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About the Degenerate Spectrum of the Tension Field for Mappings into a Symmetric Riemannian Manifold

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Abstract. Let (M,g) and (N,h) be compact Riemannian manifolds, where (N,h) is symmetric, $v \in W^{1,2}((M,g),(N,h))$, and τ is the tension field for mappings from (M,g) into (N,h). We consider the nonlinear eigenvalue problem $\tau(u) - \lambda \exp_u^{-1} v = 0$, for $u \in W^{1,2}(M,N)$ such that $u_{|\partial M} = v_{|\partial M}$, and $\lambda \in \mathbb{R}$. We prove, under some assumptions, that the set of all λ, such that there exists a solution (u,λ) of this problem and a non trivial Jacobi field V along u, is contained in $\mathbb{R}_+$, is countable, and has no accumulation point in $\mathbb{R}$. This result generalizes a well known one about the spectrum of the Laplace - Beltrami operator Δ for functions from (M,g) into $\mathbb{R}$.

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1 Introduction

Let (M,g) and (N,h) be Riemannian manifolds, and $u : M \to N$ a smooth mapping. Let $v :]-1,1[\times]-1,1[\times M \to N$, $(r,s,x) \mapsto v_{r,s}(x)$ be a C^2 - mapping such that $v_{0,0} = u$, and $v_{r,s|\partial M} = u_{|\partial M}, \forall (r,s) \in]-1,1[^2$ in case $\partial M \neq \emptyset$. The *energy* of u is

$$E(u) = \frac{1}{2}\int_M \|du\|^2(x)dx,$$

where dx is the Riemannian measure on (M,g).

In local coordinates one has

$$\|du\|^2(x) = g^{ij}(x)\frac{\partial u^\alpha}{\partial x^i}(x)\frac{\partial u^\beta}{\partial x^j}(x)h_{\alpha\beta}(u(x)).$$

It is well known, see e.g. [8], that

$$\begin{aligned}\frac{\partial^2}{\partial r \partial s}|_{r=s=0} E(v_{r,s}) &= \int_M \left\langle -[\nabla_{e_i}\nabla_{e_i} V - \nabla_{\nabla_{e_i} e_i} V] - R^N(V, d_{e_i}u)d_{e_i}u, W\right\rangle(x)dx \\ &\quad - \int_M \left\langle \nabla_r \frac{\partial v_{r,s}}{\partial s}|_{r=s=0}, \tau(u)\right\rangle(x)dx,\end{aligned}$$

where

$$V := \frac{\partial v_{r,s}}{\partial r}|_{r=s=0} \text{ and } W := \frac{\partial v_{r,s}}{\partial s}|_{r=s=0}$$

are vector fields along u,

$$\tau(u) := \text{trace } (\nabla du) = \nabla_{e_i} d_{e_i} u - d_{\nabla_{e_i} e_i} u$$

is the *tension field* of u, and $(e_i)_i$ is a local orthonormal frame. We are using the summation convention of Einstein.

If one assumes that $\tau(u) = 0$, i.e. u is *harmonic*, then one has

$$\frac{\partial^2}{\partial r \partial s}|_{r=s=0} E(v_{r,s}) = \int_M \left\langle -[\nabla_{e_i}\nabla_{e_i} V - \nabla_{\nabla_{e_i} e_i} V] - R^N(V, d_{e_i}u)d_{e_i}u, W \right\rangle (x)dx.$$

For a harmonic mapping u, some $V \in \Gamma(u^{-1}(TN))$ is called a *Jacobi field* along u when

$$\nabla_{e_i}\nabla_{e_i} V - \nabla_{\nabla_{e_i} e_i} V + R^N(V, d_{e_i}u)d_{e_i}u = 0 \text{ on } M.$$

One sees also that when $r \mapsto v_{r,0}$ is a geodesic, then even when u is not harmonic, one has

$$\frac{d^2}{dr^2}|_{r=0} E(v_{r,0}) = \int_M \left\langle -[\nabla_{e_i}\nabla_{e_i} V - \nabla_{\nabla_{e_i} e_i} V] - R^N(V, d_{e_i}u)d_{e_i}u, V \right\rangle (x)dx.$$

The existence of non vanishing Jacobi fields along a harmonic mapping u makes it difficult to know whether u is locally energy minimizing or not, and it gives informations about the uniqueness of u in its homotopy class. When (N,h) has nonpositive sectional curvature, it has been proved by Hartman in [5] that such a Jacobi field V satisfies

$$\nabla V = 0 \text{ and } \left\langle R^N(V, d_{e_i}u)d_{e_i}u, V \right\rangle = 0 \text{ on } M.$$

In our work [12] we tried to extend in some way this result of Hartman to cases where the sectional curvature of (N,h) is no more nonpositive, but (N,h) being symmetric. We proved (roughly said) in that work that given such a Jacobi field V, if it is *integrable*, i.e. there exists $v :]-1,1[\times M \to N$ a smooth mapping such that $v(0,.) = u$, $v(t,.)$ is harmonic, for any $t \in]-1,1[$ and $V(x) = \frac{\partial v(t,x)}{\partial t}|_{t=0}, \forall x \in M$, then $\nabla[||V||^{-1} V] = 0$.

In our work [9] we introduced, together with Prof. Jost, the functional

$$E_\lambda(u) = \frac{1}{2}[\int_M ||du||^2 (x)dx - \lambda \int_M d^2(u(x), w(x))dx]$$

for some fixed $\lambda \in \mathbb{R}$ and $w \in C^1(M,N)$, where $d(.,.)$ is the Riemannian distance function on (N,h). We have been motivated by the sake of developing a generalisation of the eigenvalue problem of the Laplace - Beltrami operateur Δ, as it has been done by J. Eells and J. H. Sampson in [3] to generalize the concept of harmonic functions to the one of harmonic mappings between Riemannian manifolds. For negative λ this functional generalizes also the Mumford - Shah functional (which is used in image approximation, see e.g. [1]) to the case of mappings between Riemannian manifolds.

We have from [9]

$$\frac{d}{dt}_{|t=0} E_\lambda(v_{t,0}) = -\int_M \left\langle \tau(u) - \lambda \exp_u^{-1} w, V \right\rangle dx.$$

So, u is a *critical point* of E_λ if and only if

$$L_\lambda(u) := L(\lambda, u) := \tau(u) - \lambda \exp_u^{-1} w = 0. \tag{1.1}$$

A solution u of (1.1) is called an *eigenmapping* of the tension field τ associated to the *eigenvalue* λ (for the model mapping w). In [9] we proved that the spectrum of τ in this sense may be continuous and the set of eigenvalues and eigenmappings may bifurcate, even when (N,h) has nonpositive sectional curvature. This eigenvalue problem generalizes the one for the Laplace - Beltrami operator Δ for functions defined on (M,g), since, for $(N,h)=\mathbb{R}$ and $w=0$ one has

$$\tau(u)-\lambda\exp_u^{-1}w=\Delta u+\lambda u.$$

In our work [11] we proved some first eigenvalue estimates for τ. In these studies, the case where the model mapping w is harmonic is the most close to the case of the real valued functions.

From [9] we have

$$\begin{aligned}\frac{\partial^2}{\partial r\partial s}|_{r=s=0}E_\lambda(v_{r,s}) &= \int_M[\left\langle-[\nabla_{e_i}\nabla_{e_i}V-\nabla_{\nabla_{e_i}e_i}V],W\right\rangle(x)\\ &\quad-\left\langle R^N(V,d_{e_i}u)d_{e_i}u-\lambda\nabla_V\exp_.^{-1}w,W\right\rangle(x)]dx,\end{aligned}$$

where u is a solution of (1.1).

$V\in\Gamma(u^{-1}(TN))$ is called a *Jacobi field* along a solution u of (1.1) when

$$\nabla_V L_\lambda(u):=\nabla_{e_i}\nabla_{e_i}V-\nabla_{\nabla_{e_i}e_i}V+R^N(V,d_{e_i}u)d_{e_i}u-\lambda\nabla_V\exp_.^{-1}w=0. \tag{1.2}$$

In the present work we make a qualitative study of the solutions of the equations (1.1) and (1.2).

For a fixed (λ,u) the equation (1.2) is linear in the unknown V. This equation is very close to the linear elliptic partial differential equations, at the difference that V is not here a function, but a section of some vector bundle.

Let's fix u. Then equation (1.2) gives a linear eigenvalue problem. We prove in this work that the spectrum for this problem is made of a nondecreasing sequence, which converges to $+\infty$. Furthermore, each eigenspace is a finite dimensional real vector space, the first eigenvalue is simple and the corresponding eigenvectors are positive in some sense made precise in Theorem 2. So, the spectral problem defined by (1.2) has many of the properties of the spectrum of the Laplace - Beltrami operator Δ. Actually, our proofs rely heavily on the same ideas as for the spectral problem for Δ, as they can be found in the books [4] and [7].

For the solutions of (1.1), we know already a few, as we said above. One can see that (1.1) is not linear in u. In the present work we are interested in those λ (called *degenerate eigenvalue* of τ) such that, there exists a solution u of (1.1) and a nontrivial solution V of (1.2). The set of all such λ is called the *degenerate spectrum* of τ. We will prove that the degenerate spectrum is

1°) nonnegative

2°) finite, or is made of a sequence which converges to $+\infty$. Furthermore, if λ_2^* is the smallest degenerated eigenvalue of τ, then the set of the corresponding Jacobi fields is a one dimensional real vector space, and those Jacobi fields are nonnegative in the sense of Theorem 2. So the degenerate spectrum has also many of the properties of the spectrum of Δ. One can see that, when $(N,h)=\mathbb{R}$ and $w=0$, the equations (1.1) and (1.2) are the same. We will see also that the spectrum of τ is bounded from below.

The proof that the degenerate spectrum is discrete is based on the bifurcation property which we studied in [9], and the proved fact that the sets of solutions of our equations (1.1) and (1.2) are closed under $W^{1,2}$- weak convergence. The proof that λ_2^* is simple is just an adaptation of the same proof for equation (1.2): Here, both (1.1) and (1.2) have to be satisfied at the same time. To prove the nonnegativeness of the degenerate spectrum, we show first that a Jacobi field corresponding to λ_2^* is integrable, and then use some nonexistence result of integrable Jacobi fields for negative λ

which is in [13]. Here, we must point out an important typewriting mistake in [13]: In Theorem 2.6 of that work, the assumption " $0 < K_1 \leq Riem^{(N,h)} \leq K$, should be " $-K_1 \leq Riem^{(N,h)} \leq K$, with $K > 0$ and $K_1 > 0$ ", so one must replace K_1 by $-K_1$ everywhere.

2 Definitions and results

2.1 Definitions

2.1.1. Let us assume that (N,h) is isometrically embedded into some Euclidean space $\mathbb{R}^k$. Then

$$W^{1,2}(M,N) := \{v \in W^{1,2}(M,\mathbb{R}^k) \ / \ v(x) \in N \text{ for a.e. } x \in M\}$$

where $W^{1,2}(M,\mathbb{R}^k)$ is the usual Sobolev space of all maps in $L^2(M,\mathbb{R}^k)$ whose derivative in the sense of distributions is square integrable.
2.1.2. Let $v \in W^{1,2}(M,N)$. One can extend the metric h on N to TN by using the Levi - Civita connexion of (N,h), to get a Riemannian metric h' on TN. By using the isometric embedding of (TN,h') into some Euclidean space $\mathbb{R}^q$, one can define

$$\begin{aligned} L^2(v^{-1}(TN)) \quad &: \quad = \Gamma(v^{-1}(TN)) \cap L^2(M,\mathbb{R}^q) \text{ and} \\ \Gamma_0^{1,2}(v^{-1}(TN)) \quad &: \quad = W^{1,2}(M,TN) \cap \{V \in \Gamma(v^{-1}(TN)) \ / \ V_{|\partial M} = 0\}, \end{aligned}$$

where $\Gamma(v^{-1}(TN))$ is the vector space of all sections of the pullback bundle $v^{-1}(TN)$.
2.1.3. For any $y \in N$, $inj(y)$ is the injectivity radius of the Riemannian manifold (N,h) at the point y.
For $x,y \in N$ such that $d(x,y) < inj(x)$, $\|_x^y$ will denote the parallel transport from x to y along the unique minimizing geodesic going from x to y.
2.1.4. R^N is the curvature tensor of (N,h) and $Riem^{(N,h)}$ is the sectional curvature of (N,h). ∇ designates invariably the Levi - Civita covariant derivative, and the from it defined covariant derivatives on tensors. $Ker(\nabla_{.}L)$ is the Kernel of the linear operator $\nabla_{.}L$. $\forall y \in N, \exp_y$ is the usual exponential mapping which is defined from some neighborhood of 0 in T_yN into N.
2.1.5. Let $u \in C^2(M,N)$, U an open subset of M on which there is a coordinates system $(x^1,...,x^m)$, such that there exists a coordinates system $(y^1,...,y^n)$ on some neighborhood of $u(U)$. Then: $\forall x \in U$ we have

