_p = o(h)$ and the constant part of the Puiseux series of $\mathcal{T}_p$ is

$$P(u^{1/n_p}) = \sum_{i=n_p}^{m_p-1} b_i u^{i/n_p}.$$

An equation of $\mathbf{J}(\varphi_p)$ being $x^{(n_p-1)}\partial h/\partial y = 0$, the strict transforms of the branches of $\mathbf{J}$ through p are the branches of $\partial h/\partial y = 0$ other than the germ of the second axis.

If γ is a branch of $\mathbf{J}$ whose strict transform at p is the germ of the x-axis, the Puiseux series of $\varphi_*(\gamma)$,

$$P(u^{1/n_p}) + h(u^{1/n_p}, 0),$$

has the constant part of the Puiseux series of $\mathcal{T}_p$ as a partial sum and therefore all free points of $\mathcal{T}_p$ lie on $\varphi_*(\gamma)$, as claimed.

Now, let γ be a branch of $\mathbf{J}$ with origin at O whose strict transform at p, $\tilde{\gamma}_p$ is not the x-axis, and therefore has a Puiseux parameterization $x = t^\nu$, $y = \beta t^\mu + \cdots$, $\beta \neq 0$. Then $e_{p'}(\gamma) = \nu$ and $e_p(\gamma) = \min\{\nu, \mu\}$. Let τ be the (ν, μ)-twisted order of h, namely

$$\tau = \min\{\nu i + \mu j \text{ for } a_{i,j}x^i y^j \text{ a non-zero monomial of } h\}.$$

A direct computation shows that the Puiseux series of $(\varphi_p)_*(\tilde{\gamma}_p)$ has partial sum

$$\sum_{n_p \le i < \tau/\nu} b_i u^{i/n_p}$$

and therefore, as above, the claim for γ will be satisfied if it holds $m_p - 1 < \tau/\nu$.

Assume that the monomial of higher degree in y of $\widehat{h}$ has bidegree (c_1, c_2). By the definition of h, $c_2 > 0$ and $c_1 + c_2 = o(h) = m_p$. The point $Q = (c_1, c_2)$ is a vertex of the Newton polygon $\mathbf{N}$ of h, all sides above it have slope strictly less than -1 while the remaining ones have slopes non-less than -1. By the derivation rules, the same occurs with the point $Q' = (c_1, c_2 - 1)$ and the Newton polygon $\mathbf{N}'$ of $\partial h/\partial y$.

Let us assume first that $c_2 > 1$ and so that either the x-axis is a branch of $\partial h/\partial y = 0$ or there are sides of $\mathbf{N}'$ below Q'. If $\tilde{\gamma}_p$ is the x-axis, then, clearly $e_p(\gamma)/e_{p'}(\gamma) = 1$ and the claim for γ has ben already verified. Assume that $\tilde{\gamma}_p$ corresponds to a side Γ of $\mathbf{N}'$ below Q'. The slope of Γ is then $-\nu/\mu \geq -1$, which gives $e_p(\gamma)/e_{p'}(\gamma) = 1$. Furthermore the (ν, μ)-twisted order of h is attained by the non-zero monomials on the line of equation $\nu i + \mu j = \tau$ which meets the first axis at the point $Q_1 = (\tau/\nu, 0)$. The line of slope -1 through Q meets the first axis at the point $Q_2 = (m_p, 0)$. The convexity of $\mathbf{N}$ and the inequality of slopes $-\nu/\mu \geq -1$ prevents the point Q_2 from being beyond Q_1, thus giving for γ the wanted inequality $\tau/\nu \geq m_p > m_p - 1$.

Branches $\tilde{\gamma}_p$ corresponding to sides above Q have $-\nu/\mu < -1$, after which $e_p(\gamma)/e_{p'}(\gamma) < 1$ and there is nothing to prove about them, because we have seen above that there is some branch γ with $e_p(\gamma)/e_{p'}(\gamma) = 1$.

In case $c_2 = 1$, $\mathbf{N}'$ is obtained from $\mathbf{N}$ by discarding its only possible side with upper end Q and translating the remaining sides one step downward. In particular, the lowest vertex of $\mathbf{N}'$ is $(c_1, 0) = (m_p - 1, 0)$ and the slope of any of its sides is strictly less than -1. Therefore, for any branch γ of $\mathbf{J}$, $e_p(\gamma)/e_{p'}(\gamma) = \mu/\nu < 1$. Branches with maximal $e_p(\gamma)/e_{p'}(\gamma)$ are thus those whose strict transform is associated to the side of $\mathbf{N}'$ that ends at $(m_p - 1, 0)$. Since $\mathbf{N}$ has a parallel side with lower end $(m_p - 1, 1)$, $\tau = \nu(m_p - 1) + \mu > \nu(m_p - 1)$ as wanted. ☐

