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STRONGLY NONLINEAR $p(x)$-ELLIPTIC PROBLEMS WITH L^1-DATA

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Abstract

In this paper, we will study the existence of solutions in the sense of distributions for the quasilinear $p(x)$-elliptic problem,

$$Au + g(x,u,\nabla u) = f,$$

where A is a Leray-Lions operator from $W_0^{1,p(\cdot)}(\Omega)$ into its dual, the nonlinear term $g(x,s,\xi)$ has a growth condition with respect to ξ and the sign condition with respect to s. The datum f is assumed in the dual space $W^{-1,p'(\cdot)}(\Omega)$, and then in $L^1(\Omega)$.

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1 Introduction.

Let Ω be a bounded open subset of $I\!R^N$.
In [7], *A. Bensoussan*, *L. Boccardo* and *F. Murat* have studied the nonlinear elliptic problem

$$Au + g(x,u,\nabla u) = f \quad \text{in} \quad \Omega,$$

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where A is a Leray-Lions operator from $W_0^{1,p}(\Omega)$ into $W^{-1,p'}(\Omega)$, g is a Carathéodory function satisfying the sign and growth condition, the data f belong to $W^{-1,p'}(\Omega)$, they proved the existence of solution in the sense of distributions $u \in W_0^{1,p}(\Omega)$ such that $g(x,u,\nabla u) \in L^1(\Omega)$ and $g(x,u,\nabla u)u \in L^1(\Omega)$. In the case of $f \in L^1(\Omega)$, *L. Boccardo* and *T. Gallouët* [8] have proved the existence of solutions $u \in W_0^{1,p}(\Omega)$ with $g(x,u,\nabla u) \in L^1(\Omega)$ under the additional assumption :

$$\text{there exists} \quad \sigma > 0, \gamma > 0 \quad \text{such that} \quad |g(x,s,\xi)| \geq \gamma|\xi|^p \quad \text{for} \quad |s| \geq \sigma.$$

In the recent years, variable exponent Sobolev spaces have attracted an increasing amount of attention, the impulse for this mainly comes from there physical applications, such in image processing (underline the borders, eliminate the noise) and electro-rheological fluids.
In the framework of variable exponent Sobolev spaces, M. Bendahmane and P. Wittbold **[6]** have proved the existence and uniqueness of the renormalized solutions to the nonlinear elliptic problem

$$\begin{cases} -\operatorname{div}\,(|\nabla u|^{p(x)-2}\nabla u) = f & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases}$$

where the right-hand side $f \in L^1(\Omega)$ and $p(\cdot) : \bar{\Omega} \longmapsto (1,+\infty)$ a continuous function.
M. Sanchón and J. M. Urbano have proved in **[14]** the existence and uniqueness of solution for a more general problem

$$\begin{cases} -\operatorname{div}\,(a(x,\nabla u)) = f & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases}$$

with $f \in L^1(\Omega)$ and $a : \Omega \times I\!R^N \longmapsto I\!R^N$ is a Carathéodory function which verify the natural extensions of Leray-Lions assumptions to the variable exponent case, with $p(.)$ is a measurable function such that

$$p(.) \in W^{1,\infty}(\Omega) \qquad \text{and} \qquad 1 < ess\inf_{x\in\Omega} p(x) \leq ess\sup_{x\in\Omega} p(x) < N.$$

Recently, M. B. Benboubker, E. Azroul and A. Barbara [5] have shown the existence of solutions for the $p(x)$-quasilinear elliptic problem

$$\begin{cases} -\operatorname{div} a(x,u,\nabla u) = f(x,u,\nabla u) & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases}$$

by using the calculus of variations operators method, where A is a Leray-Lions operator and f is a Carathéodory function which satisfies some growth condition.

In this paper, we will study the existence of solutions in the sense of distributions for the following quasilinear $p(x)$-elliptic problem

$$\begin{cases} -\operatorname{div} a(x,u,\nabla u) + g(x,u,\nabla u) = f & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{1.1}$$

where the second member f is first taken in $W^{-1,p'(\cdot)}(\Omega)$ and then in $L^1(\Omega)$.
Note that, in the case of $p(\cdot) = p = cte$, the problem (1.1) was studed by L. Boccardo, T.