$$\tau(u)(x) =$$

$$\begin{aligned} &= g^{ij}(x)[\frac{\partial^2 u^\alpha}{\partial x^i \partial x^j}(x) - \frac{\partial u^\alpha}{\partial x^k}(x)\ {}^M\Gamma_{ij}^k(x) + \frac{\partial u^\beta}{\partial x^i}(x)\frac{\partial u^\delta}{\partial x^j}(x)\ {}^N\Gamma_{\beta\delta}^\alpha(u(x))]\frac{\partial}{\partial y^\alpha}(u(x)) \\ &= [\Delta u^\alpha(x) + g^{ij}(x)\frac{\partial u^\beta}{\partial x^i}(x)\frac{\partial u^\delta}{\partial x^j}(x)\ {}^N\Gamma_{\beta\delta}^\alpha(u(x))]\frac{\partial}{\partial y^\alpha}(u(x)), \end{aligned}$$

where ${}^M\Gamma_{ij}^k(x)$ is the Christofell symbol, and Δ is the Laplace - Beltrami operator of (M,g).
2.1.6. Some $V \in \Gamma_0^{1,2}(v^{-1}(TN))$ is said to be *harmonic* when it is a weak solution of the equation

$$\nabla_{e_i}\nabla_{e_i}V - \nabla_{\nabla_{e_i}e_i}V = 0.$$

2.1.7. For $V,W \in \Gamma_0^{1,2}(v^{-1}(TN))$

$$\langle V,W\rangle_{L^2} := \int_M \langle V,W\rangle(x)dx.$$

For $u, v \in L^2(M,N)$,

$$d_{L^2}(u,v)^2 := \int_M d^2(u(x),v(x))dx.$$

2.1.8. Throughout this work $w \in C^1(M,N)$ and $\theta \in C^0(M,\mathbb{R})$ is such that

$$\theta(x) \in]0, \min\{inj(w(x)), \frac{\pi}{2\sqrt{K}}\}[, \forall x \in M,$$

where $K > 0$ is an upper bound for the sectional curvature of (N,h). We set

$$W^{1,2}(M,N)_w :=$$

$$\{v \in W^{1,2}(M,N) \ / \ v_{|\partial M} = w_{|\partial M} \text{ and } d(w(x),v(x)) \leq \theta(x), \forall x \in M\},$$

and for any $v \in W^{1,2}(M,N)_w$,

$$Q_v(V) := \nabla_{e_i}\nabla_{e_i}V - \nabla_{\nabla_{e_i}e_i}V + R^N(V, d_{e_i}v)d_{e_i}v, \ \forall V \in \Gamma_0^{1,2}(v^{-1}(TN)),$$

and

$$\mathcal{E}_{\lambda,v} := \{V \in \Gamma_0^{1,2}(v^{-1}(TN))\backslash\{0\} \ / \ Q_v(V) - \lambda\nabla_V \exp_{\cdot}^{-1} w = 0\}, \forall \lambda \in \mathbb{R}.$$

2.2 Results

Theorem 2.1. *Let (M,g) and (N,h) be compact Riemannian manifolds.*
Let $v \in W^{1,2}(M,N)_w$ and

$$\forall \chi > 0, \ \Lambda_\chi := \Big\{\lambda \in \mathbb{R} \ / \ \exists u \in W^{1,2}(M,N)_w \text{ such that } d_{L^2}(u,w) \leq \chi, \ L_\lambda(u) = 0, \text{ and } \mathcal{E}_{\lambda,u} \neq \emptyset\}\Big\}.$$

Then:

1°) **a)** *For any $\lambda \in \mathbb{R}$, the real vector space $\mathcal{E}_{\lambda,v} \cup \{0\}$ has finite dimension.*

b) *The set of all $\lambda \in \mathbb{R}$ such that $\mathcal{E}_{\lambda,v} \neq \emptyset$, is a nondecreasing sequence which converges to $+\infty$.*

We assume that (N,h) is symmetric. Then:

2°) *There exists $\chi > 0$ such that Λ_χ is either finite, or there exists a nondecreasing sequence $(\lambda_n)_{n\in\mathbb{N}}$ such that $\lim_{n\to+\infty} \lambda_n = +\infty$ and $\Lambda_\chi = \{\lambda_n \ / \ n \in \mathbb{N}\}$.*

3°) *If χ is as in 2°) and the model mapping w is harmonic, then Λ_χ is not finite.*

Theorem 2.2. *Let (M,g) and (N,h) be compact Riemannian manifolds with (N,h) symmetric.*
Assume $v \in W^{1,2}(M,N)_w$ and λ_1 is the infimum of all $\lambda \in \mathbb{R}$ such that $\mathcal{E}_{\lambda,v} \neq \emptyset$.
Then:

1°) $\mathcal{E}_{\lambda_1,v} \neq \emptyset$.

Furthermore, under the assumption that

$$\left\langle R^N(X,Y)Y,Z\right\rangle = 0 \text{ when } \langle X,Z\rangle = 0$$

we have:

2°) *The real vector space* $\mathcal{E}_{\lambda_1,v} \cup \{0\}$ *is one dimensional.*

3°) *For* $V \in \mathcal{E}_{\lambda_1,v}$ *and* $W \in \Gamma_0^{1,2}(v^{-1}(TN))$ *harmonic such that* $W(x) \neq 0, \forall x \in M$*, we have either* $\langle V, W\rangle \equiv 0$*,* $\langle V, W\rangle > 0$ *or* $\langle V, W\rangle < 0$*.*

Theorem 2.3. *Let* (M, g) *and* (N, h) *be compact Riemannian manifolds with* (N, h) *symmetric. Let* λ_2^* *be the infimum of all* $\lambda \in \mathbb{R}$ *such that, there exists* $v \in W^{1,2}(M, N)_w$ *such that*

$$L_\lambda(v) = 0 \text{ and } \mathcal{E}_{\lambda,v} \neq \emptyset.$$

Then:

1°) *There exists* $v \in C^2(M, N) \cap W^{1,2}(M, N)_w$ *such that*

$$L_{\lambda_2^*}(v) = 0 \text{ and } \mathcal{E}_{\lambda_2^*,v} \neq \emptyset.$$

2°) *Let's assume that*

$$\left\langle R^N(X, Y)Y, Z\right\rangle = 0 \text{ when } \langle X, Z\rangle = 0$$

Then, for $v \in W^{1,2}(M, N)_w$ *such that* $L_{\lambda_2^*}(v) = 0$*, the set* $\mathcal{E}_{\lambda_2^*,v}$ *has the same properties as* $\mathcal{E}_{\lambda_1,v}$ *in Theorem* 2.2*.*

3°) $\lambda_2^* \geq 0$*.*

Corollary 2.4. *Let* (M, g) *and* (N, h) *be compact Riemannian manifolds with* (N, h) *symmetric. If* $\lambda < 0$ *and there exists* $u \in W^{1,2}(M, N)_w$ *such that* $L_\lambda(u) = 0$ *, then, for any* $\alpha \in [\lambda, 0[$*, there exists* $u_\alpha \in W^{1,2}(M, N)_w$ *such that* $L_\alpha(u_\alpha) = 0$*.*

Remark 2.5. The assumption $d_{L^2}(u, w) \leq \chi$ in the definition of Λ_χ will be used to insure that one has bifurcation at (λ, u) as in [9].

The assumption " $\left\langle R^N(X, Y)Y, Z\right\rangle = 0$ when $\langle X, Z\rangle = 0$ " is satisfied in the case $(N, h) = \mathbb{R}^n$, the sphere S^n, or the hyperbolic space $\mathbb{H}^n$. See e.g. [8].

3 Proofs of the results

3.1 Proof of Theorem 2.1

3.1.1 Proof of 1°).

This proof follows the line given in [7] to prove the analogous assertion for the Laplace - Beltrami operator on a compact Riemannian manifold.

If $\lambda \in \mathbb{R}$ is such that $\mathcal{E}_{\lambda,v} \neq \emptyset$, then $\lambda \geq \lambda_1$ where

$$\lambda_1 := \inf_{V \in \Gamma_0^{1,2}(v^{-1}(TN))\setminus\{0\}} \left\langle \nabla_V \exp_{\cdot}^{-1} w, V\right\rangle_{L^2}^{-1} \int_M [\left\langle R^N(V, d_{e_i}v)d_{e_i}v, V\right\rangle - \|\nabla V\|^2](x)dx.$$

Since the sectional curvature of (N, h) is bounded, we have that $\lambda_1 \in \mathbb{R}$. We will see later that $\mathcal{E}_{\lambda_1,v} \neq \emptyset$.

Let's point out that

$$\int_M [\left\langle R^N(V, d_{e_i}v)d_{e_i}v, V\right\rangle - \|\nabla V\|^2](x)dx = \langle Q_v(V), V\rangle_{L^2}.$$

Let us set

$$\|V\|_{1,2} := [-\left\langle \nabla_V \exp_{\cdot}^{-1} w, V\right\rangle_{L^2} + \int_M \|\nabla V\|^2(x)dx]^{1/2}, \forall V \in \Gamma_0^{1,2}(v^{-1}(TN)).$$

From p. 156 of [8] there exists $C_2 > 0$ such that:

$$\forall V \in \Gamma_0^{1,2}(v^{-1}(TN)), \forall x \in M, -\left\langle \nabla_V \exp_{\cdot}^{-1} w, V\right\rangle(x) \geq C_2 \|V(x)\|^2.$$

It follows that $\|.\|_{1,2}$ is a norm defined by a scalar product on $\Gamma_0^{1,2}(v^{-1}(TN))$. From the row definition of the covariant derivative, one can see that this norm is equivalent to the Sobolev norm $\|.\|_{W^{1,2}}$. To see this equivalence one can proceed as follows:

Let U be the domain of some coordinates $(x^1, ..., x^m)$ of M such that $v(U)$ is contained in the domain of some coordinates $(y^1, ..., y^n)$ of N. Let $V \in \Gamma_0^{1,2}(v^{-1}(TN))$. Then, $\forall x \in U$,

$$\begin{aligned} \|\nabla V\|^2(x) &= g^{ij}(x)h_{\alpha\beta}(v(x))[\frac{\partial V^\alpha}{\partial x^i}\frac{\partial V^\beta}{\partial x^j} + 2V^\rho \frac{\partial V^\alpha}{\partial x^i}\frac{\partial v^\gamma}{\partial x^j}\, {}^N\Gamma^\beta_{\gamma\rho} \circ v \\ &\quad + V^\rho V^\sigma \frac{\partial v^\gamma}{\partial x^i}\frac{\partial v^\theta}{\partial x^j}({}^N\Gamma^\alpha_{\gamma\rho} \circ v)({}^N\Gamma^\beta_{\theta\sigma} \circ v)](x) \\ &= \|DV\|^2(x) + 2\langle DV, S(V)\rangle(x) \\ &\quad + g^{ij}(x)h_{\alpha\beta}(v(x))V^\rho V^\sigma \frac{\partial v^\gamma}{\partial x^i}\frac{\partial v^\theta}{\partial x^j}({}^N\Gamma^\alpha_{\gamma\rho} \circ v)({}^N\Gamma^\beta_{\sigma\theta} \circ v)](x) \end{aligned} \tag{3.1}$$

where the definition of $S(V)$ is obvious, and DV is the usual differential of V as a mapping from M into TN. Since v is C^1 we have that Dv is bounded. By covering N with a finite number of domains of normal coordinates, we may assume that

$$\|Dv\|(x)\left|\Gamma^k_{ij}(v(x))\right| \leq \frac{1}{4}C_2, \forall x \in M, \forall 1 \leq i, j, k \leq n.$$

Then using the inequality $ab \leq \frac{1}{2}a^2 + \frac{1}{2}b^2$, one gets that $\|V\|^2_{1,2}$ is controlled from below by $\|V\|^2_{W^{1,2}}$. The control from above is given by the Hölder's inequality.

It follows that $(\Gamma_0^{1,2}(v^{-1}(TN)), \|.\|_{1,2})$ is a Hilbert space.