Remark 17.2. Keep the assumption that p is free and non-fundamental. If $p \notin \mathbf{J}$, the trunk $\mathcal{T}_p$ has been determined from $\mathcal{T}_{p'}$ in 14.7 and 16.2 with no problem in locating the free points of $\mathcal{T}_p$ not already in $\mathcal{T}_{p'}$. Indeed, due to the equality $m_p = m_{p'} + 1$ there is at most one such point, which, if it does exist, is $\widehat{\varphi_{p'}}(p)$, by 14.6. In case $p \in \mathbf{J}$, there may be many free points of $\mathcal{T}_p$ not in $\mathcal{T}_{p'}$. After 17.1, all these points are well determined, as they lie in successive neighbourhoods of the top of $\mathcal{T}_{p'}$, belong to any branch ζ of the discriminant direct image of a branch of $\mathbf{J}$ with maximal $e_p(\gamma)/e_{p'}(\gamma)$ and their number is determined by 4.3. Obviously, ζ also determines the satellite points preceding a free point in $\mathcal{T}_p$. A possible group of satellite points at the end of $\mathcal{T}_p$ is of course determined by $m_p/n_p = m_p/n_{p'}$ by the customary division algorithm.

In the following examples we assume that $p' = O$, p is the point in its first neighbourhood on the x-axis and the morphism φ has equations $u = f(x, y)$, $v = g(x, y)$.

EXAMPLE 17.3. Take $f = x^3$ and $g = y^7 + x^{11}y + x^{13}$. Then the jacobian consists of two times the germ of the y-axis plus a further branch $\gamma : 7y^6 + x^{11} = 0$. By direct computation of the direct images of $x = t$, $y = \alpha t$ and $x = t$, $y = \alpha t^2$ one easily gets the trunks $\mathcal{T}$ and $\mathcal{T}_p$. $\mathcal{T}$ consists of the origin and two free points on the u-axis, of multiplicities $3, 3, 1$, followed by two simple satellite points, which gives $m = 7$. $\mathcal{T}_p$ has the three non-satellite points of $\mathcal{T}$, all with multiplicity 3 (case (b) of 14.6), and two further free points which still belong to the u-axis and have multiplicities $3, 1$, followed by two simple satellite points, corresponding to a single characteristic exponent $13/3$. Thus $m_p = 13$. The origin and the four free points of $\mathcal{T}_p$ lie on the $\varphi_*(\gamma)$, as claimed in 17.1, as in fact these points belong to $\varphi_*(\gamma)$ with multiplicities $18, 18, 18, 18, 5$ and are followed on $\varphi_*(\gamma)$ by the group of satellite points corresponding to the characteristic exponent $77/18$. Note that we are in case $c_2 = 1$ of the proof of 17.1. Since $\nu = 6$ and $\mu = 5$, $\tau = 77$ and $\tau/\nu = 77/6 < 13 = m_p$.

EXAMPLE 17.4. If one takes $f = x^3$ and $g = x^8 + x^4y^3 + x^{10} + x^{14}$, $\mathcal{T}$ and $\mathcal{T}_p$ have Puiseux series $\theta u^{7/3}$ and $u^{8/3} + \theta u^{10/3}$, respectively. Hence in this case the free points of $\mathcal{T}_p$ not in $\mathcal{T}$ are preceded by a satellite point that does not belong to $\mathcal{T}$ either. An easy direct computation shows that, as implied by 17.1, all these points belong to the direct image of the branch $y = 0$ of $\mathbf{J} : x^6y^2 = 0$.

EXAMPLE 17.5. By taking $f = x^3$ and $g = x^2y^2 + x^{10} + x^{14}$, one gets a case in which there are satellite points in $\mathcal{T}_p$ that do not belong to the discriminant. Indeed, the Puiseux series of $\mathcal{T}_p$ is $\theta u^{8/3}$ while the only branch of the discriminant has Puiseux series $u^{11/3} + u^{14/3}$.

18. The trunks $\mathcal{T}_q$ for q satellite. Take O and φ as above. We will show in this section that the algorithm of Section 10 may be modified in order to compute, from the equations f, g of φ, the trunks $\mathcal{T}_q$ for all points q which either lie on one of the axes of coordinates, or are satellite points of a point on an axis. This applies to the case in which a free point p infinitely near to O and the corresponding morphism φ_p are taken instead of O and φ, allowing to compute the trunks $\mathcal{T}_q$, for all satellite points q of p, from local equations of φ_p relative to coordinates for which the germ at p of the exceptional divisor is one of the axes. Furthermore, the fact that the computations for different trunks $\mathcal{T}_q$ are all performed at p allows to establish some relationship between the trunks associated to the different satellite points of p.

Once a smooth germ at O, $h = 0$, has been fixed, the point O itself, the (necessarily free) points infinitely near to O on $h = 0$ and their satellite points are called *h-satellite* points. After choosing local coordinates x, y at O, we are interested in the points that are either x-satellite or y-satellite: they will be referred to in the sequel as *frame-satellite points*. This notion obviously depends on the choice of the coordinates. Points which are not frame-satellite will be called *frame-free*.

