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Abstract

In this paper we give a generalization of the doubly $\mathcal{D}$-homothetically warped metric introduced by Blair [4], and we study the construction of Kählerian structure on the product of two almost contact metric structures. It is shown that if one factor is β-Kenmotsu, the other is β-Kenmotsu or α-Sasakian, and if one factor is cosymplectic, the other is α-Sasakian, but the product of two α-Sasakian is never Kählerian. Several examples are discussed.

AMS Subject Classification: 53C15; 53C55.

Keywords: Product manifolds, Trans-Sasakian structures, Kählerian structures.

1 Introduction

On the product of two almost contact manifolds, A. Morimoto [8] defined a natural almost complex structure (see (4.2) in this paper) and proved that this almost complex structure is integrable if and only if the two factors are normal almost contact manifolds. Later, M. Capursi [6] investigated almost Hermitian geometry of the product of two almost contact metric manifolds with the product metric, with respect to the almost complex structure defined by Morimoto. He shows that this product is Hermitian, Kählerian, almost Kählerian or nearly Kählerian, if and only if, the two factors are normal, cosymplectic, almost cosymplectic or nearly cosymplectic respectively.

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Extending ideas from Capursi and Morimoto, Blair and Oubiña [5] considered conformal and related changes of the product metric with respect to a family of almost complex structures (see relation (3.1)) containing the one of Morimoto. Under the Kähler condition on the product manifold, Blair and Oubina proved that if one factor is Sasakian, the other is not, but that locally the second factor is of the type studied by Kenmotsu. The resuls are more general and given in terms of trans-Sasakian, α-Sasakian and β-Kenmotsu structures, finally they asked the open question: What kind of change of the product metric will make both factors Sasakian?

In [10], Watanabe survey almost Hermitian, Kähler, almost quaternionic Hermitian and quaternionic Kähler structures, naturally constructed on products of manifolds with almost contact metric and Sasakian structures and open intervals, as an application of these constructions. Next, he investigated almost Hermitian structures, naturally defined on the product manifolds of two almost contact metric and Sasakian manifolds, and asked some problems related to these topics.

Here we introduce the notion of generalized doubly $\mathcal{D}$-homothetic bi-warping. We give an application to some questions of the characterization of certain geometric structures (Theorem 4.4), which is the main result, and it supports the view of Calabi-Eckmann manifold that almost Hermitian structures defined on the product of two Sasakian manifolds which are never Kählerian. This text is organized in the following way.
Section 2 is devoted to the background of the structures which will be used in the sequel.
In **section** 3 we introduce the notion of generalized doubly $\mathcal{D}$-homothetic bi-warping and prove some basic properties.
In **section** 4 we give an application to some questions of the characterization of certain geometric structures specially we study the construction of Kählerian structure on the product of two trans-Sasakian structures with examples.

2 Review of needed notions

For more background on almost complex structure manifolds, we recommend the reference [11].
An almost complex manifold with a Hermitian metric is called an almost Hermitian manifold. For an almost Hermitian manifold (M,J,g) we thus have

$$J^2 = -1, \qquad g(JX,JY) = g(X,Y).$$

An almost complex structure J is integrable, and hence the manifold is a complex manifold, if and only if its Nijenhuis tensor N_j vanishes, with

$$N_j(X,Y) = [JX,JY] - [X,Y] - J[X,JY] - J[JX,Y].$$

For an almost Hermitian manifold (M,J,g), we define the fundamental Kähler form Ω as:

$$\Omega(X,Y) = g(X,JY).$$

(M,J,g) is then called almost Kähler if Ω is closed i.e. $d\Omega = 0$. An almost Kähler manifold with integrable J is called a Kähler manifold, and thus is characterized by the conditions:

$d\Omega = 0$ and $N_J = 0$. One can prove that both of these conditions combined are equivalent with the single condition

$$\nabla J = 0.$$

An odd-dimensional Riemannian manifold (M^{2n+1}, g) is said to be an almost contact metric manifold if there exist on M a $(1,1)$ tensor field φ, a vector field ξ (called the structure vector field) and a 1-form η such that

$$\eta(\xi) = 1, \varphi^2(X) = -X + \eta(X)\xi \quad and \quad g(\varphi X, \varphi Y) = g(X,Y) - \eta(X)\eta(Y), \tag{2.1}$$

for any vector fields X,Y on M. In particular, in an almost contact metric manifold we also have $\varphi\xi = 0$ and $\eta \circ \varphi = 0$ (see [2] and [3]).

Such a manifold is said to be a contact metric manifold if $d\eta = \Phi$, where $\Phi(X,Y) = g(X, \varphi Y)$ is called the fundamental 2-form of M. If, in addition, ξ is a Killing vector field, then M is said to be a K-contact manifold. It is well-known that a contact metric manifold is a K-contact manifold if and only if $\nabla_X \xi = -\varphi X$, for any vector field X on M.
On the other hand, the almost contact metric structure of M is said to be normal if

$$N_\varphi(X,Y) = [\varphi,\varphi](X,Y) + 2d\eta\ (X,Y)\xi = 0, \tag{2.2}$$

for any X, Y, where $[\varphi,\varphi]$ denotes the Nijenhuis torsion of φ, given by

$$[\varphi,\varphi](X,Y) = \varphi^2[X,Y] + [\varphi X, \varphi Y] - \varphi[\varphi X, Y] - \varphi[X, \varphi Y].$$