Let $(V_n)_{n\in\mathbb{N}} \subseteq [\Gamma_0^{1,2}(v^{-1}(TN))]^{\mathbb{N}}$ be such that

$$-\left\langle \nabla_{V_n} \exp_{\cdot}^{-1} w, V_n\right\rangle_{L^2} =: \|V_n\|^2_{L^{v,2}} = 1, \forall n \in \mathbb{N},$$

and

$$\lambda_1 = \lim_{n\to+\infty} \int_M [-\left\langle R^N(V_n, d_{e_i}v)d_{e_i}v, V_n\right\rangle + \|\nabla V_n\|^2](x)dx.$$

Then the sequence $(\int_M \|\nabla V_n\|^2(x)dx)_{n\in\mathbb{N}}$ is bounded. The theorem of Rellich - Kondrachov then gives us the existence of a subsequence of $(V_n)_{n\in\mathbb{N}}$, which we denote again by $(V_n)_n$, which converges in $L^{v,2}$ and weakly in $(\Gamma_0^{1,2}(v^{-1}(TN)), \|.\|_{1,2})$ and in $(\Gamma_0^{1,2}(v^{-1}(TN)), \|.\|_{W^{1,2}})$ to some $Z_1 \in \Gamma_0^{1,2}(v^{-1}(TN))$. It follows that

$$\|Z_1\|_{L^{v,2}} = 1.$$

One knows that $V \mapsto \int_U \|DV\|^2(x)dx$ is lower semi - continuous w.r.t. $W^{1,2}$ - weak convergence. It follows from formula (3.1) the lower semi - continuity of the functional $V \mapsto \int_U \|\nabla V\|^2(x)dx$ w.r.t. $W^{1,2}$ - weak convergence. Since M is equal a.e. to the disjoint union of such U, we have that the functional $V \mapsto \int_M \|\nabla V\|^2(x)dx$ is lower semi - continuous w.r.t. $W^{1,2}$ - weak convergence. It follows

$$\lambda_1 = \int_M [-\left\langle R^N(Z_1, d_{e_i}v)d_{e_i}v, Z_1\right\rangle + \|\nabla Z_1\|^2](x)dx.$$

Let's prove that $Z_1 \in \mathcal{E}_{\lambda_1,v}$.
Let $V \in \Gamma_0^{1,2}(v^{-1}(TN))$. Then we have

$$\begin{aligned}
0 &= \frac{d}{dt}_{|t=0} \left\langle \nabla_{Z_1+tV} \exp_.^{-1} w, Z_1 + tV \right\rangle_{L^2}^{-1} \langle Q_v(Z_1+tV), Z_1+tV\rangle_{L^2} \\
&= 2[-\left\langle \nabla_{Z_1} \exp_.^{-1} w, Z_1\right\rangle_{L^2}^{-2} \left\langle \nabla_{Z_1} \exp_.^{-1} w, V\right\rangle_{L^2} \langle Q_v(Z_1), Z_1\rangle_{L^2} \\
&\quad + \left\langle \nabla_{Z_1} \exp_.^{-1} w, Z_1\right\rangle_{L^2}^{-1} \langle Q_v(Z_1), V\rangle_{L^2}] \\
&= 2[\left\langle \nabla_{Z_1} \exp_.^{-1} w, V\right\rangle_{L^2} \lambda_1 - \langle Q_v(Z_1), V\rangle_{L^2}] \\
&= -2\left\langle Q_v(Z_1) - \lambda_1 \nabla_{Z_1} \exp_.^{-1} w, V\right\rangle_{L^2}. \text{ So } Z_1 \in \mathcal{E}_{\lambda_1}.
\end{aligned}$$

Let's assume now that we have found $(\lambda_i, Z_i), i = 1,2,...,p-1$ such that :
* $\lambda_1 \leq \lambda_2 \leq .... \leq \lambda_{p-1}$
* $Z_i \in \mathcal{E}_{\lambda_i,v}, \forall i = 1,2,...,p-1$
*$-\left\langle \nabla_{Z_i} \exp_.^{-1} w, Z_j\right\rangle_{L^2} = \delta_{ij}, \forall i,j \in \{1,2,...,p-1\}$.

Let $\mathcal{H}_p$ be the $L^{v,2}$ - orthogonal complement of the real vector space generated by $\{Z_1,...,Z_{p-1}\}$, and

$$\lambda_p := \inf_{V \in \mathcal{H}_p \setminus \{0\}} \left\langle \nabla_V \exp_.^{-1} w, V\right\rangle_{L^2}^{-1} \langle Q_v(V), V\rangle_{L^2}.$$

Then $\mathcal{H}_p$ is a Hilbert space and $\lambda_p \geq \lambda_{p-1}$ since $\mathcal{H}_p \subseteq \mathcal{H}_{p-1}$.

In the same way as we did for Z_1, there exists $Z_p \in \mathcal{H}_p$ such that:
* $-\left\langle \nabla_{Z_p} \exp_.^{-1} w, Z_p\right\rangle_{L^2} = 1$
* $\lambda_p = -\left\langle Q_v(Z_p), Z_p\right\rangle_{L^2}$
* $Z_p \in \mathcal{E}_{\lambda_p,v}$.

In this way we have constructed a nondecreasing sequence $(\lambda_p)_{p\in\mathbb{N}^*}$ and an orthonormal $(Z_p)_{p\in\mathbb{N}^\mu}$ in $(L^2(v^{-1}(TN)), -\left\langle \nabla_. \exp_.^{-1} w, .\right\rangle_{L^2})$ such that: $\forall p \in \mathbb{N}^*$, $Z_p \in \mathcal{E}_{\lambda_p,v}$. It is easy to see that $(\lambda_p)_{p\in\mathbb{N}^*}$ is not bounded. In fact: If $(\lambda_p)_{p\in\mathbb{N}^*}$ is bounded then so will also $(\left\|Z_p\right\|_{1,2})_{p\in\mathbb{N}^*}$ and there will exist a subsequence of $(Z_p)_{p\in\mathbb{N}^*}$ which will converge in $L^{v,2}$, contradicting the facts that $\left\|Z_p\right\|_{L^{v,2}} = 1$ and $\left\langle Z_p, Z_q\right\rangle_{L^{v,2}} = 0$, $\forall p \neq q \in \mathbb{N}^*$.

3.1.2 Proof of 2°).

Let $\chi > 0$ be fixed.

Let's assume that there exists a sequence $(\lambda_k)_{k\geq 1} \subseteq \Lambda_\chi$ of pairwise distinct values which converges to some $\lambda \in \mathbb{R}$.

We will prove that $\lambda \in \Lambda_\chi$ and then we will use some results of [9] to get a contradiction. Our result will then follow.

Let, for any $k \in \mathbb{N}$:

$$\begin{aligned}
u_k &\in W^{1,2}(M,N) \text{ such that } u_{k|\partial M} = w_{|\partial M} \text{ and} \\
d(w(x), u_k(x)) &\leq \theta(x), \forall x \in M \\
\text{and } V_k &\in \Gamma_0^{1,2}(u_k^{-1}(TN))\setminus\{0\} \text{ such that } -\left\langle \nabla_{V_k} \exp_.^{-1} w, V_k\right\rangle_{L^2} = 1,
\end{aligned}$$

$$L_{\lambda_k}(u_k) = 0, \text{ and } Q_{u_k}(V_k) - \lambda_k \nabla_{V_k} \exp_.^{-1} w = 0.$$

As we already said in earlier works such as [13], the regularity theory developed in [10], although concerned by sections, applies to the solutions of (1.1). So one has that the solutions of this equation are C^3 since the model w is C^1.

Step 1: The convergence of some subsequence of $(u_k)_k$.

We have
$$L_{\lambda_k}(u_k) = 0, \forall k \in \mathbb{N}.$$

Let $k \in \mathbb{N}$ be fixed.

We have
$$\left\langle \tau(u_k), \exp_{u_k}^{-1} w \right\rangle = \left\langle \nabla_{e_i} d_{e_i} u_k - d_{\nabla_{e_i} e_i} u_k, \exp_{u_k}^{-1} w \right\rangle$$
$$= [D_{e_i} \left\langle d_{e_i} u_k, \exp_{u_k}^{-1} w \right\rangle - \left\langle d_{\nabla_{e_i} e_i} u_k, \exp_{u_k}^{-1} w \right\rangle] - \left\langle d_{e_i} u_k, \nabla_{e_i} (\exp_{u_k}^{-1} w) \right\rangle$$
$$= div(F) - \left\langle d_{e_i} u_k, \nabla_{d_{e_i} u_k} (\exp_{.}^{-1} w) \right\rangle - \left\langle d_{e_i} u_k, D_{d_{e_i} w} (\exp_{u_k}^{-1} .) \right\rangle,$$
where the definition of the vector field F is obvious.

It follows by the divergence theorem:
$$\int_M \left\langle \tau(u_k), \exp_{u_k}^{-1} w \right\rangle (x) dx = - \int_M \left\langle d_{e_i} u_k, \nabla_{d_{e_i} u_k} (\exp_{.}^{-1} w) \right\rangle (x) dx - \int_M \left\langle d_{e_i} u_k, D_{d_{e_i} w} (\exp_{u_k}^{-1} .) \right\rangle (x) dx.$$

On the other hand, since $\tau(u_k) = \lambda_k \exp_{u_k}^{-1} w$, we have
$$\int_M \left\langle \tau(u_k), \exp_{u_k}^{-1} w \right\rangle (x) dx = \lambda_k \int_M d^2(u_k, w)(x) dx,$$
and then
$$- \int_M \left\langle d_{e_i} u_k, \nabla_{d_{e_i} u_k} (\exp_{.}^{-1} w) \right\rangle (x) dx = \int_M \left\langle d_{e_i} u_k, D_{d_{e_i} w} (\exp_{u_k}^{-1} .) \right\rangle (x) dx + \lambda_k \int_M d^2(u_k, w)(x) dx. \tag{3.2}$$

We have
$$- \left\langle d_{e_i} u_k, \nabla_{d_{e_i} u_k} (\exp_{.}^{-1} w) \right\rangle \geq C_2 \left\| d_{e_i} u_k \right\|^2,$$
and
$$\left| \left\langle d_{e_i} u_k, D_{d_{e_i} w} (\exp_{u_k}^{-1} .) \right\rangle \right| \leq \left\| d_{e_i} u_k \right\| \left\| D_{d_{e_i} w} (\exp_{u_k}^{-1} .) \right\| \leq \frac{1}{2} C_2 \left\| d_{e_i} u_k \right\|^2 + \frac{1}{2C_2} \left\| D_{d_{e_i} w} (\exp_{u_k}^{-1} .) \right\|^2.$$

Putting this into (3.2) gives
$$\frac{1}{2} C_2 \int_M \left\| d_{e_i} u_k \right\|^2 (x) dx \leq \frac{1}{2C_2} \int_M \left\| D_{d_{e_i} w} (\exp_{u_k}^{-1} .) \right\|^2 (x) dx + \lambda_k \int_M d^2(u_k, w)(x) dx.$$

Since our manifolds are compact and w is C^1, we get that there exists $C_3 > 0$ such that
$$\int_M \left\| d_{e_i} u_k \right\|^2 (x) dx \leq C_3, \forall k \in \mathbb{N}.$$

It follows by Rellich - Kondrachov that $(u_k)_k$ admits a subsequence, denoted again by $(u_k)_k$, which converges in L^2 and weakly in $W^{1,2}$ to some $u \in W^{1,2}(M, N)$.

The lower semi - continuity of the energy w.r.t. the weak convergence in $W^{1,2}$ gives us
$$E(u) \leq \lim \inf_{k \to +\infty} E(u_k) \leq 2C_3.$$

From [13] there is a subsequence of $(u_k)_k$, denoted again by $(u_k)_k$, which converges in $W^{1,2}$ to u, and
$$\tau(u) = \lambda \exp_u^{-1} w.$$

Step 2: The convergence of the sequence $(V_k)_k$.

We have

$$Q_{u_k}(V_k) = \lambda_k \nabla_{V_k} \exp_{\cdot}^{-1} w, \forall k \in \mathbb{N}.$$

Let $k \in \mathbb{N}$ be fixed in a first time.

By taking the scalar product of the last equation with V_k we get

$$\int_M \|\nabla V_k\|^2 (x) dx =$$

$$\int_M \left\langle R^N(V_k, d_{e_i} u_k) d_{e_i} u_k, V_k \right\rangle (x) dx - \lambda_k \int_M \left\langle \nabla_{V_k} \exp_{\cdot}^{-1} w, V_k \right\rangle (x) dx.$$

From [13], there exists $C_4 > 0$ such that

$$\|du_k\| (x) \leq C_4, \forall k \in \mathbb{N}, \forall x \in M.$$

It follows that, there exists $C_5 > 0$ such that

$$\int_M \|\nabla V_k\|^2 (x) dx \leq C_5, \forall k \in \mathbb{N}$$

by using the assumption that

$$-\left\langle \nabla_{V_k} \exp_{\cdot}^{-1} w, V_k \right\rangle_{L^2} = 1, \forall k \in \mathbb{N}.$$

Let $[0,1] \ni t \mapsto V_t(x)$ be the parallel transport of $V_k(x)$ along the unique minimizing geodesic $u_t(x)$ from $u_k(x)$ to $u(x)$ which is parametrized on $[0,1]$.

Let $W_k(x)$ be the value of $V_t(x)$ at $t = 1$, for any $x \in M$.