It is well known (see [7], Chapter 5, for instance) that all Puiseux series $y = s(x)$ of all irreducible germs at O going through a fixed frame-satellite point q and having a frame-free point in its first neighbourhood have the same order $\sigma \in \mathbb{Q}$, which in turn determines q as the last frame-satellite point on any germ with Puiseux series $s(x) = ax^\sigma + \cdots$, $a \neq 0$. We will call σ the *slope* of q and will often write $\sigma = \sigma(q)$ and $q = q(\sigma)$, the maps $q \mapsto \sigma(q)$ and $\sigma \mapsto q(\sigma)$ being reciprocal bijections between the set of frame-satellite points and the set of positive rational numbers. As the reader may notice, $O = q(1)$ and $q(\sigma)$ is y-satellite if $\sigma \geq 1$, while it is x-satellite if $\sigma \leq 1$. Also $q(\sigma)$ belongs to the x-axis (resp. y-axis) if and only if σ (resp. $1/\sigma$) is an integer. It follows from the first step of the Newton-Puiseux algorithm that a germ of curve ξ at O goes through $q(\sigma)$ and has a frame free point in its first neighbourhood if and only if the Newton polygon of ξ has a side with slope $-1/\sigma$. In the sequel such a side will be said to be *orthogonal* to σ.

If σ is any positive rational number, write it as a continued fraction

$$\sigma = d_0 + \cfrac{1}{d_1 + \cfrac{1}{\ddots \, \cfrac{}{\frac{1}{d_s}}}},$$

take

$$\sigma' = d_0 + \cfrac{1}{d_1 + \cfrac{1}{\ddots \cfrac{1}{d_{s-1}}}}$$

and then write σ and σ' as irreducible fractions: $\sigma = \sigma_1/\sigma_2$, $\sigma' = \sigma_1'/\sigma_2'$.

LEMMA 18.1. *There are local coordinates $\tilde{x}, \tilde{y}$ at $q(\sigma)$ for which each coordinate axis is either the strict transform of a coordinate axis at O or a component of the exceptional divisor, and such that the equations of the blowing-up giving rise to $q(\sigma)$ are*

$$\begin{aligned} x =& \tilde{x}^{\sigma_2 - \sigma_2'} \tilde{y}^{\sigma_2'} \\ y =& \tilde{x}^{\sigma_1 - \sigma_1'} \tilde{y}^{\sigma_1'}. \end{aligned}$$

Proof. Follows from the elementary properties of continued fractions by direct computation, if one takes as new coordinates after each blowing-up the pull-backs of z_1 and z_2/z_1, z_1, z_2 being the former coordinates taken in a suitable order. □

In the sequel we take at $q(\sigma)$ the local coordinates $\tilde{x}, \tilde{y}$ of Lemma 18.1. Take the *σ-order* of $h = \sum a_{i,j} x^i y^j$ to be

$$o_\sigma h = \min\{\sigma_2 i + \sigma_1 j | a_{i,j} \neq 0\}$$

and its *σ-initial form*

$$in_\sigma(h) = \sum_{\sigma_2 i + \sigma_1 j = o_\sigma(h)} a_{i,j} x^i y^j.$$

Two σ-initial forms I, I' are called *homothetical* if and only if $I^\delta = aI'$ for a non-zero complex number a and a positive rational number δ. We will call δ the *ratio* of I, I'.

LEMMA 18.2. *If a bar denotes the pull-back by the composition of blowing-ups giving rise to $q(\sigma)$, then for any $h, h' \in \mathcal{O}_{S,O}$:*

(i) $o(\bar{h}) = o_\sigma(h)$

(ii) $\hat{\bar{h}} = \overline{in_\sigma(h)}$

(iii) *The initial forms of $\bar{h}$ and $\bar{h}'$ are homothetical if and only if so are the σ-initial forms of h and h'*

Proof. By 18.1, the σ-order of any monomial is the ordinary order of its pull-back, after which all three claims directly follow. □

PROPOSITION 18.3. *If the algorithm of Section 10 is applied to equations of φ using σ-orders and σ-initial forms instead of the ordinary ones, then it gives rise to germs of analytic function $P_0, \ldots P_k$ at O' and a pencil $\mathcal{Q}$ such that $\{o_\sigma(P_i)\}_{i=0,\ldots,k}$ is a minimal system of generators of $\Gamma(\varphi_{q(\sigma)})$ and $BP(\mathcal{Q}) = (\mathcal{T}_{q(\sigma)})_{red}$*

Proof. Obvious from 18.2 and 10.5, as the pull-backs of all intermediate germs of function at O given rise to by the algorithm of the claim are just those produced by the ordinary algorithm performed at $q(\sigma)$ from the pull-backs of the equations of φ.□

Next we will relate the above σ-algorithms for different values of σ. Let h be a non-zero germ of analytic function at O and $\mathbf{N}(h)$ its Newton polygon. For a given positive rational number σ two possibilities may occur, namely:

(a) There is a side $\Gamma_\sigma(h)$ of $\mathbf{N}(h)$ orthogonal to σ. Then the monomials of highest and lowest degree in y of $in_\sigma(h)$ are different and correspond to the ends of Γ_σ. The remaining monomials of $in_\sigma(h)$ correspond to points lying on Γ_σ.