In [9], the author proves that (φ,ξ,η,g) is trans-Sasakian structure if and only if it is normal and

$$\mathrm{d}\eta = \alpha\Phi, \qquad \mathrm{d}\Phi = 2\beta\eta\wedge\Phi, \tag{2.3}$$

where $\alpha = \frac{1}{2n}\delta\Phi(\xi)$, $\beta = \frac{1}{2n} div\xi$ and δ is the codifferential of g.
It is well known that the trans-Sasakian condition may be expressed as an almost contact metric structure satisfying

$$(\nabla_X\varphi)Y = \alpha\big(g(X,Y)\xi - \eta(Y)X\big) + \beta\big(g(\varphi X, Y)\xi - \eta(Y)\varphi X\big). \tag{2.4}$$

From this formula one easily obtains

$$\nabla_X\xi = -\alpha\varphi X - \beta\varphi^2 X, \tag{2.5}$$

$$(\nabla_X\eta)Y = \alpha g(X,\varphi Y) + \beta g(\varphi X, \varphi Y), \tag{2.6}$$

It is clear that a trans-Sasakian manifold of type $(1,0)$ is a Sasakian manifold and a trans-Sasakian manifold of type $(0,1)$ is a Kenmotsu manifold. A trans-Sasakian manifold of type $(0,0)$ is called a cosymplectic manifold. More generally, a trans-Sasakian structure (φ,ξ,η,g) on M is said to be

$$\begin{cases} (a): \ \alpha - Sasaki \quad if \quad \beta = 0, \\ (b): \ \beta - Kenmotsu \quad if \quad \alpha = 0, \\ (c): \ Cosymplectic \quad if \quad \alpha = \beta = 0. \end{cases} \tag{2.7}$$

where α and β are two constants (see [3] and [11]).

The relation between trans-Sasakian, α-Sasakian and, β-Kenmotsu structures was discussed by Marrero [7].

Proposition 2.1. *(Marrero [7]) A trans-Sasakian manifold of dimension ≥ 5 is either α-Sasakian, β-Kenmotsu or cosymplectic.*

Proposition 2.2. *(Marrero [7]) Let M be a 3-dimensional Sasakian manifold with structure tensors (φ,ξ,η,g) , $f > 0$ a non-constant function on M and $\overline{g} = fg + (1-f)\eta\otimes\eta$. Then $(\varphi,\xi,\eta,\overline{g})$ is a trans-Sasakian structure of type $(\frac{1}{f}, \frac{1}{2}\xi(\ln f))$.*

We can give a generalisation for the above proposition as follows:

Proposition 2.3. *Let $(M\varphi,\xi,\eta,g)$ be a 3-dimensional trans-Sasakian manifold of type (α,β), $f > 0$ a non-constant function on M and $\overline{g} = fg + (1-f)\eta\otimes\eta$. Then $(\varphi,\xi,\eta,\overline{g})$ is a trans-Sasakian structure of type $(\frac{\alpha}{f}, \beta + \frac{1}{2}\xi(\ln f))$.*

Proof Let (M,φ,ξ,η,g) be a 3-dimensional trans-Sasakian manifold of type (α,β), $f > 0$ a non-constant function on M and $\overline{g} = fg + (1-f)\eta\otimes\eta$. First, note that $(M,\varphi,\xi,\eta,\overline{g})$ is an almost contact metric manifold.
We have $\mathrm{d}\eta = \alpha\Phi$ and $\overline{\Phi} = f\Phi$, where Φ and $\overline{\Phi}$ are the fundamental 2-forms of g and $\overline{g}$ respectively, so

$$\begin{aligned} \mathrm{d}\overline{\eta} &= d\eta \\ &= \frac{\alpha}{f}\overline{\Phi}, \end{aligned}$$

and

$$\begin{aligned} \mathrm{d}\overline{\Phi} &= d(f\Phi) \\ &= 2\Big(\frac{1}{2}d(\ln f) + \beta\eta\Big)\wedge\overline{\Phi} \\ &= 2\Big(\frac{1}{2}\xi(\ln f) + \beta\Big)\eta\wedge\overline{\Phi} \end{aligned}$$

putting $\overline{\alpha} = \frac{\alpha}{f}$ and $\overline{\beta} = \frac{1}{2}\xi(\ln f) + \beta$, the proof is completed.

Proposition 2.4. *If $(M^{2n+1},\varphi,\xi,\eta,g)$ is a Kenmotsu manifold with η exact (i.e. $\eta = \mathrm{d}\rho$ where $\rho \in C^\infty(M)$) and $\tilde{g} = e^{-2\rho}g + (1-e^{-2\rho})\eta\otimes\eta$. Then $(\varphi,\xi,\eta,\tilde{g})$ is a cosymplectic structure.*

Proof Let $(M^{2n+1},\varphi,\xi,\eta,g)$ be a Kenmotsu manifold i.e.

$$\mathrm{d}\eta = 0, \qquad \mathrm{d}\Phi = 2\eta\wedge\Phi \qquad \textit{and} \qquad (\varphi,\xi,\eta) \textit{ is normal}.$$

For all X and Y vectors fields on M, we have

$$\begin{aligned} \tilde{\Phi}(X,Y) &= \tilde{g}(X,\varphi Y) \\ &= e^{-2\rho}g(X,\varphi Y) \\ &= e^{-2\rho}\Phi(X,Y) \end{aligned}$$

Then,

$$\begin{aligned} d\tilde{\Phi} &= d(e^{-2\rho}\Phi) \\ &= -2e^{-2\rho}d\rho\wedge\Phi + e^{-2\rho}d\Phi \\ &= 2e^{-2\rho}(\eta - d\rho)\wedge\Phi, \end{aligned}$$

so, if $\eta = d\rho$ then $d\tilde{\Phi} = 0$ and $(M^{2n+1},\varphi,\xi,\eta,g)$ is a cosymplectic manifold.