The parallel transport is an isometry, and the mapping $M \times N \ni (x,y) \mapsto \exp_y^{-1} w(x)$ is uniformly continuous in the C^1 - norm because of the dependence of the geodesics on their endpoints. It follows that there exists $C_6 > 0$ such that

$$\left| \left\langle \nabla_{V_t} \exp_{\cdot}^{-1} w, V_t \right\rangle_{L^2} \right| \leq C_6, \forall k \in \mathbb{N}, \forall t \in [0,1].$$

We have

$$\begin{aligned} \frac{d}{dt} \int_M \|\nabla V_t\|^2 (x) dx &= 2 \int_M [\left\langle \nabla_{e_i} \nabla_{\frac{\partial}{\partial t}} V_t, \nabla_{e_i} V_t \right\rangle + \left\langle R^N(\frac{\partial u_t}{\partial t}, d_{e_i} u_t) V_t, \nabla_{e_i} V_t \right\rangle] dx \\ &= 2 \int_M \left\langle R^N(\frac{\partial u_t}{\partial t}, d_{e_i} u_t) V_t, \nabla_{e_i} V_t \right\rangle dx. \end{aligned}$$

It follows that, there exists $C_7 > 0$ such that

$$\int_M \|\nabla W_k\|^2 (x) dx \leq C_7, \forall k \in \mathbb{N}.$$

So the sequence $(W_k)_k$ is bounded in $\Gamma_0^{1,2}(u^{-1}(TN))$. The theorem of Rellich - Kondrachov then gives the existence of a subsequence of $(W_k)_k$, denoted again by $(W_k)_k$, which converges in L^2 and weakly in $W^{1,2}$ to some $V \in \Gamma_0^{1,2}(u^{-1}(TN))$.

We want now to prove that $V \neq 0$.

The L^2 - convergence of $(W_k)_k$ to V gives that

$$\lim_{k \to +\infty} \left\langle \nabla_{W_k} \exp_{\cdot}^{-1} w, W_k \right\rangle_{L^2} = \left\langle \nabla_V \exp_{\cdot}^{-1} w, V \right\rangle_{L^2}.$$

Let $W \in \Gamma_0^{1,2}(u^{-1}(TN))$ and Z_k its parallel transport to u_k, for any $k \in \mathbb{N}$.

Since $(u_k)_{k\in\mathbb{N}}$ converges to u in L^2, one theorem of Riesz in integration theory insures the existence of a subsequence of $(u_k)_k$ which converges simply a.a. to u. The mapping $(x,y) \mapsto (\nabla \exp_\cdot^{-1} w(x))(y)$ is uniformly continuous on $M \times N$. It follows that

$$\lim_{k\to+\infty} \|_{u_k}^{u} [(\nabla \exp_\cdot^{-1} w(x))(u_k(x))] = (\nabla \exp_\cdot^{-1} w(x))(u(x)), \text{ for a.a. } x \in M.$$

So

$$\lim_{k\to+\infty} \|_{u_k}^{u} [\nabla_{Z_k} \exp_\cdot^{-1} w(x)] = \nabla_W \exp_\cdot^{-1} w(x), \text{ for a.a. } x \in M,$$

and then

$$\lim_{k\to+\infty} \left\langle \|_{u_k}^{u} [\nabla_{Z_k} \exp_\cdot^{-1} w(x)], W \right\rangle_{L^2} = \left\langle \nabla_W \exp_\cdot^{-1} w(x), W \right\rangle_{L^2}.$$

It follows

$$\lim_{k\to+\infty} \left\langle \nabla_{Z_k} \exp_\cdot^{-1} w(x), Z_k \right\rangle_{L^2} = \left\langle \nabla_W \exp_\cdot^{-1} w(x), W \right\rangle_{L^2}.$$

As a consequence we have

$$\lim_{k\to+\infty} \left\langle \nabla_{V_k} \exp_\cdot^{-1} w, V_k \right\rangle_{L^2} = \lim_{k\to+\infty} \left\langle \nabla_{W_k} \exp_\cdot^{-1} w, W_k \right\rangle_{L^2}.$$

Since $\left\langle \nabla_{V_k} \exp_\cdot^{-1} w, V_k \right\rangle_{L^2} = 1, \forall k \in \mathbb{N}$, we get

$$\left\langle \nabla_V \exp_\cdot^{-1} w, V \right\rangle_{L^2} = 1, \text{ and then } V \neq 0.$$

Step 3: We prove that V satisfies $Q_u(V) = \lambda \nabla_V \exp_\cdot^{-1} w$.

Let $k \in \mathbb{N}$ be fixed in a first time.

For any $t \in [0,1]$, let

$$B_t := \nabla_{e_i}\nabla_{e_i} V_t - \nabla_{\nabla_{e_i}e_i} V_t + R^N(V_t, d_{e_i}u_t)d_{e_i}u_t - \lambda_k \nabla_{V_t} \exp_\cdot^{-1} w \in \Gamma(u_t^{-1}(TN)).$$

We assume that things are regular enough to allow the following computations. That is no restriction since we will use only the weak version of these equations.

We have

$$\begin{aligned}
\nabla_t B_t &= \nabla_{e_i}\nabla_t\nabla_{e_i}V_t + R^N(\frac{\partial u_t}{\partial t}, d_{e_i}u_t)\nabla_{e_i}V_t - \nabla_{\nabla_{e_i}e_i}\nabla_t V_t - R^N(\frac{\partial u_t}{\partial t}, d_{\nabla_{e_i}e_i}u_t)V_t \\
&\quad + (\nabla_t(R^N \circ u_t))(V_t, d_{e_i}u_t)d_{e_i}u_t + R^N(\nabla_t V_t, d_{e_i}u_t)d_{e_i}u_t \\
&\quad + R^N(V_t, \nabla_t d_{e_i}u_t)d_{e_i}u_t + R^N(V_t, d_{e_i}u_t)\nabla_t d_{e_i}u_t - \lambda_k \nabla_t \nabla_{V_t} \exp_\cdot^{-1} w \\
&= \nabla_{e_i}\nabla_{e_i}\nabla_t V_t + \nabla_{e_i}[R^N(\frac{\partial u_t}{\partial t}, d_{e_i}u_t)V_t] + R^N(\frac{\partial u_t}{\partial t}, d_{e_i}u_t)\nabla_{e_i}V_t - \nabla_{\nabla_{e_i}e_i}\nabla_t V_t \\
&\quad - R^N(\frac{\partial u_t}{\partial t}, d_{\nabla_{e_i}e_i}u_t)V_t + (\nabla_t(R^N \circ u_t))(V_t, d_{e_i}u_t)d_{e_i}u_t \\
&\quad + R^N(\nabla_t V_t, d_{e_i}u_t)d_{e_i}u_t + R^N(V_t, \nabla_{e_i}\frac{\partial u_t}{\partial t})d_{e_i}u_t + R^N(V_t, d_{e_i}u_t)\nabla_{e_i}\frac{\partial u_t}{\partial t} \\
&\quad - \lambda_k \nabla_t \nabla_{V_t} \exp_\cdot^{-1} w.
\end{aligned}$$

Since $\nabla_t V_t = 0$ and $\nabla R^N = 0$, we get

$$\begin{aligned}
\nabla_t B_t &= [\nabla_{e_i}[R^N(\frac{\partial u_t}{\partial t}, d_{e_i}u_t)V_t] - R^N(\frac{\partial u_t}{\partial t}, d_{\nabla_{e_i}e_i}u_t)V_t] + R^N(\frac{\partial u_t}{\partial t}, d_{e_i}u_t)\nabla_{e_i}V_t \\
&\quad + R^N(V_t, \nabla_{e_i}\frac{\partial u_t}{\partial t})d_{e_i}u_t + R^N(V_t, d_{e_i}u_t)\nabla_{e_i}\frac{\partial u_t}{\partial t} - \lambda_k \nabla_t \nabla_{V_t} \exp_{\cdot}^{-1} w \\
&= [\nabla_{e_i}[R^N(\frac{\partial u_t}{\partial t}, d_{e_i}u_t)V_t] - R^N(\frac{\partial u_t}{\partial t}, d_{\nabla_{e_i}e_i}u_t)V_t] + R^N(\frac{\partial u_t}{\partial t}, d_{e_i}u_t)\nabla_{e_i}V_t \\
&\quad + \nabla_{e_i}[R^N(V_t, \frac{\partial u_t}{\partial t})d_{e_i}u_t)] - R^N(\nabla_{e_i}V_t, \frac{\partial u_t}{\partial t})d_{e_i}u_t - R^N(V_t, \frac{\partial u_t}{\partial t})\nabla_{e_i}d_{e_i}u_t \\
&\quad + \nabla_{e_i}[R^N(V_t, d_{e_i}u_t)\frac{\partial u_t}{\partial t}] - R^N(\nabla_{e_i}V_t, d_{e_i}u_t)\frac{\partial u_t}{\partial t} - R^N(V_t, \nabla_{e_i}d_{e_i}u_t)\frac{\partial u_t}{\partial t} \\
&\quad - \lambda_k \nabla_t \nabla_{V_t} \exp_{\cdot}^{-1} w \\
&= [\nabla_{e_i}[R^N(\frac{\partial u_t}{\partial t}, d_{e_i}u_t)V_t] - R^N(\frac{\partial u_t}{\partial t}, d_{\nabla_{e_i}e_i}u_t)V_t] + [\nabla_{e_i}[R^N(V_t, \frac{\partial u_t}{\partial t})d_{e_i}u_t] \\
&\quad - R^N(V_t, \frac{\partial u_t}{\partial t})d_{\nabla_{e_i}e_i}u_t] + [\nabla_{e_i}[R^N(V_t, d_{e_i}u_t)\frac{\partial u_t}{\partial t}] - R^N(V_t, d_{\nabla_{e_i}e_i}u_t)\frac{\partial u_t}{\partial t}] \\
&\quad - R^N(V_t, \frac{\partial u_t}{\partial t})\tau(u_t) - R^N(V_t, \tau(u_t))\frac{\partial u_t}{\partial t} + R^N(\frac{\partial u_t}{\partial t}, d_{e_i}u_t)\nabla_{e_i}V_t - \\
&\quad R^N(\nabla_{e_i}V_t, \frac{\partial u_t}{\partial t})d_{e_i}u_t - R^N(\nabla_{e_i}V_t, d_{e_i}u_t)\frac{\partial u_t}{\partial t} - \lambda_k \nabla_t \nabla_{V_t} \exp_{\cdot}^{-1} w \\
&= 2[\nabla_{e_i}[R^N(V_t, d_{e_i}u_t)\frac{\partial u_t}{\partial t}] - R^N(V_t, d_{\nabla_{e_i}e_i}u_t)\frac{\partial u_t}{\partial t}] + [\nabla_{e_i}[R^N(V_t, \frac{\partial u_t}{\partial t})d_{e_i}u_t] \\
&\quad - R^N(V_t, \frac{\partial u_t}{\partial t})d_{\nabla_{e_i}e_i}u_t] - 2R^N(\nabla_{e_i}V_t, \frac{\partial u_t}{\partial t})d_{e_i}u_t \\
&\quad - \lambda_k \nabla_t \nabla_{V_t} \exp_{\cdot}^{-1} w - R^N(V_t, \frac{\partial u_t}{\partial t})\tau(u_t) - R^N(V_t, \tau(u_t))\frac{\partial u_t}{\partial t},
\end{aligned}$$

where we used the first Bianchi identity.

Let $Z \in \Gamma_0^{1,2}(u^{-1}(TN))$, and Z_t its parallel transport along $t \mapsto u_{1-t}$. Then

$$\frac{d}{dt}\langle B_t, Z_{1-t}\rangle = \langle \nabla_t B_t, Z_{1-t}\rangle$$

$$\begin{aligned}
&= 2[D_{e_i}\left\langle R^N(V_t, d_{e_i}u_t)\frac{\partial u_t}{\partial t}, Z_{1-t}\right\rangle - \left\langle R^N(V_t, d_{\nabla_{e_i}e_i}u_t)\frac{\partial u_t}{\partial t}, Z_{1-t}\right\rangle] \\
&\quad - 2\left\langle R^N(V_t, d_{e_i}u_t)\frac{\partial u_t}{\partial t}, \nabla_{e_i}Z_{1-t}\right\rangle - 2\left\langle R^N(\nabla_{e_i}V_t, \frac{\partial u_t}{\partial t})d_{e_i}u_t, Z_{1-t}\right\rangle \\
&\quad - \lambda_k \frac{d}{dt}\left\langle \nabla_{V_t}\exp_{\cdot}^{-1} w, Z_{1-t}\right\rangle - \left\langle R^N(V_t, \frac{\partial u_t}{\partial t})\tau(u_t), Z_{1-t}\right\rangle \\
&\quad - \left\langle R^N(V_t, \tau(u_t))\frac{\partial u_t}{\partial t}, Z_{1-t}\right\rangle + [D_{e_i}\left\langle R^N(V_t, \frac{\partial u_t}{\partial t})d_{e_i}u_t, Z_{1-t}\right\rangle \\
&\quad - \left\langle R^N(V_t, \frac{\partial u_t}{\partial t})d_{\nabla_{e_i}e_i}u_t, Z_{1-t}\right\rangle] - \left\langle R^N(V_t, \frac{\partial u_t}{\partial t})d_{e_i}u_t, \nabla_{e_i}Z_{1-t}\right\rangle.
\end{aligned}$$