Gallouët and F. Murat in [9], when the variational case is treated.
This paper can be seen as a generalization to the variable exponent of the work of Y. Akdim, E. Azroul and A. Benkirane [2] and of [1, 3, 9], and as a continuation of the works [5, 6, 14]. The paper is organized as follows. In section 2, we recall some important definitions and results of variable exponent Lebesgue and Sobolev spaces. In section 3 we give the assumptions on $a(x,s,\xi)$ and $g(x,s,\xi)$ for which our problem has a solutions in the sense of distributions. We introduce in the section 4 some important lemmas useful to prove our main results. In the first part of the section 5, we study the problem (1.1) for $f \in W^{-1,p'(\cdot)}(\Omega)$, the second part of the section 5 will be devoted to the study of the problem (1.1) in the case of $f \in L^1(\Omega)$, we will need the assumption

$$\exists \rho_1, \rho_2 > 0, \text{ such that : } \quad \text{if} \quad |s| > \rho_1 \quad \Rightarrow \quad |g(x,s,\xi)| \geq \rho_2 \, |\xi|^{p(x)}.$$

2 Preliminaries.

Let Ω be a bounded open subset of $\mathbb{R}^N$ $(N \geq 2)$. The continuous real-valued function $p(\cdot)$ is log-Hölder continuous in Ω if

$$|p(x) - p(y)| \leq \frac{C}{|log|x-y||} \qquad \forall x, y \in \overline{\Omega} \quad \text{such that} \quad |x-y| < \frac{1}{2},$$

with possible different constant C. We denote

$$C_+(\overline{\Omega}) = \{\text{log-Hölder continuous function} \quad p(\cdot) : \overline{\Omega} \longrightarrow \mathbb{R} \quad \text{such that} \quad 1 < p_- \leq p_+ < N\},$$

where

$$p_- = \min\{p(x) \,/\, x \in \overline{\Omega}\} \qquad \text{and} \qquad p_+ = \max\{p(x) \,/\, x \in \overline{\Omega}\}.$$

Note that the log-Hölder continuity of the exponent $p(\cdot)$ is necessary to obtain the generalized Poincaré and Sobolev-Poincaré type inequality (see [10], [13]).

We define the variable exponent Lebesgue space for $p(\cdot) \in C_+(\overline{\Omega})$ by

$$L^{p(\cdot)}(\Omega) = \{u : \Omega \longrightarrow \mathbb{R} \quad \text{measurable} \,/\, \int_\Omega |u(x)|^{p(x)}\, dx < \infty\},$$

the space $L^{p(\cdot)}(\Omega)$ under the norm

$$||u||_{p(\cdot)} = \inf\left\{\lambda > 0, \quad \int_\Omega \left|\frac{u(x)}{\lambda}\right|^{p(x)} dx \leq 1\right\}$$

is a uniformly convex Banach space, and therefore reflexive. We denote by $L^{p'(\cdot)}(\Omega)$ the conjugate space of $L^{p(\cdot)}(\Omega)$ where $\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$ (see [11], [15]).

Proposition 2.1. ***(see [11], [15]) (Generalized Hölder Inequality)***

(i) *For any* $u \in L^{p(\cdot)}(\Omega)$ *and* $v \in L^{p'(\cdot)}(\Omega)$, *we have*

$$\left|\int_\Omega u\, v\, dx\right| \leq \Big(\frac{1}{p_-} + \frac{1}{p'_-}\Big)||u||_{p(\cdot)}\, ||v||_{p'(\cdot)}\,.$$

(ii) *For all $p_1,\, p_2 \in C_+(\overline{\Omega})$ such that $p_1(x) \leq p_2(x)$ a.e in Ω, then $L^{p_2(x)}(\Omega) \hookrightarrow L^{p_1(x)}(\Omega)$ and the embedding is continuous.*

Proposition 2.2. ***(see [11], [15])***
We denote the modular

$$\rho(u) = \int_\Omega |u|^{p(x)}\,dx, \quad \forall u \in L^{p(\cdot)}(\Omega),$$

then, the following assertions hold

(i) $\|u\|_{p(\cdot)} < 1 \quad (resp, = 1, > 1) \quad \Longleftrightarrow \quad \rho(u) < 1 \quad (resp, = 1, > 1)$,

(ii) $\|u\|_{p(\cdot)} > 1 \implies \|u\|^{p_-}_{p(\cdot)} \leq \rho(u) \leq \|u\|^{p_+}_{p(\cdot)}$ *and* $\|u\|_{p(\cdot)} < 1 \implies \|u\|^{p_+}_{p(\cdot)} \leq \rho(u) \leq \|u\|^{p_-}_{p(\cdot)}$,

(iii) $\|u_n\|_{p(\cdot)} \to 0 \iff \rho(u_n) \to 0,$ *and* $\|u_n\|_{p(\cdot)} \to \infty \iff \rho(u_n) \to \infty.$

Which implies that the norm convergence and the modular convergence are equivalent.