3 Generalized doubly $\mathcal{D}$-homothetic bi-warping

In [4], Blair introduced the notion of doubly $\mathcal{D}$-homothetically warped metric on $\tilde{M} = M'\times M$ where M' and M are two almost contact metric manifolds by

$$\tilde{g} = Fg' + F(F-1)\eta'\otimes\eta' + fg + f(f-1)\eta\otimes\eta,$$

where f is a positive function on M' and F is a positive function on M.

Our idea is to generalize this notion by putting

$$\tilde{g} = F^2g' + F^2(H^2-1)\eta'\otimes\eta' + f^2g + f^2(h^2-1)\eta\otimes\eta.$$

with f,h,F and H are smooth functions on $\tilde{M} = M'\times M$ such that $fh \neq 0$ and $FH \neq 0$ everywhere. We refer to this construction as generalized doubly $\mathcal{D}$-homothetic bi-warping. In particular, if $H = h = \pm 1$ then we have a generalized doubly warped product metric [5] and if $H = F$ and $h = f$ we get the generalized doubly $\mathcal{D}$-homothetically warped metric [4], but if $F = H = \pm 1$ we get a $\mathcal{D}$-homothetic bi-warped metric [1].

Using the Koszul formula for the Levi-Civita connection of a Riemannian metric, one can obtain the following:

Proposition 3.1. *Let $\tilde{\nabla},\nabla'$ and ∇ denote the Riemannian connections of $\tilde{g},g'$, and g respectively. For all X',Y' vector fields tangent to M' and independent of M and similarly for X,Y, we have*

$$\begin{aligned} 2\tilde{g}(\tilde{\nabla}_{X'}Y',Z') = {} & 2\tilde{g}(\nabla_{X'}Y',Z') \\ & + F^2(H^2-1)\Big((g'(\nabla'_{X'}\xi',Y') + g'(\nabla'_{Y'}\xi',X'))\eta'(Z') \\ & \qquad + 2d\eta'(X',Z')\eta'(Y') + 2d\eta'(Y',Z')\eta'(X')\Big) \\ & - \tilde{g}(X'(\ln F^2)\varphi'^2Y' + Y'(\ln F^2)\varphi'^2X',Z') \\ & + \tilde{g}\Big(X'(\ln(F^2H^2))\eta'(Y')\xi' + Y'(\ln F^2H^2)\eta'(X')\xi',Z'\Big) \\ & + \tilde{g}\Big(g'(\varphi'X',\varphi'Y')(\varphi'^2\text{grad}'(\ln F^2) - \frac{1}{H^2}\eta'(\text{grad}'(\ln F^2))\xi'),Z'\Big) \\ & + \tilde{g}\Big(\eta'(X')\eta'(Y')(\varphi'^2\text{grad}'(\ln F^2H^2) - \frac{1}{H^2}\eta'(\text{grad}'(\ln F^2H^2))\xi'),Z'\Big), \end{aligned}$$

$$\begin{aligned} 2\tilde{g}(\tilde{\nabla}_XY',Z') &= 2\tilde{g}(\tilde{\nabla}_{Y'}X,Z') = -2\tilde{g}(\tilde{\nabla}_{Z'}Y',X) \\ &= \tilde{g}\Big(-X(\ln F^2)\varphi'^2Y' + X(\ln(F^2H^2)\eta'(Y')\xi'),Z'\Big) \end{aligned}$$

$$\begin{aligned} 2\tilde{g}(\tilde{\nabla}_{X'}Y,Z) &= 2\tilde{g}(\tilde{\nabla}_Y X',Z) = -2\tilde{g}(\tilde{\nabla}_Z Y,X') \\ &= \tilde{g}\Big(-X'(\ln f^2)\varphi^2 Y + X'(\ln(f^2h^2))\eta(Y)\xi, Z\Big) \end{aligned}$$

$$\begin{aligned} 2\tilde{g}(\tilde{\nabla}_X Y,Z) = 2\tilde{g}(\nabla_X Y,Z) & \\ &+ f^2(h^2-1)\Big((g(\nabla_X\xi,Y)+g(\nabla_Y\xi,X))\eta(Z) \\ &\qquad + 2d\eta(X,Z)\eta(Y) + 2d\eta(Y,Z)\eta(X)\Big) \\ &- \tilde{g}(X(\ln f^2)\varphi^2 Y + Y(\ln f^2)\varphi^2 X, Z) \\ &+ \tilde{g}\Big(X(\ln(f^2h^2))\eta(Y)\xi + Y(\ln f^2h^2)\eta(X)\xi, Z\Big) \\ &+ \tilde{g}\Big(g(\varphi X,\varphi Y)(\varphi^2\mathrm{grad}(\ln f^2) - \frac{1}{h^2}\eta(\mathrm{grad}(\ln f^2))\xi), Z\Big) \\ &+ \tilde{g}\Big(\eta(X)\eta(Y)(\varphi^2\mathrm{grad}(\ln f^2h^2) - \frac{1}{h^2}\eta(\mathrm{grad}(\ln f^2h^2))\xi), Z\Big), \end{aligned}$$