By integrating this on M and after on $[0,1]$ we get

$$\langle B_1, Z_1\rangle_{L^2} - \langle B_0, Z_0\rangle_{L^2} =$$

$$-2\int_0^1 [\left\langle R^N(V_t, d_{e_i}u_t)\frac{\partial u_t}{\partial t}, \nabla_{e_i}Z_{1-t}\right\rangle_{L^2} + \frac{1}{2}\left\langle R^N(V_t, \frac{\partial u_t}{\partial t})d_{e_i}u_t, \nabla_{e_i}Z_{1-t}\right\rangle_{L^2}$$
$$+\left\langle R^N(\nabla_{e_i}V_t, \frac{\partial u_t}{\partial t})d_{e_i}u_t, Z_{1-t}\right\rangle_{L^2}]dt$$
$$-\int_0^1 \left\langle R^N(V_t, \frac{\partial u_t}{\partial t})\tau(u_t) + R^N(V_t, \tau(u_t))\frac{\partial u_t}{\partial t}, Z_{1-t}\right\rangle_{L^2} dt$$
$$-\lambda_k[\left\langle \nabla_{V_1}\exp_{\cdot}^{-1} w, Z_1\right\rangle_{L^2} - \left\langle \nabla_{V_0}\exp_{\cdot}^{-1} w, Z_0\right\rangle_{L^2}].$$

Since

$$Q_{u_k}(V_k) = \lambda_k \nabla_{V_k}\exp_{\cdot}^{-1} w, \text{ we have } B_0 = 0.$$

It follows

$$\langle B_1, Z_1\rangle_{L^2} =$$

$$-2\int_0^1 [\left\langle R^N(V_t, d_{e_i}u_t)\frac{\partial u_t}{\partial t}, \nabla_{e_i}Z_{1-t}\right\rangle_{L^2} + \frac{1}{2}\left\langle R^N(V_t, \frac{\partial u_t}{\partial t})d_{e_i}u_t, \nabla_{e_i}Z_{1-t}\right\rangle_{L^2}$$
$$+\left\langle R^N(\nabla_{e_i}V_t, \frac{\partial u_t}{\partial t})d_{e_i}u_t, Z_{1-t}\right\rangle_{L^2}]dt$$
$$-\int_0^1 \left\langle R^N(V_t, \frac{\partial u_t}{\partial t})\tau(u_t) + R^N(V_t, \tau(u_t))\frac{\partial u_t}{\partial t}, Z_{1-t}\right\rangle_{L^2} dt$$
$$-\lambda_k[\left\langle \nabla_{V_1}\exp_{\cdot}^{-1} w, Z_1\right\rangle_{L^2} - \left\langle \nabla_{V_0}\exp_{\cdot}^{-1} w, Z_0\right\rangle_{L^2}].$$

Since our geodesics depend smoothly on their endpoints, we have

$$\lim_{k\to+\infty}[\left\langle \nabla_{V_1}\exp_{\cdot}^{-1} w, Z_1\right\rangle - \left\langle \nabla_{V_0}\exp_{\cdot}^{-1} w, Z_0\right\rangle](x) = 0, \forall x \in M,$$

and also, there exists $C_8 > 0$ such that

$$\|\tau(u_t)(x)\| \le C_8, \forall x \in M, \forall t \in [0,1].$$

We have

$$\left\|\frac{\partial u_t}{\partial t}(x)\right\| = d(u_k(x), u(x)), \forall x \in M.$$

Since

$$\|du\|(x), \|du_k\|(x) \le 1 + C_4, \forall x \in M, \forall k \in \mathbb{N},$$

there exists $C_9 > 0$ such that

$$\|du_t\|(x) \le C_9, \forall x \in M, \forall t \in [0,1], \forall k \in \mathbb{N}.$$

It follows that there exists $C_{10} > 0, C_{11} > 0$ such that: $\forall k \in \mathbb{N}$,

$$\begin{aligned}\left|\langle B_1, Z_1\rangle_{L^2}\right| &\le C_{10}[\int_M d(u_k,u)\|V_k\|\|\nabla Z\|dx + \int_M d(u_k,u)\|Z\|\|\nabla V_k\|dx] \\ &\quad + C_{11}\int_M d(u_k,u)\|Z\|\int_0^1 \|V_t\|dtdx,\end{aligned}$$

where

$$\begin{aligned}
&\left|\int_0^1 \left\langle R^N(V_t,\frac{\partial u_t}{\partial t})\tau(u_t)+R^N(V_t,\tau(u_t))\frac{\partial u_t}{\partial t},Z_{1-t}\right\rangle_{L^2} dt\right| \\
\leq\;& C_{11}\int_M d(u_k,u)\,\|Z\|\int_0^1 \|V_t\|\,dtdx\,.
\end{aligned}$$

From Hölder's inequality we have

$$\int_M d(u_k,u)\,\|Z\|\,\|V_k\|\,dx \leq \|V_k\|_{L^2}\,[\int_M d^2(u_k,u)\,\|Z\|^2\,dx]^{1/2},$$

$$\int_M d(u_k,u)\,\|V_k\|\,\|\nabla Z\|\,dx \leq \|V_k\|_{L^2}\,[\int_M d^2(u_k,u)\,\|\nabla Z\|^2\,dx]^{1/2},$$

and

$$\int_M d(u_k,u)\,\|Z\|\,\|\nabla V_k\|\,dx \leq \|\nabla V_k\|_{L^2}\,[\int_M d^2(u_k,u)\,\|Z\|^2\,dx]^{1/2}.$$

Since $(u_k)_{k\in\mathbb{N}}$ converges a.e. to u, the usual Lebesgue's dominated convergence theorem from integration theory then gives that

$$\lim_{k\to+\infty}[\int_M d(u_k,u)\,\|Z\|\,\|V_k\|\,dx+\int_M d(u_k,u)\,\|V_k\|\,\|\nabla Z\|\,dx$$

$$+\int_M d(u_k,u)\,\|Z\|\,\|\nabla V_k\|\,dx]=0,$$

and then

$$\lim_{k\to+\infty}\langle B_1,Z_1\rangle_{L^2}=0.$$

We have

$$\langle B_1,Z_1\rangle_{L^2}=\int_M[\langle\nabla_{e_i}W_k,\nabla_{e_i}Z_1\rangle+\left\langle R^N(W_k,d_{e_i}u)d_{e_i}u,Z_1\right\rangle-\lambda_k\left\langle\nabla_{W_k}\exp_{.}^{-1}w,Z_1\right\rangle]dx.$$

Since $(W_k)_{k\in\mathbb{N}}$ converges in L^2 and weakly in $W^{1,2}$ to V, and Z_1 converges in $W^{1,2}$ to Z, we get

$$\begin{aligned}
0 &= \lim_{k\to+\infty}\langle B_1,Z_1\rangle_{L^2}\\
&= \int_M[\langle\nabla_{e_i}V,\nabla_{e_i}Z_1\rangle+\left\langle R^N(V,d_{e_i}u)d_{e_i}u,Z_1\right\rangle-\lambda\left\langle\nabla_V\exp_{.}^{-1}w,Z_1\right\rangle]dx,
\end{aligned}$$

i.e. V is a weak solution of $Q_u(V)=\lambda\nabla_V\exp_{.}^{-1}w$. From the regularity of such solutions, we have that V is a strong solution of $Q_u(V)=\lambda\nabla_V\exp_{.}^{-1}w$.

Step 4: The assumption on χ. We will assume in a first time that

$$\dim Ker(\nabla_{.}L_{\lambda_k})(u_k)=1,\forall k\in\mathbb{N}.$$

Let $\xi_k\in\Gamma_0^{1,2}(u_k^{-1}(TN))$ be such that

$$Ker(\nabla_{.}L_{\lambda_k})(u_k)=\mathbb{R}\xi_k \text{ with } \|\xi_k\|_{L^2}=1,\forall k\in\mathbb{N}.$$

We are going to prove that, there exists $\chi > 0$ such that the two assumptions used in [9] to have bifurcation, namely

$$\pi(D_\lambda D_\alpha v) > 0 \text{ and } \pi(D_\lambda^2 v)\pi(D_\alpha^2 v) - [\pi(D_\lambda D_\alpha v)]^2 < 0$$

are satisfied for $v = v_k$ the solution of the bifurcation equation at (λ_k, u_k), where π is the L^2 - orthogonal projection onto $\mathbb{R}\xi_k$, see Step 5 for details.

From equation (3.11) and the formula just after in [9] we have

$$\pi(D_\lambda D_\alpha v_k) = -\pi[\nabla_{\xi_k} \exp_.^{-1} w + (\nabla_{D_\lambda v_k} \nabla_{\xi_k} L_{\lambda_k})](u_k), \tag{3.3}$$

with

$$\begin{gathered} -\left\langle \nabla_{\xi_k} \exp_.^{-1} w, \xi_k \right\rangle \geq C_2 \|\xi_k\|^2, \\ \left| \left\langle (\nabla_{D_\lambda v_k} \nabla_{\xi_k} L_{\lambda_k})(u_k), \xi_k \right\rangle (x) \right| \leq \\ d(w(x), u_k(x)) \left\| [(\nabla_. L_{\lambda_k})(u_k) + \pi]^{-1}(x) \right\| \left\| (\nabla_.^2 L)(\lambda_k, u_k) \right\| (x) \, \|\xi_k(x)\|^2, \forall x \in M. \end{gathered} \tag{3.4}$$

From the last formula at page 94 of [9] we have: $\forall W_k \in \Gamma_0^{1,2}(u_k^{-1}(TN))$,

$$\begin{aligned} (\nabla_.^2 L_{\lambda_k})(u_k)(W_k, W_k) &= -4R^N(d_{e_i}u_k, W_k)\nabla_{e_i} W_k + R^N(W_k, \tau(u_k))W_k \\ &\quad + \lambda_k R^N(W_k, \exp_{u_k}^{-1} w)W_k \\ &= -4R^N(d_{e_i}u_k, W_k)\nabla_{e_i} W_k, \text{ since } \tau(u_k) - \lambda_k \exp_{u_k}^{-1} w = 0. \end{aligned} \tag{3.5}$$

Let

$$W_k = W_{k,1} + W_{k,2} \in \mathbb{R}\xi_k \oplus (\mathbb{R}\xi_k)^\perp = \Gamma_0^{1,2}(u_k^{-1}(TN)).$$

Then

$$\begin{aligned} [(\nabla_. L_{\lambda_k})(u_k) + \pi](W_k) &= (\nabla_{W_{k,2}} L_{\lambda_k})(u_k) + W_{k,1}, \\ \langle [(\nabla_. L_{\lambda_k})(u_k) + \pi](W_k), W_k \rangle_{L^2} &= \left\langle (\nabla_{W_{k,2}} L_{\lambda_k})(u_k), W_{k,2} \right\rangle_{L^2} + \left\| W_{k,1} \right\|_{L^2}^2, \end{aligned}$$

since $(\nabla_. L_{\lambda_k})(u_k)$ is self adjoint,

$$\begin{aligned} &= \left\| \nabla W_{k,2} \right\|_{L^2}^2 + \int_M [\left\langle R^N(d_{e_i}u_k, W_{k,2}) d_{e_i}u_k, W_{k,2} \right\rangle \\ &\quad -\lambda \left\langle \nabla_{W_{k,2}} \exp_.^{-1} w, W_{k,2} \right\rangle] dx + \left\| W_{k,1} \right\|_{L^2}^2. \end{aligned}$$

Let $W_{k,i,t}$ ($i = 1, 2$) be the parallel transport of $W_{k,i}$ along the geodesic $u_t, t \in [0,1]$, $H_{k,i}$ its value for $t = 1$, and

$$W_{k,t} := W_{k,1,t} + W_{k,2,t}, \quad H_k := H_{k,1} + H_{k,2}, \forall (t,k) \in [0,1] \times \mathbb{N}.$$

It is clear that

$$\lim_{k \to +\infty} \int_M \left\langle R^N(d_{e_i}u_k, W_{k,2}) d_{e_i}u_k, W_{k,2} \right\rangle dx = \int_M \left\langle R^N(d_{e_i}u, H_{(2)}) d_{e_i}u, H_{(2)} \right\rangle dx,$$

where $H_{(i)}$ is the L^2 - limit and weak $W^{1,2}$ - limit of $(H_{k,i})_{k\in\mathbb{N}}$. Then, $\forall k \in \mathbb{N}$ we have

$$\begin{aligned} \nabla_{H_{k,2}} \exp_.^{-1} w - \|_{u_k}^u \nabla_{W_{k,2}} \exp_.^{-1} w &= \nabla_{H_{k,2}} [\exp_.^{-1} w - \|_{u_k}^u \exp_.^{-1} w] \\ &\quad + \nabla_{H_{k,2}} [\|_{u_k}^u \exp_.^{-1} w] - \|_{u_k}^u \nabla_{W_{k,2}} \exp_.^{-1} w. \end{aligned}$$