(b) No side of $\mathbf{N}(h)$ is orthogonal to σ. Then $in_\sigma(h)$ is a monomial and its corresponding point is a vertex $\Gamma_\sigma(h)$ of $\mathbf{N}(h)$.

The next lemma is elementary and needs no proof:

LEMMA 18.4. *Let h_1 and h_2 be non-zero germs of analytic function at O. We have:*

(1) *If $in_\sigma(h_1)$, $in_\sigma(h_2)$ are homothetical of ratio δ, then the homothety of center the origin and ratio δ maps $\Gamma_\sigma(h_1)$ onto $\Gamma_\sigma(h_2)$*

(2) *$in_\sigma(h_1)$, $in_\sigma(h_2)$ are homothetical monomials and have ratio δ if and only if $\Gamma_\sigma(h_1)$ and $\Gamma_\sigma(h_2)$ are vertices aligned with the origin and their ratio of distances to it is δ.*

LEMMA 18.5. *If h_1 and h_2 are as above and $in_{\sigma_0}(h_1)$ $in_{\sigma_0}(h_2)$ are homothetical of ratio δ for a certain σ_0, then there exists an open interval (σ_1, σ_2), $0 \leq \sigma_1 < \sigma_0 < \sigma_2 \leq \infty$, such that $in_\sigma(h_1)$ $in_\sigma(h_2)$ are homothetical of ratio δ for any $\sigma \in (\sigma_1, \sigma_2)$, while they are not homothetical if $\sigma = \sigma_i \neq 0, \infty$, $i = 1, 2$. Each end σ_i is, if finite and non-zero, orthogonal to a side of either $\mathbf{N}(h_1)$ or $\mathbf{N}(h_2)$.*

Proof. Take σ' to be the maximum of 0 and the slopes $\sigma < \sigma_0$ orthogonal to a side of $\mathbf{N}(h_1)$ or $\mathbf{N}(h_2)$ and let p_i denote the upper end of $\Gamma_{\sigma_0}(h_i)$ if it is a side, or the vertex $\Gamma_{\sigma_0}(h_i)$ itself otherwise. By Lemma 18.4, p_1, p_2 are aligned with the origin and their ratio of distances to it is δ. Therefore, by the choice of σ' and again 18.4, $in_\sigma(h_1)$, $in_\sigma(h_2)$ are homothetical monomials of ratio δ for $\sigma' < \sigma < \sigma_0$. If either $\sigma' = 0$ or $in_{\sigma'}(h_1)$, $in_{\sigma'}(h_2)$ are not homothetical, then we take $\sigma_1 = \sigma'$. Otherwise $in_{\sigma'}(h_1)$ and $in_{\sigma'}(h_2)$ are homothetical, and their ratio is δ because their corresponding sides $\Gamma_\sigma(h_1)$, $\Gamma_\sigma(h_2)$ have lower ends p_1, p_2. Then we restart from σ' till getting σ_1 after finitely many steps. A similar argument gives σ_2. □

In the sequel open and closed intervals in the set of positive real numbers $\mathbb{R}^+$ will be allowed to have infinite length, that is, to be open or closed half-lines, $[a, \infty)$, (a, ∞), respectively. Intervals $(0, a]$ will be taken as (relatively) closed. An homographic function is a real function δ of the form $\delta(\sigma) = (a\sigma + b)/(c\sigma + d)$, for a, b, c, d real and $ac - bd \neq 0$. If $\{H_i\}$ is a family of closed intervals, no two sharing an interior point, the functions defined in $\bigcup_i H_i$ whose restrictions to each H_i are homographical will be said to be piecewise homographical in $\bigcup_i H_i$.