Theorem 3.2. *Let $(M'\varphi',\xi',\eta',,g')$ and $(M,,\varphi,\xi,\eta,g)$ be two almost contact metric manifolds and $\tilde{g}$ the doubly $\mathcal{D}$-homothetically warped metric on $\tilde{M} = M'\times M$ so, we have the following assertions:*

1. The submanifold M is quasi umbilical if $grad'h = grad'f$ in which case its second fundamental form σ is given by

$$\sigma(X,Y) = -\frac{1}{2F^2}\Big(g(X,Y) + (h^2+fh-1)\eta(X)\eta(Y)\Big)\Big(grad' f^2 + \frac{1-H^2}{H^2}\xi'(f^2)\xi'\Big).$$

2. The submanifold M is minimal if and only if $h^2 = \frac{c}{f^2} - 2n$ where c is a positive constant in which case its second fundamental form σ is given by

$$\sigma(X,Y) = -\frac{1}{2F^2}\Big(g(X,Y) - (2n+1)\eta(X)\eta(Y)\Big)\Big(grad' f^2 + \frac{1-H^2}{H^2}\xi'(f^2)\xi'\Big).$$

3. If $\nabla_\xi\xi = 0$ then,

$$\tilde{\nabla}_\xi\xi = -\frac{1}{2F^2}\Big(grad'(f^2h^2) + \frac{1-H^2}{H^2}\xi'(f^2h^2)\xi'\Big).$$

Proof 1. Let σ the second fundamental form of M, we have

$$\begin{aligned} \tilde{g}(\tilde{\nabla}_X Y, Z') &= -fZ'(f)g(X,Y) - f\Big((h^2-1)Z'(f) + fhZ'(h)\Big)\eta(X)\eta(Y) \\ &= -\frac{1}{2}g'\Big(g(\varphi X,\varphi Y)grad' f^2 + \eta(X)\eta(Y)grad'(f^2h^2), Z'\Big) \\ &= -\frac{1}{2F^2}\tilde{g}\Big(g(\varphi X,\varphi Y)grad' f^2 + \eta(X)\eta(Y)grad'(f^2h^2), Z'\Big) \\ &\quad + \frac{1}{2}(H^2-1)\eta'\Big(g(\varphi X,\varphi Y)grad' f^2 + \eta(X)\eta(Y)grad'(f^2h^2)\Big)\eta'(Z') \\ &= -\frac{1}{2F^2}\tilde{g}\Big(g(\varphi X,\varphi Y)grad' f^2 + \eta(X)\eta(Y)grad'(f^2h^2) \\ &\qquad + \frac{1-H^2}{H^2}(g(\varphi X,\varphi Y)\xi'(f^2) + \eta(X)\eta(Y)\xi'(f^2h^2))\xi', Z'\Big) \end{aligned}$$

since $\tilde{g}(\nabla_X Y, Z') = 0$ and knowing that $\sigma = \tilde{\nabla}_X Y - \nabla_X Y$ then

$$\begin{aligned}\sigma(X,Y) &= \frac{1}{2F^2}\Big(g(\varphi X,\varphi Y)grad' f^2 + \eta(X)\eta(Y)grad'(f^2h^2) \\ &\quad + \frac{1-H^2}{H^2}(g(\varphi X,\varphi Y)\xi'(f^2) + \eta(X)\eta(Y)\xi'(f^2h^2))\xi'\Big) ...(*)\end{aligned}$$

If $grad'h = grad'f$ then we obtain

$$\sigma(X,Y) = -\frac{1}{2F^2}\Big(g(X,Y) + (h^2 + fh - 1)\eta(X)\eta(Y)\Big)\Big(grad' f^2 + \frac{1-H^2}{H^2}\xi'(f^2)\xi'\Big).$$

2. From the equation (*) we have

$$\mathcal{H} = \frac{1}{2n+1} tr_g\, \sigma = \frac{1}{2n+1}\sum_{i=1}^{2n+1}\sigma(e_i, e_i)$$

where $\{e_i\}_{i=1,2n+1}$ is an orthonormal basis on M so,

$$\begin{aligned}\mathcal{H} &= \frac{-1}{2(2n+1)F^2}\Big(2ngrad' f^2 + grad'(f^2h^2) + \frac{1-H^2}{H^2}(2n\xi'(f^2) + \xi'(f^2h^2))\xi'\Big) \\ &= \frac{-1}{2(2n+1)F^2}\Big(grad'((2n+h^2)f^2) + \frac{1-H^2}{H^2}\xi'((2n+h^2)f^2)\xi'\Big)\end{aligned}$$

we can notice easily that

$$\mathcal{H} = 0 \Leftrightarrow (2n+h^2)f^2 = c \Leftrightarrow h^2 = \frac{c}{f^2} - 2n,$$

with c is a positive constant. In this case, replacing $h^2 = \frac{c}{f^2} - 2n$ in equation (*) we obtain

$$\sigma(X,Y) = -\frac{1}{2F^2}\Big(g(X,Y) - (2n+1)\eta(X)\eta(Y)\Big)\Big(grad' f^2 + \frac{1-H^2}{H^2}\xi'(f^2)\xi'\Big).$$