Let $(Z_{k,l,t})_{1\leq l\leq n}$ be an pointwise orthonormal family of vector fields along u_t which are parallel along the geodesic $t\mapsto u_t$. We have

$$\exp_{u_t}^{-1} w = \left\langle \exp_{u_t}^{-1} w, Z_{k,l,t}\right\rangle Z_{k,l,t}, \forall t\in[0,1],$$

and then

$$\nabla_{W_{k,2}} \exp_{.}^{-1} w = D_{W_{k,2}}\left\langle \exp_{u_k}^{-1} w, Z_{k,l,0}\right\rangle Z_{k,l,0} + \left\langle \exp_{u_k}^{-1} w, Z_{k,l,0}\right\rangle \nabla_{W_{k,2}} Z_{k,l,0}, \forall k\in\mathbb{N}.$$

It follows: $\forall k\in\mathbb{N}$,

$$\begin{aligned} ||_{u_k}^{u} \exp_{u_k}^{-1} w &= \left\langle \exp_{u_k}^{-1} w, Z_{k,l,0}\right\rangle Z_{k,l,1}, \\ ||_{u_k}^{u} \nabla_{W_{k,2}} \exp_{.}^{-1} w &= [D_{W_{k,2}}\left\langle \exp_{u_k}^{-1} w, Z_{k,l,0}\right\rangle] Z_{k,l,1} \\ &\quad + \left\langle \exp_{u_k}^{-1} w, Z_{k,l,0}\right\rangle ||_{u_k}^{u} \nabla_{W_{k,2}} Z_{k,l,0}. \end{aligned}$$

There is no problem with the injectivity of u. We have: $\forall (t,k)\in[0,1]\times\mathbb{N}$,

$$\nabla_{H_{k,2}}[||_{u_k}^{u} \exp_{u_k}^{-1} w] = [D_{W_{k,2}}\left\langle \exp_{u_k}^{-1} w, Z_{k,l,0}\right\rangle] Z_{k,l,1} + \left\langle \exp_{u_k}^{-1} w, Z_{k,l,0}\right\rangle \nabla_{H_{k,2}} Z_{k,l,1},$$

and

$$\nabla_t \nabla_{W_{k,2,t}} Z_{k,l,t} = R^N(\frac{\partial u_t}{\partial t}, W_{k,2,t}) Z_{k,l,t}, \text{ since } \nabla_t Z_{k,l,t} = 0.$$

By integrating this in the basis $(Z_{k,l,t})_{1\leq l\leq n}$ we get

$$\nabla_{H_{k,2}} Z_{k,l,1} - ||_{u_k}^{u} \nabla_{W_{k,2}} Z_{k,l,0} = [\int_0^1 \left\langle R^N(\frac{\partial u_t}{\partial t}, W_{k,2,t}) Z_{k,l,t}, Z_{k,p,t}\right\rangle dt] Z_{k,p,1}.$$

And finally: $\forall k\in\mathbb{N}$,

$$\begin{aligned} \nabla_{H_{k,2}}[||_{u_k}^{u} \exp_{u_k}^{-1} w] &= ||_{u_k}^{u} \nabla_{W_{k,2}} \exp_{.}^{-1} w \\ &+ \left\langle \exp_{u_k}^{-1} w, Z_{k,l,0}\right\rangle [\int_0^1 \left\langle R^N(\frac{\partial u_t}{\partial t}, W_{k,2,t}) Z_{k,l,t}, Z_{k,p,t}\right\rangle dt] Z_{k,p,1}. \end{aligned}$$

It follows: $\forall k\in\mathbb{N}$,

$$\begin{aligned} \nabla_{H_{k,2}} \exp_{.}^{-1} w - ||_{u_k}^{u} \nabla_{W_{k,2}} \exp_{.}^{-1} w &= \nabla_{H_{k,2}}[\exp_{.}^{-1} w - ||_{u_k}^{u} \exp_{.}^{-1} w] \\ &+ \left\langle \exp_{u_k}^{-1} w, Z_{k,l,0}\right\rangle [\int_0^1 \left\langle R^N(\frac{\partial u_t}{\partial t}, W_{k,2,t}) Z_{k,l,t}, Z_{k,p,t}\right\rangle dt] Z_{k,p,1}, \end{aligned}$$

and then

$$\lim_{k\to+\infty} \int_M \left\| \nabla_{H_{k,2}} \exp_{.}^{-1} w - ||_{u_k}^{u} \nabla_{W_{k,2}} \exp_{.}^{-1} w \right\|^2 dx = 0.$$

We then have proved that

$$\lim_{k\to+\infty} \langle [(\nabla_{.} L_{\lambda_k})(u_k) + \pi](W_k), W_k\rangle_{L^2} = \langle [(\nabla_{.} L_{\lambda})(u) + \pi](H), H\rangle_{L^2},$$

where $H := H_{(1)} + H_{(2)}$, and that there exists a subsequence of $(u_k)_k$ such that

$$\lim_{k\to+\infty} \left\| (\nabla_{.} L_{\lambda_k})(u_k) + \pi \right\|(x) = \|(\nabla_{.} L_{\lambda})(u) + \pi\|(x) \text{ for a.a. } x\in M.$$

It follows

$$\lim_{k\to+\infty} \left\| [(\nabla_{.} L_{\lambda_k})(u_k) + \pi]^{-1}(x) \right\| = \left\| [(\nabla_{.} L_{\lambda})(u) + \pi]^{-1}(x) \right\|, \text{ for a.a. } x\in M.$$

One has from (3.5) that there exists $C_{12} > 0$ such that

$$\left\|(\nabla^2_. L)(\lambda_k, u_k)\right\|(x) \leq C_{12}, \forall x \in M, \forall k \in \mathbb{N}.$$

From the regularity theory for linear elliptic pde's, $\exists C_{13} > 0$ such that

$$\|\xi_k(x)\| \leq C_{13}, \forall x \in M, \forall k \in \mathbb{N}.$$

Using the Lebesgue's dominated convergence theorem we then have

$$\lim_{k\to+\infty} \int_M d(w(x), u_k(x)) \left\|[(\nabla_. L_{\lambda_k})(u_k) + \pi]^{-1}(x)\right\| \left\|(\nabla^2_. L)(\lambda_k, u_k)\right\|(x) \, \|\xi_k(x)\|^2 \, dx =$$

$$\int_M d(w(x), u(x)) \left\|[(\nabla_. L_\lambda)(u) + \pi]^{-1}(x)\right\| \left\|(\nabla^2_. L)(\lambda, u)\right\|(x) \, \|\xi(x)\|^2 \, dx,$$

where ξ is the a.e. pointwise limit of $(\xi_k)_{k\in\mathbb{N}}$.

It follows from (3.4) and (3.3) that, for χ small enough, we have: $\exists k_0 \in \mathbb{N}$ such that $\pi(D_\lambda D_\alpha v_k) > 0, \forall k \geq k_0$. From the formulae (3.6), (3.8) and (3.13) of [9] one has:

$$\pi D_\lambda v_k = -\pi \exp^{-1}_{u_k} w,$$

$$\pi D^2_\lambda v_k = -2\pi \nabla_{D_\lambda v_k}(\exp^{-1}_. w) - \pi[(\nabla^2_. L_{\lambda_k})(u_k)(D_\lambda v_k, D_\lambda v_k)],$$

and

$$\pi D^2_\alpha v_k = -\pi[(\nabla^2_. L_{\lambda_k})(u_k)(\xi_k, \xi_k)].$$

It follows that: for χ small enough, there $k_0 \in \mathbb{N}$ such that: $\forall k \geq k_0$

$$\pi(D^2_\lambda v_k)\pi(D^2_\alpha v_k) - [\pi(D_\lambda D_\alpha v_k)]^2 < 0.$$

Step 5: The conclusion

We want to conclude by using the bifurcation behaviour of solutions of (1.1) at which one has a non trivial Jacobi field, which we studied in [9].

Since $\dim[Ker(\nabla_. L_{\lambda_k})(u_k)] = 1, \forall k$, we get from Step 3 that $\dim[Ker(\nabla_. L_{\tilde{\lambda}})(u)] \geq 1$ where $\tilde{\lambda} := \lambda$ in this part of the work.

We have seen in [9] that in any case (u_k, λ_k) (for $k \geq k_0$) and $(u, \tilde{\lambda})$ are bifurcation points, and we gave in each case the precise bifurcation behaviour. Actually, we studied only the case $\dim[Ker(\nabla_. L_{\tilde{\lambda}})(u)] = 1$ in [9]. But one can check that the higher dimensional case is similar. From the general bifurcation theory in [15], for $\dim[Ker(\nabla_. L_{\tilde{\lambda}})(u)] \geq 1$, $(\tilde{\lambda}, u)$ is a bifurcation point in any case and the number of solutions of our equation (1.1) is finite and is given by $(u, \tilde{\lambda})$ and $L_{\tilde{\lambda}}$. One uses the implicit function theorem to find the solutions of the equation (1.1) as in [9].

For sake of completeness we want to precise a little what happens at $(\tilde{\lambda}, u)$ under the assumption that $\dim[Ker(\nabla_. L_{\tilde{\lambda}})(u)] = 2$. For more details, see [9].

Let $\xi_1, \xi_2 \in Ker(\nabla_. L_{\tilde{\lambda}})(u)$ be such that

$$Ker(\nabla_. L_{\tilde{\lambda}})(u) = \mathbb{R}\xi_1 + \mathbb{R}\xi_2.$$

Let's set

$$\tilde{L}_{\tilde{\lambda}}(v) := L_{\tilde{\lambda}}(v) + \|^v_u \, [\pi_1(\exp^{-1}_u v)],$$

for v as in the theorem, where π_1 is the L^2 - orthogonal projection onto $Ker(\nabla_. L_{\tilde{\lambda}})(u)$. Then we have

$$L_{\tilde{\lambda}}(v) = 0 \Leftrightarrow \tilde{L}_{\tilde{\lambda}}(v) = \|^v_u \, [\pi_1(\exp^{-1}_u v)],$$

and

$$\nabla_V \tilde{L}_{\tilde{\lambda}} = \nabla_V L_{\tilde{\lambda}} + \pi_1(V), \forall V \in \Gamma_0^{1,2}(u^{-1}(TN))$$

and then $(\nabla_. \tilde{L}_{\tilde{\lambda}})(u)$ is a continuous linear isomorphism of $\Gamma_0^{1,2}(u^{-1}(TN))$.

For $\alpha_1, \alpha_2 \in \mathbb{R}$ we consider the equation

$$\tilde{L}_{\tilde{\lambda}}(v) = \|_u^v (\alpha_1\xi_1 + \alpha_2\xi_2) \Leftrightarrow \|_v^u L_{\tilde{\lambda}}(v) + \pi_1(\exp_u^{-1} v) = \alpha_1\xi_1 + \alpha_2\xi_2. \tag{3.6}$$

Since $(\nabla_. \tilde{L}_{\tilde{\lambda}})(u)$ is an isomorphism, the implicit function theorem applied to (3.6) gives: there exists $I \times J \times \mathcal{V}$ a neighborhood of $(\tilde{\lambda}, (0,0), u)$ in $\mathbb{R} \times \mathbb{R}^2 \times C^2(M,N)$, and a mapping $v : I \times J \to \mathcal{V}$ such that: $v(\tilde{\lambda}, (0,0)) = u$ and $\forall (\tilde{\lambda}, (\alpha_1, \alpha_2), w) \in I \times J \times \mathcal{V}$: $(\tilde{\lambda}, (\alpha_1, \alpha_2), w)$ solution of (3.6) iff $w = v(\tilde{\lambda}, (\alpha_1, \alpha_2))$, i.e. $L_{\tilde{\lambda}}(w) = 0$ iff $\exists (\alpha_1, \alpha_2) \in J$ such that $w = v(\tilde{\lambda}, (\alpha_1, \alpha_2))$.