Now, if still $u = f(x, y), v = g(x, y)$ are local equations of φ, by Lemma 18.5 there are open intervals H_j, $j = 1, \ldots, \ell$, $\ell \geq 0$, so that $in_\sigma(f)$, $in_\sigma(g)$ are homothetical if and only if $\sigma \in H_j$, their ratio being constant for $\sigma \in H_j$. Take $\Xi = \mathbb{R}^+ - \bigcup_j H_j$. For $\sigma \in \Xi$, $in_\sigma(f)$, $in_\sigma(g)$ are not homothetical and therefore the algorithm for $\mathcal{T}_{q(\sigma)}$ ends after a single step, giving as Puiseux series of $\mathcal{T}_{q(\sigma)}$ $\mathcal{S}_\sigma = \theta u^{\rho(\sigma)}$, where $\rho(\sigma)$ is a piecewise homographical function in Ξ. Indeed, fix a slope $\sigma_0 \in \Xi$ and take σ_1 (σ_2) to be the highest (lowest) slope below (above) σ orthogonal to a side of either

$\mathbf{N}(f)$ or $\mathbf{N}(g)$, or 0 (∞) if there is no such. By its own definition, Ξ contains $[\sigma_1, \sigma_2]$ and for $\sigma \in (\sigma_1, \sigma_2)$ the σ-initial forms $in_\sigma(f)$, $in_\sigma(g)$ are monomials independent of σ, corresponding to certain vertices (c, d) and (a, b) of $\mathbf{N}(f)$ and $\mathbf{N}(g)$, respectively. Then

$$\rho(\sigma) = \frac{o_\sigma(g)}{o_\sigma(f)} = \frac{a + b\sigma}{c + d\sigma}.$$

Furthermore $ac - bd \neq 0$, as the vertices are not aligned with the origin due to Lemma 18.4. For $\sigma = \sigma_i \neq 0, \infty$, $i = 1, 2$, one of the initial forms is no longer a monomial, but still the σ-orders of f and g may be computed using the vertices (c, d) and (a, b). Therefore the above expression for $\rho(\sigma)$ still holds.

Now take one of the open intervals H_j and call it just H. By 18.5, the first step of the algorithm is equally performed for all $\sigma \in H$, giving rise to a new function $h \in \mathcal{O}_{T,O'}$, independent of σ. The above argument may then be repeated by taking h^* and H instead of g and $\mathbb{R}^+$: we get a set Ξ_1 which is the union of finitely many relatively closed intervals in H, so that for $\sigma \in \Xi_1$ the algorithm ends at this step and the Puiseux series of the trunks $\mathcal{T}_{q(\sigma)}$ have all the same constant part. As above, Ξ_1 splits in finitely many intervals so that in each interval

$$\frac{\mathcal{T}^2_{q(\sigma)}}{r_{q(\sigma)} n_{q(\sigma)}} = \frac{o_\sigma(h)}{o_\sigma(f)} = \frac{a' + b'\sigma}{c + d\sigma},$$

(a', b') a vertex of $\mathbf{N}(h)$ and $a'c - b'd \neq 0$. Then, using the equality (2) of Section 5, the degree of the variable part of the Puiseux series of the trunks $\mathcal{T}_{q(\sigma)}$ is again given by a piecewise homographical function in Ξ_1. The argument may be repeated within each of the finitely many open intervals $H - \Xi_1$ is composed of, and so on, in a procedure that need not to be finite and whose infiniteness may depend on the coordinates, as shown by the easy example $u = x + y$, $v = x/(1 - x)$. Summarizing, we have:

PROPOSITION 18.6. *There is a finite or countable family of open intervals with rational or infinite ends, $I_j = (\sigma_j, \sigma'_j)$, pairwise disjoint and such that $\mathbb{R}^+ \cup \{0, \infty\} = \bigcup_j [\sigma_j, \sigma'_j]$, for which we have:*

(1) *For all rational $\sigma \in I_j$, the Puiseux series of $\mathcal{T}_{q(\sigma)}$ may be written in the form*

$$\mathcal{S}_\sigma = \bar{\mathcal{S}}_j + \theta u^{\delta_j(\sigma)},$$

where their constant part $\bar{\mathcal{S}}_j$ is independent of σ and δ_j is a piecewise homographical function in $[\sigma_j, \sigma'_j]$.

(2) *But for ends equal to 0 or ∞, the constant parts of $\mathcal{S}_{\sigma_j}$ and $\mathcal{S}_{\sigma'_j}$ are partial sums of $\bar{\mathcal{S}}_j$.*

EXAMPLE 18.7. Take $f = x^9 - x^7 y - x^2 y^2 + y^3$, $g = -x^{10} + x^2 y^2$. The Puiseux series $\mathcal{S}_\sigma$ of the trunks $\mathcal{T}_{q(\sigma)}$ turn out to be:

$$\mathcal{S}_\sigma = \theta u^{\frac{2\sigma+2}{3\sigma}} \text{ for } \sigma \in (0,2]$$

$$\mathcal{S}_\sigma = -u + \theta u^{\delta(\sigma)} \text{ for } \sigma \in (2,7/2), \quad \delta(\sigma) = \begin{cases} \frac{3\sigma}{2\sigma+2} & \text{for } \sigma \in (2,3] \\ \frac{9}{2\sigma+2} & \text{for } \sigma \in [3,7/2) \end{cases}$$

$$\mathcal{S}_\sigma = \theta u^{\frac{2\sigma+2}{9}} \text{ for } \sigma \in [7/2,4]$$

$$\mathcal{S}_\sigma = -u^{\frac{10}{9}} + \theta u^{\delta'(\sigma)} \text{ for } \sigma \in (4,\infty), \quad \delta'(\sigma) = \begin{cases} \frac{2\sigma+2}{9} & \text{for } \sigma \in (4,6] \\ \frac{\sigma+8}{9} & \text{for } \sigma \in [6,\infty) \end{cases}.$$