3. From proposition (3.1), we have

$$\tilde{g}(\tilde{\nabla}_X Y, Z') = -fZ'(f)g(X,Y) - f((h^2-1)Z'(f) - fhZ'(h))\eta(X)\eta(Y),$$

so,

$$\begin{aligned}\tilde{g}(\tilde{\nabla}_\xi \xi, Z') &= -fZ'(f) - f((h^2-1)Z'(f) - fhZ'(h)) \\ &= -\frac{1}{2}\Big(Z'(f^2) + (h^2-1)Z'(f^2) + f^2Z'(h^2)\Big) \\ &= -\frac{1}{2}Z'(f^2h^2) \\ &= -\frac{1}{2}g'(grad'(f^2h^2), Z') \\ &= -\frac{1}{2F^2}\Big(\tilde{g}(grad'(f^2h^2), Z') - F^2(H^2-1)\eta'(grad'(f^2h^2))\eta'(Z')\Big) \\ &= -\frac{1}{2F^2}\tilde{g}\Big(grad'(f^2h^2) + \frac{1-H^2}{H^2}\xi'(f^2h^2)\xi', Z'\Big).\end{aligned}$$

On the other hand, we can easily prove that if $\nabla_\xi \xi = 0$ then

$$\tilde{g}(\tilde{\nabla}_\xi \xi, Z) = 0,$$

hence

$$\tilde{\nabla}_\xi \xi = -\frac{1}{2F^2}\Big(grad'(f^2h^2) + \frac{1-H^2}{H^2}\xi'(f^2h^2)\xi'\Big),$$

This completes the proof.

Next, we introduce a class of almost complex structure $\tilde{J}$ on the product manifold $\tilde{M}$:

$$\tilde{J}(X',X) \;=\; \Big(\varphi' X' - \frac{fh}{FH}\eta(X)\xi' \;,\; \varphi X + \frac{FH}{fh}\eta'(X')\xi\Big), \tag{3.1}$$

for any vector filds X' of M' and any vector filds X of M.

For $\frac{FH}{fh} = e^{2\mu}$ where μ is a function on $M' \times M$, we notice that J is the family of almost complex structures introduced by Blair-Oubiña [5] and for $\frac{FH}{fh} = 1$ we observe that the case corresponds to Morimotos almost complex structure [8].

That $J^2 = -I$ is easily checked and for all $\tilde{X} = (X',X), \tilde{Y} = (Y',Y)$ on $\tilde{M}$ we can see that $\tilde{g}$ is almost Hermitian with respect to $\tilde{J}$ i.e.

$$\tilde{g}(\tilde{J}(X',X), \tilde{J}(Y',Y)) = \tilde{g}((X',X),(Y',Y)).$$

Note that the fundamental 2-form of $(\tilde{J}, \tilde{g})$ is very simply as follows:

$$\tilde{\Omega} = F^2\phi' + f^2\phi + 2fhFH(\eta' \wedge \eta)$$

where we denote by $\phi'(X',Y') = g'(X',\varphi' Y')$ and $\phi(X,Y) = g(X,\varphi Y)$ for any vector fields X', Y' of M' and any vector fields X, Y of M.

It is easily observed that,

$$\begin{aligned} d\tilde{\Omega} \;&=\; 2FdF \wedge \phi' + F^2 d\phi' + 2f df \wedge \phi + f^2 d\phi \\ &+\; 2d(fhFH)(\eta' \wedge \eta) + 2fhFH(d\eta' \wedge \eta - \eta' \wedge d\eta). \end{aligned}$$

Remark 3.3. If ϕ', ϕ, η' and η are closed with f,h,F and H are constants then the 2-form $\tilde{\Omega}$ is closed and the structure $(\tilde{J}, \tilde{g})$ is almost Kählerian.

4 Application to geometric structures

In the remaining part of the paper, we consider M' and M as two almost contact metric manifolds and f,h,F and H as functions on $\tilde{M} = M' \times M$ such that $fh = FH = \pm 1$, i.e. we have .

$$\tilde{g} = F^2 g' + (1-F^2)\eta' \otimes \eta' + f^2 g + (1-f^2)\eta \otimes \eta, \tag{4.1}$$

and

$$\tilde{J}(X',X) \;=\; \Big(\varphi' X' - \eta(X)\xi' \;,\; \varphi X + \eta'(X')\xi\Big). \tag{4.2}$$

Using the proposition (3.1), one can obtain the following:

Proposition 4.1. *Let ∇', ∇ and $\tilde{\nabla}$ denote the Riemannian connections of $g', g,$ and $\tilde{g}$ respectively. For all X', Y' vector fields tangent to M' and independent of M and similarly for X, Y, we give the connection $\tilde{\nabla}$ explicitly:*

$$\begin{aligned}
\tilde{\nabla}_{X'}Y' &= \nabla'_{X'}Y' - \frac{\alpha'}{F^2}(1-F^2)(\eta'(Y')\varphi'X' + \eta'(X')\varphi'Y') \\
&- \frac{1}{2}\Big(Y'(\ln F^2)\varphi'^2X' + X'(\ln F^2)\varphi'^2Y'\Big) \\
&+ g'(\varphi'X', \varphi'Y')\Big(\beta'(1-F^2)\xi' + \frac{1}{2}(\varphi'^2 grad'(\ln F^2) \\
&\qquad - F^2\eta'(grad'(\ln F^2))\xi'\Big) \\
&+ \frac{1}{2f^2}\Big(\varphi^2 gradF^2 - f^2\eta(gradF^2)\xi\Big)g'(\varphi'X', \varphi'Y').
\end{aligned}$$