We assume that $(\tilde{\lambda}, (\alpha_1, \alpha_2)) \in I \times J$. Then

$$L_{\tilde{\lambda}}(v(\tilde{\lambda}, (\alpha_1, \alpha_2))) = 0 \Leftrightarrow \pi_1(\exp_u^{-1} v(\tilde{\lambda}, (\alpha_1, \alpha_2))) = \alpha_1\xi_1 + \alpha_2\xi_2.$$

So solving the equation

$$\pi_1(\exp_u^{-1} v(\tilde{\lambda}, (\alpha_1, \alpha_2))) = \alpha_1\xi_1 + \alpha_2\xi_2, \text{ for } (\tilde{\lambda}, (\alpha_1, \alpha_2)) \in I \times J, \tag{3.7}$$

is equivalent to solve the equation

$$L_{\tilde{\lambda}}(w) = 0, \text{ for } (\tilde{\lambda}, w) \in I \times \mathcal{V}.$$

To find the number of solutions of equation (3.7), one sets $\tilde{\lambda} = \hat{\lambda} + \beta$, and then takes the asymptotic expansion near $(\beta, \alpha := (\alpha_1, \alpha_2)) = (0, (0,0))$ of equation (3.7). Then using the fact (proved in [9]) that

$$\pi_1[D_\alpha v(\tilde{\lambda}, (0,0))] = Id_{Ker(\nabla_. L_{\tilde{\lambda}})(u)},$$

one gets

$$\begin{aligned} 0 \;=\; & \beta\pi_1(D_{\tilde{\lambda}}v) + \frac{1}{2}\beta^2\pi_1(D^2_{\tilde{\lambda}}v) + \beta[\alpha_1\pi_1(D_{\tilde{\lambda}}D_{\alpha_1}v) + \alpha_2\pi_1(D_{\tilde{\lambda}}D_{\alpha_2}v)] \\ & + \frac{1}{2}[\alpha_1^2\pi_1(D^2_{\alpha_1}v) + \alpha_2^2\pi_1(D^2_{\alpha_2}v) \\ & + 2\alpha_1\alpha_2\pi_1(D_{\alpha_1}D_{\alpha_2}v) + \text{ higher order terms in } \beta, \alpha_1 \text{ and } \alpha_2, \end{aligned}$$

where $D_{\tilde{\lambda}}v$ is the partial derivative of $v(\tilde{\lambda}, (\alpha_1, \alpha_2))$ w.r.t $\tilde{\lambda}$ at $(\hat{\lambda}, (0,0))$ and $\pi_1(D_{\tilde{\lambda}}v) \in \mathbb{R}\xi_1 + \mathbb{R}\xi_2$. One can then discuss the number of solutions for a given β. In any case the number of solutions is finite and given by informations at $(\tilde{\lambda}, u)$

Let's go back to our sequence (λ_k, u_k). There exists $k_1 \geq k_0 \in \mathbb{N}$ such that: $\forall k \geq k_1, (\lambda_k = \tilde{\lambda} + \beta_k, u_k) \in I \times \mathcal{V}$ and $L_{\lambda_k}(u_k) = 0$. It follows that $u_k = v(\lambda_k, (\alpha_{1,k}, \alpha_{2,k}))$ with $(\alpha_{1,k}, \alpha_{2,k}) \in J$, for $k \geq k_1$. That is impossible since the set of solutions bifurcates also at (λ_k, u_k).

We conclude at this stage that Λ is either finite, or is made of a sequence which converges to $+\infty$ or $-\infty$.

By taking also the infimum over the set of all admissible v in the definition of λ_1 in the proof of $1°$), one sees that the sequence in Λ cannot converge to $-\infty$.

3.1.3 Proof of $3°$)

Let's assume now that w is harmonic and Λ is bounded from above by some $\alpha \in \mathbb{R}$.

For any $\lambda \in]\alpha, +\infty[$, w is a solution of (1.1). From $1°$), there exists $\beta > \alpha$ such that $Ker(\nabla_. L_\beta)(u) \neq \{0\}$. That is a contradiction since (w, β) is a solution of (1.1).

We conclude that Λ is not bounded from above.

In this way our theorem is proved

3.2 Proof of Theorem 2.2

Let $v \in W^{1,2}(M,N)_w$ and $V \in \Gamma_0^{1,2}(v^{-1}(TN))\backslash\{0\}$ be such that

$$Q_v(V) - \lambda \nabla_V \exp_{.}^{-1} w = 0. \tag{3.8}$$

We have

$$\left\langle Q_v(V) - \lambda \nabla_V \exp_{.}^{-1} w, V \right\rangle_{L^2} =$$

$$-\int_M [\left\|\nabla_{e_i} V\right\|^2 - \left\langle R^N(V, d_{e_i} v) d_{e_i} v, V \right\rangle + \lambda \left\langle \nabla_V \exp_{.}^{-1} w, V \right\rangle] dx,$$

so

$$\lambda = \mathcal{R}(v,V) := -\left\langle \nabla_V \exp_{.}^{-1} w, V \right\rangle_{L^2}^{-1} \int_M [\left\|\nabla_{e_i} V\right\|^2 - \left\langle R^N(V, d_{e_i} v) d_{e_i} v, V \right\rangle] dx. \tag{3.9}$$

It follows that

$$\lambda_1 = \inf_{W \in \Gamma_0^{1,2}(v^{-1}(TN))\backslash\{0\}} \mathcal{R}(v,W).$$

Then, by taking a minimizing sequence, one knows from the proof of 1°) in theorem 1 that there exists $V \in \Gamma_0^{1,2}(v^{-1}(TN))\backslash\{0\}$, such that $\lambda_1 = \mathcal{R}(v,V)$ and V is a solution of (3.8) for $\lambda = \lambda_1$.

Let $W \in \Gamma_0^{1,2}(v^{-1}(TN))$ be harmonic such that $W(x) \neq 0, \forall x \in M$. Then we have

$$\begin{aligned}
\Delta \langle V, W \rangle &= D_{e_i} D_{e_i} \langle V, W \rangle - D_{\nabla_{e_i} e_i} \langle V, W \rangle \\
&= D_{e_i} [\langle \nabla_{e_i} V, W \rangle + \langle V, \nabla_{e_i} W \rangle] - \left\langle \nabla_{\nabla_{e_i} e_i} V, W \right\rangle - \left\langle V, \nabla_{\nabla_{e_i} e_i} W \right\rangle \\
&= \left\langle \nabla_{e_i} \nabla_{e_i} V - \nabla_{\nabla_{e_i} e_i} V, W \right\rangle + 2 \langle \nabla_{e_i} V, \nabla_{e_i} W \rangle \\
&\quad + \left\langle V, \nabla_{e_i} \nabla_{e_i} W - \nabla_{\nabla_{e_i} e_i} W \right\rangle \\
&= -\left\langle R^N(V, d_{e_i} v) d_{e_i} v - \lambda \nabla_V \exp_{.}^{-1} w, W \right\rangle + 2 \langle \nabla_{e_i} V, \nabla_{e_i} W \rangle.
\end{aligned}$$

Because of our assumption on the curvature tensor of (N,h), if $\langle V, W \rangle (x) = 0$ for some $x \in M$, then the RHS of our last equation vanishes at x, by taking normal coordinates centered at x. We may then assume that

$$V = \langle V, W \rangle \|W\|^{-2} W.$$

It follows

$$\begin{aligned}
\nabla_{e_i} V &= [[D_{e_i} \langle V, W \rangle] \|W\|^{-2} - \langle V, W \rangle \|W\|^{-4} D_{e_i} \|W\|^2] W \\
&\quad + \langle V, W \rangle \|W\|^{-2} \nabla_{e_i} W,
\end{aligned}$$

and then

$$\begin{aligned}
\langle \nabla_{e_i} V, \nabla_{e_i} W \rangle &= [D_{e_i} \langle V, W \rangle] \|W\|^{-2} \langle W, \nabla_{e_i} W \rangle \\
&\quad - \frac{1}{2} \|W\|^{-4} [D_{e_i} \|W\|^2]^2 \langle V, W \rangle + \frac{1}{2} \|W\|^{-2} \left\|\nabla_{e_i} W\right\|^2 \langle V, W \rangle
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{2} [D_{e_i} \langle V, W \rangle] \|W\|^{-2} D_{e_i} \|W\|^2 - \frac{1}{2} \|W\|^{-4} [D_{e_i} \|W\|^2]^2 \langle V, W \rangle \\
&\quad + \frac{1}{2} \|W\|^{-2} \left\|\nabla_{e_i} W\right\|^2 \langle V, W \rangle.
\end{aligned}$$

By using these last informations we have

$$\begin{aligned}\Delta\langle V,W\rangle &= \|W\|^{-2}[[D_{e_i}\|W\|^2]D_{e_i}\langle V,W\rangle + \\ &[-\left\langle R^N(W,d_{e_i}v)d_{e_i}v,W\right\rangle + \frac{\lambda}{2}D^2(d^2(w,.))(W,W) \\ &-\|W\|^{-2}(D_{e_i}\|W\|^2)^2 + \left\|\nabla_{e_i}W\right\|^2]\langle V,W\rangle].\end{aligned}$$

Let's assume that $\langle V,W\rangle(x_0) > 0$ for some x_0. By multiplying $\langle V,W\rangle$ by -1 where ever it is necessary, one gets some $V_1 \in \Gamma_0^{1,2}(v^{-1}(TN))\backslash\{0\}$ such that $\langle V_1,W\rangle \geq 0$ and $\lambda_1 = \mathcal{R}(v,V_1)$. It follows that V_1 is a solution of (3.8) for $\lambda = \lambda_1$, and then $\langle V_1,W\rangle$ is a solution of

$$\begin{aligned}\Delta f &= \|W\|^{-2}[[D_{e_i}\|W\|^2]D_{e_i}f + \\ &[-\left\langle R^N(W,d_{e_i}v)d_{e_i}v,W\right\rangle + \frac{\lambda}{2}D^2(d^2(w,.))(W,W) \\ &-\|W\|^{-2}(D_{e_i}\|W\|^2)^2 + \left\|\nabla_{e_i}W\right\|^2]f].\end{aligned} \tag{3.10}$$

From the Harnack inequality for nonnegative solutions of such linear elliptic partial differential equations, we have that $\langle V_1,W\rangle > 0$ in the interior of M. It follows that actually $\langle V,W\rangle > 0$ in the interior of M, because of the local uniqueness of the solution of (3.10).

It remains to prove the simpleness of λ_1.

Let V_2 be another solution of (3.8) which is orthogonal to V in L^2.

Let's assume that there exists $x_3 \in M$ such that $\langle V,V_2\rangle(x_3) \neq 0$.

Then we have $\langle V_2,V\rangle > 0$ in the interior of M, or $\langle V_2,V\rangle < 0$ in the interior of M. It follows that V_2 cannot be orthogonal to V in L^2. In fact $\langle V,V_2\rangle$ is a solution of an equation similar to (3.10). So we must have $\langle V_2,V\rangle \equiv 0$. Since the set of our solutions is a vector space, there exist $x \in M$, and $\alpha,\beta \in \mathbb{R}$ such that $\alpha V + \beta V_2$ is a solution and $\langle \alpha V + \beta V_2, V\rangle(x) \neq 0$. So V_2 doesn't exist.

3.3 Proof of Theorem 2.3

3.3.1 Proof of 1°) :

For any $n \in \mathbb{N}$, let $\lambda^{(n)} \in \mathbb{R}$ be such that, there exists $(v_n,V_n) \in W^{1,2}(M,N)_w \times \mathcal{E}_{\lambda^{(n)},v_n}$ such that $L_{\lambda^{(n)}}(v_n) = 0$ and $\lim_{n\to+\infty}\lambda^{(n)} = \lambda_2^*$. Then, in the same way as in the proof of 2°) of Theorem 1, there exists a subsequence of (v_n,V_n) and $(v,V) \in W^{1,2}(M,N)_w \times \mathcal{E}_{\lambda_2^*,v}$ such that $(v_n)_{n\in\mathbb{N}}$ converges to v in $W^{1,2}$, $(V_n)_{n\in\mathbb{N}}$ converges to V in $W^{1,2}$, and $L_{\lambda_2^*}(v) = 0$.

3.3.2 Proof of 2°) :

Let $v \in W^{1,2}(M,N)_w, \lambda \in \mathbb{R}$ and $V \in \Gamma_0^{1,2}(v^{-1}(TN))\backslash\{0\}$ be such that

$$L_\lambda(v) = 0 \text{ and } Q_v(V) - \lambda\nabla_V \exp_.^{-1} w = 0.$$

We have

$$\begin{aligned}\left\langle \tau(v) - \lambda\exp_v^{-1}w, \exp_v^{-1}w\right\rangle_{L^2} &= -\int_M [\left\langle \nabla_{d_{e_i}v}\exp_.^{-1}w, d_{e_i}v\right\rangle \\ &+ \left\langle d_{e_i}v, D_{d_{e_i}w}(\exp_v^{-1}.)\right\rangle]dx - \lambda d_{L^2}^2(v,w),\end{aligned}$$

by the divergence theorem, so

$$\lambda = \mathcal{F}(v) := -d_{L^2}^{-2}(v,w)\int_M [\left\langle \nabla_{d_{e_i}v}\exp_{.}^{-1}w, d_{e_i}v\right\rangle + \left\langle d_{e_i}v, D_{d_{e_i}w}(\exp_v^{-1}.)\right\rangle]dx.$$

From (3.9) we have also

$$\lambda = \mathcal{R}(v,V) = -\left\langle \nabla_V \exp_{.}^{-1}w, V\right\rangle_{L^2}^{-1}\int_M [\left\|\nabla_{e_i}V\right\|^2 - \left\langle R^N(V,d_{e_i}v)d_{e_i}v, V\right\rangle]dx.$$

Let's set

$$\begin{aligned}\mathcal{A} &: = \{v \in W^{1,2}(M,N)_w \ / \ L_{\mathcal{F}(v)}(v) = 0, \mathcal{F}(v) \text{ is degenerated }\},\\ \mathcal{B}_v &: = \{V \in \Gamma_0^{1,2}(v^{-1}(TN)) \ / \mathcal{R}(v,V) \geq \mathcal{F}(v)\},\end{aligned}$$

and

$$\lambda_3^* := \inf_{v\in\mathcal{A}, V\in\mathcal{B}_v} \mathcal{R}(v,V).$$

We are going next to prove that there exists $v \in W^{1,2}(M,N)_w$ such that $L_{\lambda_3^*}(v) = 0$, and there exists $V \in \Gamma_0^{1,2}(v^{-1}(TN))$ such that $\mathcal{R}(v,V) = \lambda_3^*$.