The reader may consider the case in which $f = x$ and g is the equation of an irreducible germ having maximal contact with the x-axis. Then the jacobian is just the x-polar of $g = 0$ and the σ-algorithms eliminate from g monomials in x till getting a σ-initial form involving y. The slopes σ for which this σ-initial form is not a monomial correspond to branches of the x-polar of $g = 0$, as described in [5] and [6].

19. Infinitely near points the jacobian is going through . The points belonging to the jacobian germ $\mathbf{J}(\varphi)$ of a morphism φ have been characterized in terms of trunks in preceding Section 14. Next we will show other conditions for an infinitely near point to belong to the jacobian, mainly based on Theorem 12.2. Recall that in examples 10.10 and 12.7 we have seen that the jacobian of a morphism may miss some of its fundamental points, even if they are free and multiple.

PROPOSITION 19.1. *Let p be any point equal or infinitely near to O and denote by $\tilde{n}$ the top twist of $\mathcal{T}_p$, as already defined in 10.6. Then:*

(1) *A free non-fundamental point p' in the first neighbourhood of p belongs to $\mathbf{J}(\varphi)$ if and only if either p' is a multiple point of a non-special fibre of $\hat{\varphi}_p$, or it belongs with multiplicity higher than $\tilde{n}$ to the second special fibre of $\hat{\varphi}_p$*

(2) *All free fundamental points in the first neighbourhood of p and not in the first special fibre of $\hat{\varphi}_p$, do belong to $\mathbf{J}(\varphi)$*

(3) *All points on the contracted germ belong to $\mathbf{J}(\varphi)$*

Proof. By 14.2, the strict transform of $\mathbf{J}(\varphi)$ and $\mathbf{J}(\varphi_p)$ differ by germs of components of the exceptional divisor only. Thus claims (1) and (2) directly follow form 12.2 applied to φ_p. Claim (3) is obvious and has been included for the sake of completeness, as a direct computation easily shows that the contracted germ is a component of the jacobian. ◻

The case of a dicritical p is easy. Then $(\mathcal{T}_p)_{red} = \{O'\}$, thus any pencil of lines at O' goes sharply through $(\mathcal{T}_p)_{red}$ and therefore the pull-back at p of the pencil $\mathcal{P} : \alpha_1 f + \alpha_2 g = 0$, f, g the equations of φ, cuts on the first neighbourhood of p a linear series with fixed part B and, by 8.3, variable part the linear series of the fibres of $\hat{\varphi}_p$. Furthermore in this case there are no special fibres and therefore Proposition 19.1 gives the next corollary, whose first part reproves theorem 3 of [9].

COROLLARY 19.2. *If φ has equations f, g and p is a dicritical fundamental point of φ, then:*

1 *A free non-fundamental point in the first neighbourhood of p belongs to $\mathbf{J}(\varphi)$ if and only if it is a singular point of the only germ $\alpha_1 f + \alpha_2 g = 0$ going through it.*

2 *All free fundamental points in the first neighbourhood of p belong to $\mathbf{J}(\varphi)$.*

In the easiest non-dicritical cases we have:

COROLLARY 19.3. *Assume that $\mathcal{T}_p$ has either no characteristic exponent or a single characteristic exponent and satellite end. Write B the fundamental divisor of φ_p. Let ζ be any smooth germ going through the last free point of $\mathcal{T}_p$ and write $B' = TC(\varphi_p^*(\zeta))$. Then a free point p' in the first neighbourhood of p belongs to $\mathbf{J}(\varphi)$ if and only if it is multiple either in $B + B'$ or in its fibre $F_{p'} = \hat{\varphi}^*(\hat{\varphi}_*(p'))$.*

Proof. As above the free points on $\mathbf{J}(\varphi)$ are those on $\mathbf{J}(\varphi_p)$. Just take the local coordinates at O' with first coordinate axis ζ to have φ_p in the conditions of Example 10.8 and then use 12.3. □

When all the free points of $\mathcal{T}_p$ belong to one of the coordinate axes, the case of the above corollary has been already considered in [12],[13].