$$\tilde{\nabla}_{X'}Y = \tilde{\nabla}_Y X' = -\frac{1}{2}\Big(Y(\ln F^2)\varphi'^2X' + X'(\ln f^2)\varphi^2Y\Big),$$

$$\begin{aligned}
\tilde{\nabla}_X Y &= \nabla_X Y - \frac{\alpha}{f^2}(1-f^2)(\eta(Y)\varphi X + \eta(X)\varphi Y) \\
&- \frac{1}{2}\Big(Y(\ln f^2)\varphi^2X + X(\ln f^2)\varphi^2Y\Big) \\
&+ g(\varphi X, \varphi Y)\Big(\beta(1-f^2)\xi + \frac{1}{2}(\varphi^2 grad(\ln f^2) \\
&\qquad - f^2\eta(grad(\ln f^2))\xi\Big) \\
&+ \frac{1}{2F^2}\Big(\varphi'^2 grad' f^2 - F^2\eta'(grad' f^2)\xi'\Big)g(\varphi X, \varphi Y).
\end{aligned}$$

Knowing that $(\tilde{\nabla}_{\tilde{X}}\tilde{J})\tilde{Y} = \tilde{\nabla}_{\tilde{X}}(\tilde{J}\tilde{Y}) - \tilde{J}\tilde{\nabla}_{\tilde{X}}\tilde{Y}$ for all $\tilde{X}$ and $\tilde{Y}$ vectors fields on $\tilde{M}$ with using the proposition (4.1), we compute the covariant derivative of $\tilde{J}$.

$$\begin{aligned}
(\tilde{\nabla}_{X'}\tilde{J})Y' &= (\nabla'_{X'}\varphi')Y' - (\nabla'_{X'}\eta')(Y')\xi - \frac{1}{2}Y'(\ln F^2)\varphi'X' \\
&+ \frac{1}{2}\Big((\xi(\ln F^2) + \frac{2\alpha'}{F^2}(1-F^2))\eta'(Y') - \varphi'Y'(\ln F^2)\Big)\varphi'^2X' \\
&- \frac{1}{2}g'(\varphi'X', Y')\Big(2\beta'(1-F^2)\xi' + \varphi'\mathrm{grad}'(\ln F^2) \\
&\qquad - F^2\eta'(\mathrm{grad}'(\ln F^2))\xi'\Big) \\
&- \frac{1}{2f^2}g(\varphi'X', Y')\Big(\varphi\mathrm{grad}F^2 - f^2\eta(\mathrm{grad}(F^2))\xi\Big) \\
&+ \frac{1}{2}g'(\varphi'X', \varphi'Y')\Big(2\beta'(1-F^2)\xi + \varphi'\mathrm{grad}'(\ln F^2) \\
&\qquad - F^2\eta'(\mathrm{grad}'(\ln F^2))\xi\Big) \\
&+ \frac{1}{2f^2}g'(\varphi'X', \varphi'Y')\Big(\varphi\mathrm{grad}F^2 + f^2\eta(\mathrm{grad}(F^2))\xi'\Big),
\end{aligned}$$

$$\begin{aligned}(\tilde{\nabla}_X\tilde{J})Y' &= \eta'(Y')\nabla_X\xi+\Big(\frac{\alpha}{f^2}(1-f^2)\eta'(Y')+\frac{1}{2}Y'(\ln f^2)\Big)\varphi X\\ &- \frac{1}{2}\Big((\varphi'Y')(\ln f^2)-\xi(\ln f^2)\eta'(Y')\Big)\varphi^2X,\end{aligned}$$

$$\begin{aligned}(\tilde{\nabla}_{X'}\tilde{J})Y &= -\eta(Y)\nabla_{X'}\xi'-\Big(\frac{\alpha'}{F^2}(1-F^2)\eta(Y)-\frac{1}{2}Y(\ln F^2)\Big)\varphi'X'\\ &- \frac{1}{2}\Big((\varphi Y)(\ln F^2)-\xi'(\ln F^2)\eta(Y)\Big)\varphi'^2X',\end{aligned}$$

$$\begin{aligned}(\tilde{\nabla}_X\tilde{J})Y &= (\nabla_X\varphi)Y-(\nabla_X\eta)(Y)\xi'-\frac{1}{2}Y(\ln f^2)\varphi X\\ &+ \frac{1}{2}\Big((\xi'(\ln f^2)+\frac{2\alpha}{f^2}(1-f^2))\eta(Y)-\varphi Y(\ln f^2)\Big)\varphi^2X\\ &- \frac{1}{2}g(\varphi X,Y)\Big(2\beta(1-f^2)\xi+\varphi\mathrm{grad}(\ln f^2)-f^2\eta(\mathrm{grad}(\ln f^2))\xi\Big)\\ &- \frac{1}{2F^2}g(\varphi X,Y)\Big(\varphi'\mathrm{grad}'f^2-F^2\eta'(\mathrm{grad}'(f^2))\xi'\Big)\\ &+ \frac{1}{2}g(\varphi X,\varphi Y)\Big(2\beta(1-f^2)\xi'+\varphi\mathrm{grad}(\ln f^2)-f^2\eta(\mathrm{grad}(\ln f^2))\xi'\Big)\\ &+ \frac{1}{2F^2}g(\varphi X,\varphi Y)\Big(\varphi'\mathrm{grad}'f^2+F^2\eta'(\mathrm{grad}'(f^2))\xi\Big),\end{aligned}$$