Now, we define the variable exponent Sobolev spaces by

$$W^{1,p(\cdot)}(\Omega) = \{\, u \in L^{p(\cdot)}(\Omega) \quad \text{and} \quad |\nabla u| \in L^{p(\cdot)}(\Omega)\},$$

normed by

$$\|u\|_{1,p(\cdot)} = \|u\|_{p(\cdot)} + \|\nabla u\|_{p(\cdot)} \qquad \forall u \in W^{1,p(\cdot)}(\Omega).$$

We denote by $W_0^{1,p(\cdot)}(\Omega)$ the closure of $C_0^\infty(\Omega)$ in $W^{1,p(\cdot)}(\Omega)$ and we define the Sobolev exponent by $p^*(x) = \dfrac{N\,p(x)}{N - p(x)}$ for $p(x) < N$.

Proposition 2.3. ***(see [11, 13])***

(i) *Assuming $1 < p_- \leq p_+ < \infty$, the spaces $W^{1,p(x)}(\Omega)$ and $W_0^{1,p(x)}(\Omega)$ are separable and reflexive Banach spaces.*

(ii) *If $q(\cdot) \in C_+(\bar{\Omega})$ and $q(x) < p^*(x)$ for any $x \in \Omega$, then embedding $W_0^{1,p(x)}(\Omega) \hookrightarrow\hookrightarrow L^{q(x)}(\Omega)$ is continuous and compact.*

(iii) *Poincaré type inequality : there exists a constant $C > 0$, such that*

$$\|u\|_{p(\cdot)} \leq C\,\|\nabla u\|_{p(\cdot)} \qquad \forall u \in W_0^{1,p(\cdot)}(\Omega).$$

(vi) *Sobolev type inequality : there exists an other constant $C > 0$, such that*

$$\|u\|_{p*(\cdot)} \leq C\,\|\nabla u\|_{p(\cdot)} \qquad \forall u \in W_0^{1,p(\cdot)}(\Omega).$$

Definition 2.4. By (iii) of the proposition 2.3, we deduce that $\|\nabla u\|_{p(\cdot)}$ and $\|u\|_{1,p(\cdot)}$ are equivalent norms in $W_0^{1,p(\cdot)}(\Omega)$.

Theorem 2.5. ***(see [10])*** *We denote the dual space of the Sobolev space* $W_0^{1,p(\cdot)}(\Omega)$ *by* $W^{-1,p'(\cdot)}(\Omega)$, *and for each* $F \in W^{-1,p'(\cdot)}(\Omega)$ *there exists* $f_0, f_1, \ldots, f_N \in L^{p'(\cdot)}(\Omega)$ *such that* $F = f_0 + \sum_{i=1}^{N} \frac{\partial f_i}{\partial x_i}$, *and for all* $u \in W_0^{1,p(\cdot)}(\Omega)$ *we have*

$$\langle F, u\rangle_{W^{-1,p'(\cdot)}(\Omega), W_0^{1,p(\cdot)}(\Omega)} = \int_\Omega f_0 u\, dx - \sum_{i=1}^{N} \int_\Omega f_i \frac{\partial u}{\partial x_i}\, dx.$$

Moreover,

$$||F||_{-1,p'(\cdot)} = \sum_{i=0}^{N} ||f_i||_{p'(\cdot)}.$$

Definition 2.6. For all $k > 0$ and $s \in \mathbb{R}$, the truncation function $T_k(\cdot)$ can be defined by

$$T_k(s) = \begin{cases} s & \text{if} \quad |s| \le k, \\ k\dfrac{s}{|s|} & \text{if} \quad |s| > k. \end{cases}$$

and we define

$$T_0^{1,p(\cdot)}(\Omega) := \{\text{measurable function} \quad u \quad \text{such that} \qquad T_k(u) \in W_0^{1,p(\cdot)}(\Omega) \quad \forall k > 0\}.$$

Proposition 2.7. *Let* $u \in T_0^{1,p(\cdot)}(\Omega)$, *there exists a unique measurable function* $v : \Omega \longmapsto \mathbb{R}^N$ *such that*

$$v.\chi_{\{|u| \le k\}} = \nabla T_k(u) \qquad \text{for} \quad a.e.\ x \in \Omega \qquad \text{and} \qquad \text{for all} \quad k > 0.$$

We will define the gradient of u as the function v, and we will denote it by $v = \nabla u$. *Moreover, if* $u \in W_0^{1,1}(\Omega)$, *then* v *coincides with the standard distributional gradient of u, (see. [14]).*

Lemma 2.8. *Let* $\lambda \in \mathbb{R}$ *and let u and v be two functions which are finite almost everywhere, and belong to* $\mathcal{T}_0^{1,p(\cdot)}(\Omega)$, *then*

$$\nabla(u + \lambda v) = \nabla u + \lambda \nabla v \quad a.e.\ in\ \Omega,$$

where ∇u, ∇v *and* $\nabla(u + \lambda v)$ *are the gradients of u, v and* $u + \lambda v$ *introduced in the Proposition 2.7.*