On the other hand, the case of a free non-fundamental point p may be easily handled using suitable coordinates at p, as the ones already used in sections 15 and 16. Indeed, if u, v are local coordinates at O', p being non-fundamental, all pull-backs $\varphi_p^*(\alpha_1 u + \alpha_2 v)$, $\alpha = \alpha_1/\alpha_2 \in \mathbb{C} \cup \{\infty\}$, define the same multiple of the exceptional divisor at p but for a single one, which defines a germ with higher multiplicity at p. Up to a linear change of coordinates at O', one may assume that the exceptional pull-back is $\varphi_p^*(v)$, after which, by choosing the first coordinate x at p to be a suitable equation of the germ of the exceptional divisor, the equations of φ_p take the form:

$$u = x^n, \quad v = x^{n'} h(x, y),$$

$x \not| h$, $n' \geq n$ and $n' > n$ if $h(0,0) \neq 0$. Then $\mathbf{J}(\varphi_p) : x^{n+n'-1} \partial h/\partial y = 0$ and since x does not divide $\partial h/\partial y$ either, the strict transform of the jacobian is $\widetilde{\mathbf{J}(\varphi)}_p : \partial h/\partial y = 0$. Thus p belongs to $\mathbf{J}(\varphi)$ if and only if $\partial h/\partial y$ vanishes at $p = (0,0)$.

As above, call $\mathcal{P}$ the pencil of the inverse images at O of the lines $\alpha_1 u + \alpha_2 v = 0$: if f, g are the equations of φ relative to any choice of local coordinates at O, then $\mathcal{P} : \alpha_1 f + \alpha_2 g = 0$. If p belongs to a germ $\xi \in \mathcal{P}$, this germ is necessarily unique, as p is not fundamental. In such a case, after our choice of coordinates at p, the strict transform of ξ is $\tilde{\xi}_p : h = 0$ and so the strict transform of the jacobian is a polar of the non-empty germ $\tilde{\xi}_p = 0$ relative to x, $\widetilde{\mathbf{J}(\varphi)}_p = P_x(\tilde{\xi}_p)$. As it is well known ([7], 6.1.7, for instance), the polar is empty if and only if $\tilde{\xi}_p$ is transverse to the exceptional divisor $x = 0$. Thus:

PROPOSITION 19.4. *Let φ have equations f, g and p be a free non-fundamental point that lies on one germ $\xi : \alpha_1 f + \alpha_2 g = 0$. Then the jacobian germ of φ goes through p if and only if p is a singular point of ξ.*

Note that if $\mathbf{J}(\varphi)$ goes through p, then the germ ξ is special in $\mathcal{P}$. One may even go a bit further. We have seen that the strict transform of $\mathbf{J}(\varphi)$ is a polar of $\tilde{\xi}_p$, although not necessarily a transverse one, as $\tilde{\xi}_p$ may be tangent to the exceptional divisor. Thus, the splitting of the polar germs in terms of the singularity of the germ (see [7], section 6.10) applies to $\widetilde{\mathbf{J}(\varphi)}_p$, which gives a partial splitting of $\mathbf{J}(\varphi)$ in terms of the singularities of the special germs of the pencil $\mathcal{P} : \alpha_1 f + \alpha_2 g = 0$ beyond the fundamental points. The reader may have noticed that the particular case in which ξ is either $f = 0$ or $g = 0$ is dealt with in [1].

A slightly more general reformulation of the above arguments may be obtained by modifying the choice of the coordinates at O' and hence the pencil $\mathcal{P}$. Let ζ be any smooth germ at O' and assume that the free non-fundamental point p belongs to

$\varphi^*(\zeta)$. Then the reader may easily check that a choice of coordinates as above may be made with v an equation of ζ. Then 19.4 gives:

PROPOSITION 19.5. *If a free non-fundamental point p lies on the inverse image of a smooth germ ζ, then the jacobian germ of φ goes through p if and only if p is a singular point of $\varphi^*(\zeta)$.*

Still in this case the strict transform of the jacobian is a polar of $\varphi^*(\zeta)$ and therefore splitting of polars again gives a partial splitting of the jacobian, this time in terms of the singularities of $\varphi^*(\zeta)$ at p and its infinitely near points.

It is worth noting that the condition of a non-fundamental point to belong to the inverse image of a smooth germ is a rather strong one, as we will show next. Recall that the intersection multiplicities of a fixed irreducible singular germ γ with smooth germs reach a maximum which is usually named the *maximal contact* of γ ([7], 5.2.4, for instance). Obviously the same occurs if either a multiple of an irreducible singular germ or a multiple of an irreducible cluster containing some satellite point is taken instead of γ. We will keep calling maximal contact the maximal intersection multiplicity with an smooth germ in these cases. We have:

LEMMA 19.6. *Assume that γ is an irreducible germ at O whose direct image contains some satellite point (i.e., $\varphi_*(\gamma)_{red}$ is singular). If κ is the maximal contact of $\varphi_*(\gamma)$, then no point on γ in the i-th neighbourhood of O, $i \geq \kappa$, has the inverse image of an smooth germ going through.*

Proof. For any smooth germ ξ at O', $[\gamma.\varphi^*(\xi)] = [\varphi_*(\gamma).\xi] \leq \kappa$ and the Noether formula for the intersection multiplicity ([7], 3.3.1) gives the claim. ☐