Now, we can declare the following proposition:

Proposition 4.2. *Let $(M',\varphi',\xi',\eta',g')$ and (M,φ,ξ,η,g) be two almost contact metric manifolds. Consider the almost Hermitian structure $(\tilde{g},\tilde{J})$ on $M'\times M$ given in (4.1) and (4.2) with*

$$\begin{cases}\mathrm{grad}'f^2=-2\alpha\xi', & \mathrm{grad}\ln f^2=-2\beta\xi,\\ \mathrm{grad}F^2=2\alpha'\xi, & \mathrm{grad}'\ln F^2=-2\beta'\xi'.\end{cases}$$

Then, $(M'\times M,\tilde{g},\tilde{J})$ is Kählerian if and only if $(M',\varphi',\xi',\eta',g')$ and (M,φ,ξ,η,g) are two trans-Sasakian manifolds of type (α',β') and (α,β) respectively.

Proof. Replacing the formulas

$$\begin{cases}\mathrm{grad}'f^2=-2\alpha\xi', & \mathrm{grad}\ln f^2=-2\beta\xi,\\ \mathrm{grad}F^2=2\alpha'\xi, & \mathrm{grad}'\ln F^2=-2\beta'\xi'.\end{cases}$$

in the components of $\tilde{\nabla}\tilde{J}$, we get

$$\begin{aligned}(\tilde{\nabla}_{X'}\tilde{J})Y' =&(\nabla'_{X'}\varphi')Y'-\alpha'(g'(X',Y')\xi'-\eta'(Y')X')\\ &-\beta'(g'(\varphi'X',Y')\xi'-\eta'(Y')\varphi'X')\\ &+g'(\nabla'_{X'}\xi'-\alpha'\varphi'X'-\beta\varphi'^2X',Y')\xi,\end{aligned}$$

$$(\tilde{\nabla}_{X'}\tilde{J})Y = \eta(Y)\Big(\nabla'_{X'}\xi' - \alpha'\varphi' X' - \beta'\varphi'^2 X'\Big),$$

$$(\tilde{\nabla}_{X}\tilde{J})Y' = \eta'(Y')\Big(\nabla_{X}\xi - \alpha\varphi X - \beta\varphi^2 X\Big),$$

$$\begin{aligned}(\tilde{\nabla}_{X}\tilde{J})Y = &(\nabla_X\varphi)Y - \alpha(g(X,Y)\xi - \eta(Y)X)\\ &-\beta(g(\varphi X,Y)\xi - \eta(Y)\varphi X)\\ &+ g(\nabla_X\xi - \alpha\varphi X - \beta\varphi^2 X, Y)\xi'.\end{aligned}$$

Suppose that $(\tilde{M},\tilde{J},\tilde{g})$ is Kählerian i.e. $\tilde{\nabla}\tilde{J} = 0$, we get

$$(\nabla'_{X'}\varphi')Y' = \alpha'(g'(X',Y')\xi' - \eta'(Y')X') - \beta'(g'(\varphi' X',Y')\xi' - \eta'(Y')\varphi' X'),$$

$$\nabla'_{X'}\xi' = \alpha'\varphi' X' - \beta'\varphi'^2 X',$$

$$\nabla_X\xi = \alpha\varphi X - \beta\varphi^2 X,$$

$$(\nabla_X\varphi)Y = \alpha(g(X,Y)\xi - \eta(Y)X) - \beta(g(\varphi X,Y)\xi - \eta(Y)\varphi X),$$

i.e. M' and M are two trans-Sasakian manifolds of type (α',β') and (α,β) respectively.

Conversely, suppose that M' and M are two trans-Sasakian manifolds of type (α',β') and (α,β) respectively. It is clear that if we use the formulas (2.4)-(2.6) in the components of $\tilde{\nabla}\tilde{J}$ above, we get directly $\tilde{\nabla}\tilde{J} = 0$. □

Remark 4.3. A similar result was obtained by Blair and Oubiña [5] using the generalized doubly warped product.

Now suppose that M' is β'-Kenmotsu, then from the above proposition we get $\mathrm{grad}' \ln F^2 = -2\beta'\xi'$ and F is independent of M. Knowing that, if η' is exact i.e. $\eta' = d\rho'$ where ρ' is a function on M' then $\xi' = \mathrm{grad}'\rho'$.
In addition, if M' is a connected manifold then

$$F^2 = e^{-2\beta'\rho'}.$$

On the other hand, if M is α-Sasakian then from the proposition (4.2) we obtain $\mathrm{grad}' f^2 = -2\alpha\xi'$ and f is independent of M. Thus we obtain

$$f^2 = -2\alpha\rho',$$

and we have the following theorem:

Theorem 4.4. *Let M' and M be almost contact metric manifolds. Consider the almost Hermitian structure $(\tilde{g},\tilde{J})$ on $M'\times M$ given in (4.1) and (4.2). Then,*

(1): If $F^2 = e^{-2\beta'\rho'}$ and $f^2 = e^{-2\beta\rho}$ then,
$M'\times M$ is Kählerian if and only if M' is β'-Kenmotsu connected manifold and M is β-Kenmotsu connected manifold.