Let (v_n, V_n) be a minimizing sequence for $\mathcal{R}(v,V)$ in the considered set. We may assume w.l.o.g. that V_n minimizes $\mathcal{R}(v_n,.)$ under the constraint $V \in \Gamma_0^{1,2}(v_n^{-1}(TN))$ and $\mathcal{R}(v_n,V) \geq \mathcal{F}(v_n)$, and

$$-\left\langle \nabla_{V_n}\exp_{.}^{-1}w, V_n\right\rangle_{L^2} = 1, \forall n \in \mathbb{N}. \tag{3.11}$$

Since $\mathcal{F}(v_n)$ is degenerated, we have $\mathcal{F}(v_n) \geq \mathcal{R}(v_n,V_n), \forall n \in \mathbb{N}$. So $\mathcal{F}(v_n) = \mathcal{R}(v_n,V_n), \forall n \in \mathbb{N}$. It follows that $V_n \in \mathcal{E}_{\mathcal{F}(v_n),v_n}, \forall n \in \mathbb{N}$.

We have

$$\mathcal{R}(v_n,V_n) = \mathcal{F}(v_n), \forall n \in \mathbb{N}, \text{ and } \lim_{n\to+\infty}\mathcal{R}(v_n,V_n) = \lambda_3^* \in \mathbb{R}.$$

It follows from [13] that, there exists $v \in W^{1,2}(M,N)_w$ and a subsequence of $(v_n)_{n\in\mathbb{N}}$ which converges in $W^{1,2}$ to v and $L_{\lambda_3^*}(v) = 0$. From the same work, we have that there exists $C_{14} > 0$ such that

$$\|dv_n\|(x) \leq C_{14}, \forall x \in M, \forall n \in \mathbb{N}.$$

Using this, and since our manifolds are compact, there exists $C_{15} > 0$ such that

$$\left|\left\langle R^N(V_n, d_{e_i}v_n)d_{e_i}v_n, V_n\right\rangle(x)\right| \leq C_{15}\|V_n\|^2(x), \forall x \in M, \forall n \in \mathbb{N}.$$

From (3.11), there exists $C_{16} > 0$ such that

$$\left|\int_M \left\langle R^N(V_n, d_{e_i}v_n)d_{e_i}v_n, V_n\right\rangle(x)dx\right| \leq C_{16}, \forall n \in \mathbb{N}.$$

Since the sequence $(\mathcal{R}(v_n,V_n))_n$ is bounded, there exists $C_{17} > 0$ such that

$$\int_M \|\nabla V_n\|^2(x)dx \leq C_{17}, \forall n \in \mathbb{N}.$$

The theorem of Rellich - Kondrachov then gives the existence of a subsequence of $(V_n)_{n\in\mathbb{N}}$ which converges in L^2 and weakly in $W^{1,2}$ to some $V \in \Gamma_0^{1,2}(v^{-1}(TN))$. It follows that

$$\|V\|_{L^2}^2 := -\left\langle \nabla_V \exp_{.}^{-1}w, V\right\rangle_{L^2} = 1.$$

And, as in the proof of Theorem 2, we have that $V \in \mathcal{E}_{\lambda_3^*,v}$, since $V_n \in \mathcal{E}_{\mathcal{F}(v_n),v_n}, \forall n \in \mathbb{N}$.

From the definition of λ_3^* we have $\lambda_3^* \leq \lambda_2^*$. Since $\mathcal{F}(v) = \mathcal{R}(v,V) = \lambda_3^*$, $L_{\lambda_3^*}(v) = 0$ and $V \in \mathcal{E}_{\lambda_3^*,v}$, we have $\lambda_2^* \leq \lambda_3^*$. We conclude that $\lambda_2^* = \lambda_3^*$.

From this point the remaining part of this proof is the same as in the proof of Theorem 2.

3.3.3 Proof of 3°) :

Let $v \in W^{1,2}(M,N)_w$ and $V \in \Gamma_0^{1,2}(v^{-1}(TN))\backslash\{0\}$ be such that

$$L_{\lambda_2^*}(v) = 0 \text{ and } Q_v(V) - \lambda_2^* \nabla_V \exp_.^{-1} w = 0.$$

We have seen in 2°) that $Ker(\nabla_. L_{\lambda_2^*})(v) = \mathbb{R}V$. We are going to use this fact to prove that V is integrable. Once that is proved, our result will follow directly from the nonexistence result for integrable Jacobi field, for $\lambda < 0$, which is in [13].

Let $\varepsilon > 0$ be such that

$$\theta(x) + \varepsilon < \min\{inj(w(x)), \frac{\pi}{2\sqrt{K}}\}, \forall x \in M.$$

Let

$$W^{1,2}(M,N)'_w :=$$
$$\{v \in W^{1,2}(M,N) \ / \ u_{|\partial M} = w_{|\partial M} \text{ and } d(w(x), v(x)) < \theta(x) + \varepsilon, \forall x \in M\}.$$

We are going to look at the structure near v of the set

$$\mathcal{G}_w := \{u \in W^{1,2}(M,N)'_w \ / \ L_{\lambda_2^*}(u) = 0\}.$$

The mapping $\psi : u \mapsto \exp_w^{-1} u$ is a bijection from $W^{1,2}(M,N)'_w$ onto an open subset of $\Gamma_0^{1,2}(w^{-1}(TN))$. The structure of manifold we consider on $W^{1,2}(M,N)'_w$ is the one defined by this bijection and the structure of $\mathbb{R}$ - Banach space of $\Gamma_0^{1,2}(w^{-1}(TN))$.

Let's set

$$\mathcal{H}_w := \cup_{u \in W^{1,2}(M,N)'_w} \Gamma_0^{1,2}(u^{-1}(TN)).$$

Then $\mathcal{H}_w$ is a vector bundle over $W^{1,2}(M,N)'_w$, $u \mapsto L_{\lambda_2^*}(u)$ is a section of this bundle, and, by using the parallel transport from each $u \in W^{1,2}(M,N)'_w$ to w, we have

$$\mathcal{H}_w \simeq W^{1,2}(M,N)'_w \times \Gamma_0^{1,2}(w^{-1}(TN)).$$

It follows that: $\forall u \in W^{1,2}(M,N)'_w$, we have

$$L_{\lambda_2^*}(u) = (u, \tilde{L}_{\lambda_2^*}(u)) \text{ where } \tilde{L}_{\lambda_2^*}(u) := \|_u^v \circ L_{\lambda_2^*}(u) \in \Gamma_0^{1,2}(v^{-1}(TN)).$$

So, we have $L_{\lambda_2^*}(u) = 0 \Longleftrightarrow \tilde{L}_{\lambda_2^*}(u) = 0$.

Let v_t be a variation of v in $W^{1,2}(M,N)'_w$, i.e. a differentiable mapping $]-1,1[\to W^{1,2}(M,N)'_w, t \mapsto v_t$ such that $v_0 = v$. Let $V_1 := \frac{\partial v_t}{\partial t}_{|t=0}$.

In local coordinates one has

$$\tilde{L}_{\lambda_2^*}(v_t)(x) = [L_{\lambda_2^*}(v_t)(x)]^\alpha \ \|_{v_t}^v \ [\frac{\partial}{\partial y^\alpha}(v_t(x))], \forall t \in]-1,1[.$$

Since $[\nabla_t \ \|_{v_t}^v]_{|t=0} = 0$, we have

$$\frac{d}{dt}_{|t=0} \tilde{L}_{\lambda_2^*}(v_t)(x) = \nabla_{V_1} L_{\lambda_2^*},$$

and then: $\forall W \in \Gamma_0^{1,2}(w^{-1}(TN))$,

$$\begin{aligned} \frac{d}{dt}_{|t=0} \left\langle \tilde{L}_{\lambda_2^*}(v_t), W \right\rangle_{L^2} &= \left\langle \frac{d}{dt}_{|t=0} \tilde{L}_{\lambda_2^*}(v_t), W \right\rangle_{L^2} = \left\langle D_{V_1} \tilde{L}_{\lambda_2^*}, W \right\rangle_{L^2} \\ &= \left\langle \nabla_{V_1} L_{\lambda_2^*}, \|_w^v \ W \right\rangle_{L^2}. \end{aligned}$$

Let's assume that $\nabla_W L_{\lambda_2^*} \neq 0$. Then there exists some variation v_t of v such that $\frac{d}{dt}_{|t=0} \left\langle \tilde{L}_{\lambda_2^*}(v_t), W \right\rangle_{L^2} \neq 0$. It follows that the function $u \mapsto \left\langle \tilde{L}_{\lambda_2^*}(u), W \right\rangle_{L^2}$ is of rank one at the point v, and then the set

$$\{u \in W^{1,2}(M,N)'_w \ / \ \left\langle \tilde{L}_{\lambda_2^*}(u), W_n \right\rangle_{L^2} = 0\}$$

is a submanifold of $W^{1,2}(M,N)'_w$. Let $(W_n)_{n\in\mathbb{N}}$ be an orthonormal basis (w.r.t. L^2 - product) of $\Gamma_0^{1,2}(w^{-1}(TN))$. We may assume that $W_n \neq V, \forall n \in \mathbb{N}$. Then

$$\begin{aligned} \mathcal{G}_w &= \cap_{n\in\mathbb{N}}\{u \in W^{1,2}(M,N)'_w \ / \ \left\langle \tilde{L}_{\lambda_2^*}(u), W_n \right\rangle_{L^2} = 0\} \\ &= \cap_{n\in\mathbb{N}} \cap_{0\leq m\leq n} \{u \in W^{1,2}(M,N)'_w \ / \ \left\langle \tilde{L}_{\lambda_2^*}(u), W_m \right\rangle_{L^2} = 0\}. \end{aligned}$$

$(\cap_{0\leq m\leq n}\{u \in W^{1,2}(M,N)'_w \ / \ \left\langle \tilde{L}_{\lambda_2^*}(u), W_m \right\rangle_{L^2} = 0\})_{n\geq 1}$ is a nonincreasing sequence of submanifolds of $W^{1,2}(M,N)'_w$, so $\mathcal{G}_w$ is a submanifold of $W^{1,2}(M,N)'_w$.

We have $v \in \mathcal{G}_w$ and the tangent space of $\mathcal{G}_w$ at v is

$$\begin{aligned} T_v\mathcal{G}_w &= \cap_{n\in\mathbb{N}} T_v\{u \in W^{1,2}(M,N)'_w \ / \ \left\langle \tilde{L}_{\lambda_2^*}(u), W_n \right\rangle_{L^2} = 0\} \\ &= \cap_{n\in\mathbb{N}}\{V_1 \in \Gamma_0^{1,2}(v^{-1}(TN)) \ / \ \left\langle \nabla_{V_1} L_{\lambda_2^*}, \|_w^v \ W_n \right\rangle_{L^2} = 0\}. \end{aligned}$$

It is obvious that $V \in T_v\mathcal{G}_w$ since $\nabla_V L_{\lambda_2^*} = 0$. It follows that V is integrable. In this way 3°) is proved

3.4 Proof of Corollary 2.4

Let's assume that

$$\beta := \sup\{\alpha \in [\lambda, 0[\ / \ \exists u_\alpha \ \in W^{1,2}(M,N)_w \text{ satisfying } L_\alpha(u_\alpha) = 0\} < 0.$$

Let $(\alpha_n)_{n\in\mathbb{N}} \in [\lambda, 0[^{\mathbb{N}}$ be converging to β such that: For any $n \in \mathbb{N}$, there exists $u_{\alpha_n} \in W^{1,2}(M,N)_w$ satisfying $L_{\alpha_n}(u_{\alpha_n}) = 0$ and the same assumptions as u. We have seen in the proof of 2°) of Theorem 1 that there exists $v \in W^{1,2}(M,N)_w$ such that $L_\beta(v) = 0$. Since $\beta < 0$ we have that $Ker(\nabla_. L_\beta)(v) = \{0\}$. From the implicit function theorem as we used it in the description of the bifurcation behaviour, there exists $r > 0$ such that $]\beta, \beta + r[\subseteq [\lambda, 0[$ and, for any $\alpha \in]\beta, \beta + r[$, there exists $u_\alpha \in W^{1,2}(M,N)_w$ satisfying $L_\alpha(u_\alpha) = 0$. This contradicts the definition of β. We conclude that $\beta = 0$

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