The next lemma may give an idea of how scarce are the points lying on inverse images of smooth germs:

LEMMA 19.7. *Let q be equal or infinitely near to O, assume that the trunk $\mathcal{T}_q$ contains some satellite point and call κ its maximal contact. For all but finitely many free points q' in the first neighbourhood of q, no inverse image of a smooth germ is going through the i-th point of a path $\{q, q', \dots\}$ if $i > \kappa$.*

Proof. By Theorem 11.1, for all but finitely many free q' in the first neighbourhood of q, all smooth germs with origin at q and going through q' have their direct images going through $\mathcal{T}_q$ with effective multiplicities equal to the virtual ones. This obviously implies that all these direct images have maximal contact κ, after which the claim follows from 19.6 above. ☐

The next corollaries give necessary conditions for a fundamental point to be missed by the jacobian germ. The first one is a direct consequence of 19.2, 19.4 and 19.5.

COROLLARY 19.8. *Assume that φ has equations f, g and take $\mathcal{P} = \{\xi : \alpha_1 f + \alpha_2 g = 0, \quad \alpha_1/\alpha_2 \in \mathbb{C} \cup \{\infty\}\}$. If p is a fundamental point of φ the jacobian germ is not going through, then:*

(1) *no dicritical fundamental point of φ, equal or infinitely near to p, has a free fundamental point in its first neighbourhood, and*

(2) *no inverse image of a smooth germ, and in particular no germ of $\mathcal{P}$, has a free non-fundamental singular point infinitely near to p.*

COROLLARY 19.9. *If p is a free fundamental point $\mathbf{J}(\varphi)$ does not go through, then no satellite fundamental point infinitely near to p is dicritical.*

Proof. Assume that q is a satellite dicritical fundamental point infinitely near to p. By taking the last free point preceding q instead of p, it is not restrictive to assume that q is a satellite of p. Take local coordinates x, y at p so that the germ of the exceptional divisor is the second axis $x = 0$. Then q is a x-satellite point and therefore, as explained in Section 18, it has associated slope $\sigma = \sigma(q) < 1$. Assume that f, g are the equations of φ_p and take $\xi_\lambda : h_\lambda = \lambda_1 f + \lambda_2 g = 0$, $\lambda = \lambda_1/\lambda_2 \in \mathbb{C} \cup \{\infty\}$. Take at q the coordinates of 18.1 and use a bar to denote pull-backs with origin at q. The point q being dicritical, by 19.8, neither a free point in the first neighbourhood of q is a fixed point of the linear series of the tangent cones to the germs $\bar{\xi}_\lambda$, nor may a such free point be multiple in one of the groups of the series. This easily implies that, up to a linear substitution of the parameters λ_1, λ_2, the initial forms of the $\bar{h}_\lambda$ have the form

$$\widehat{\bar{h}}_\lambda = \tilde{x}^a \tilde{y}^b (\lambda_1 \tilde{x}^d + \lambda_2 \tilde{y}^d),$$

$d > 0$. Then, by 18.2 we may perform a linear change of coordinates on the target to have

$$\begin{aligned} in_\sigma(f) &= x^\alpha y^\beta \\ in_\sigma(g) &= x^{\alpha'} y^{\beta'}, \end{aligned}$$

both monomials having the same σ-order and $\alpha\beta' - \alpha'\beta \neq 0$. Then

$$J(in_\sigma(f), in_\sigma(g)) = (\alpha\beta' - \alpha'\beta) x^{\alpha+\alpha'-1} y^{\beta+\beta'-1} \neq 0$$

and therefore

$$in_\sigma J(f, g) = (\alpha\beta' - \alpha'\beta) x^{\alpha+\alpha'-1} y^{\beta+\beta'-1}$$

Since $\mathbf{J}(\varphi)$ is assumed to miss p, $J(f, g)$ has the equation x of the exceptional divisor as its only factor, which forces $\beta + \beta' - 1 = 0$ and so, up to swapping over β, β', we may take $\beta = 0$, $\beta' = 1$. Now, the equality of σ-orders gives

$$\alpha = \alpha + \sigma\beta = \alpha' + \sigma\beta' = \alpha' + \sigma$$

against the already noted inequality $\sigma < 1$. ☐

To close, next is an easy example showing that a free multiple fundamental point missed by the jacobian may be followed by satellite fundamental (necessarily non-dicritical) points.

Example 19.10. Take φ to have equations $f = y^2 + 2x^5$, $g = y^3 + 3x^5 y$. Its fundamental points are the points on the germ $\gamma : y^2 + 2x^5 = 0$ up to the 8-th neighbourhood, with the same multiplicities they have on γ. The jacobian $\mathbf{J}(\varphi) : x^9 = 0$ misses the fundamental point in the first neighbourhood, which has multiplicity two and is followed by a satellite fundamental point in the third neighbourhood of the origin.

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