(2): If $F^2 = e^{-2\beta'\rho'}$ and $f^2 = -2\alpha\rho'$ then,
$M'\times M$ is Kählerian if and only if M' is β'-Kenmotsu connected manifold and M is α-Sasakian manifold.

(3): If F = constant and $f^2 = -2\alpha\rho'$ then,
$M'\times M$ is Kählerian if and only if M' is cosymplectic connected manifold and M is α-Sasakian manifold.

(4): If F and f are constant then,
$M'\times M$ is Kählerian if and only if M' and M are cosymplectic connected manifolds.

Proof. We can draw α, α', β and β' from the initial conditions, then using proposition (4.2) and the different cases given in (2.7) we obtain directly the four results of the theorem. □

Remark 4.5. In this theorem, the result in the case (4) was gotten by M. Capursi ([6]).

Proposition 4.6. *Let $(M',\varphi',\xi',\eta',g')$ and (M,φ,ξ,η,g) be two Sasakian manifolds. the manifold $(M'\times M,\tilde{J},\tilde{g})$ equipped with the Hermitian structure $(\tilde{J},\tilde{g})$ defined by (4.1) and (4.2) is never Kähler.*

Proof Suppose that M' and M are two Sasakian manifolds i.e. $\alpha' = \alpha = 1$, then from proposition (4.2), we get

$$\mathrm{grad}' f^2 = -2\xi', \quad \textit{and} \quad \mathrm{grad} F^2 = 2\xi,$$

with f is independent of M and F is independent of M'.
But knowing that ξ is Killing i.e.

$$g(\nabla_X\xi,Y)+g(\nabla_Y\xi,X) = 0,$$

the following relation holds for all X,Y on M

$$g(\nabla_X \mathrm{grad} F^2,Y) = 0,$$

giving

$$\nabla_X \mathrm{grad} F^2 = 0 \Rightarrow \nabla_X\xi = 0,$$

and since M supposed Sasakian so, $\varphi X = 0$ for all X on M i.e. $\varphi = 0$, a contradiction.

Exercise 4.7. Let (x,y,z) be cartesian coordinates on E^3 and put

$$\xi = \frac{\partial}{\partial z}, \qquad \eta = -\tau dx + dz,$$

$$\varphi = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & -\tau & 0 \end{pmatrix}, \quad g = \begin{pmatrix} \rho^2+\tau^2 & 0 & -\tau \\ 0 & \rho^2 & 0 \\ -\tau & 0 & 1 \end{pmatrix}$$

where ρ and τ are functions on E^3 such that $\rho \neq 0$ everywhere.
Then the structure (φ,ξ,η,g) is a:

(1) Sasaki when $\tau_2 = -2\rho^2$ and $\tau_3 = 0$,

(2) Cosymplectic when $\rho_3 = 0$, $\tau_2 = 0$, and $\tau_3 = 0$,

(3) Kenmotsu when $\rho_3 = \rho$, $\tau_2 = 0$ and $\tau_3 = 0$.

where $\rho_i = \frac{\partial\rho}{\partial x_i}$ and $\tau_i = \frac{\partial\tau}{\partial x_i}$ [1].
Now, we use the product of the almost contact metric manifold (E^3,φ,ξ,η,g) by itself, i.e. using (4.1) and (4.2) with a straightforward computation we can get the associated matrices of $\tilde{g}$ and $\tilde{J}$ on $E^6 = E'^3 \times E^3$

$$\tilde{g} = \begin{pmatrix} F^2\rho'^2+\tau'^2 & 0 & -\tau' & 0 & 0 & 0 \\ 0 & F^2\rho'^2 & 0 & 0 & 0 & 0 \\ -\tau' & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & f^2\rho^2+\tau^2 & 0 & -\tau \\ 0 & 0 & 0 & 0 & f^2\rho^2 & 0 \\ 0 & 0 & 0 & -\tau & 0 & 1 \end{pmatrix},$$

$$\tilde{J} = \begin{pmatrix} 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\tau' & 0 & \tau & 0 & -1 \\ 0 & 0 & 0 & 0 & 0-1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ -\tau' & 0 & 1 & 0 & -\tau & 0 \end{pmatrix}.$$

Using the above cases in theorem (4.4), the manifold $(E^6,\tilde{g},\tilde{J})$ is Kählerian if and only if one of these assertions is satisfied:

(1): $\rho'_3 = \rho', \quad \tau'_2 = \tau'_3 = 0, \quad F^2 = e^{-2\beta'(z'-\int\tau' dx')},$
$\rho_3 = \rho, \quad \tau_2 = \tau_3 = 0, \quad f^2 = e^{-2\beta(z-\int\tau dx)}.$

(2): $\rho'_3 = \rho', \quad \tau'_2 = \tau'_3 = 0, \quad F^2 = e^{-2\beta'(z'-\int\tau' dx')},$
$\tau_2 = -2\rho^2, \quad \tau_3 = 0 \quad f^2 = -2\alpha(z' - \int\tau' dx').$

(3): $\rho'_3 = \tau'_2 = \tau'_3 = 0, \quad F = constant,$
$\tau_2 = -2\rho^2, \quad \tau_3 = 0 \quad f^2 = -2\alpha(z' - \int\tau' dx').$

(4): $\rho'_3 = \tau'_2 = \tau'_3 = \rho_3 = \tau_2 = \tau_3 = 0, \quad F$ and f are constants.

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