Proof. Let $E_n = \{|u| \le n\} \cap \{|v| \le n\}$. We have $T_n(u) = u$ and $T_n(v) = v$ in E_n, then for every $k > 0$

$$T_k(T_n(u) + \lambda T_n(v)) = T_k(u + \lambda v) \ \text{ a.e. in } E_n,$$

and therefore, since both functions belong to $W_0^{1,p(\cdot)}(\Omega)$,

$$\nabla T_k(T_n(u) + \lambda T_n(v)) = \nabla T_k(u + \lambda v) \ \text{ a.e. in } E_n. \tag{2.1}$$

Since $T_n(u)$ and $T_n(v)$ belong to $W_0^{1,p(\cdot)}(\Omega)$, we have by using a classical property of the truncates functions in $W_0^{1,p(\cdot)}(\Omega)$, and the definition of ∇u and ∇v,

$$\begin{aligned} \nabla T_k(T_n(u) + \lambda T_n(v)) &= \chi_{\{|T_n(u)+\lambda T_n(v)| \le k\}}(\nabla T_n(u) + \lambda \nabla T_n(v)) \\ &= \chi_{\{|T_n(u)+\lambda T_n(v)| \le k\}}(\nabla u.\chi_{\{|u| \le n\}} + \lambda \nabla v.\chi_{\{|v| \le n\}}) \ \text{ a.e. in } \ \Omega. \end{aligned}$$

Therefore

$$\nabla T_k(T_n(u)+\lambda T_n(v)) = \chi_{\{|u+\lambda v|\le k\}}(\nabla u+\lambda\nabla v) \text{ a.e. in } E_n. \tag{2.2}$$

On the other hand, by definition of $\nabla(u+\lambda v)$,

$$\nabla T_k(u+\lambda v) = \chi_{\{|u+\lambda v|\le k\}}\nabla(u+\lambda v) \text{ a.e. in } E_n. \tag{2.3}$$

Putting together (2.1), (2.2) and (2.3), we obtain

$$\chi_{\{|u+\lambda v|\le k\}}\nabla(u+\lambda v) = \chi_{\{|u+\lambda v|\le k\}}(\nabla u+\lambda\nabla v) \text{ a.e. in } E_n. \tag{2.4}$$

We have $\bigcup_{n\in N} E_n$ (resp. $\bigcup_{k\in\mathbb{N}}\{|u+\lambda v|\le k\}$) differs at most from Ω by a set of zero Lebesgue measure, since u and v are almost everywhere finite, then (2.4) holds almost everywhere in Ω. which conclude the proved of Lemma 2.8.

3 Main assumptions.

Let Ω be a bounded open subset of $\mathbb{R}^N$ $(N\ge 2)$ and $p(\cdot)\in C_+(\bar\Omega)$, we consider a Leray-Lions operator A from $W_0^{1,p(\cdot)}(\Omega)$ into its dual $W^{-1,p'(\cdot)}(\Omega)$, defined by

$$Au = -\text{div } a(x,u,\nabla u) \tag{3.1}$$

where $a:\Omega\times\mathbb{R}\times\mathbb{R}^N \longmapsto \mathbb{R}^N$ is a *Carathéodory* function (measurable with respect to x in Ω for every (s,ξ) in $\mathbb{R}\times\mathbb{R}^N$, and continuous with respect to (s,ξ) in $\mathbb{R}\times\mathbb{R}^N$ for almost every x in Ω) satisfying the following conditions

$$|a(x,s,\xi)| \le \beta(K(x)+|s|^{p(x)-1}+|\xi|^{p(x)-1}), \tag{3.2}$$

$$a(x,s,\xi)\cdot\xi \ge \alpha|\xi|^{p(x)}, \tag{3.3}$$

$$(a(x,s,\xi)-a(x,s,\overline{\xi}))\cdot(\xi-\overline{\xi})>0 \quad \text{for all } \xi\ne\overline{\xi} \text{ in } \mathbb{R}^N, \tag{3.4}$$

for a.e. $x\in\Omega$, all $(s,\xi)\in\mathbb{R}\times\mathbb{R}^N$, where $K(x)$ is a nonnegative function lying in $L^{p'(\cdot)}(\Omega)$ and $\alpha,\beta>0$.
The nonlinear term $g(x,s,\xi)$ is a Carathéodory function which satisfies

$$g(x,s,\xi)s \ge 0, \tag{3.5}$$

$$|g(x,s,\xi)| \le b(|s|)(c(x)+|\xi|^{p(x)}), \tag{3.6}$$

where $b:\mathbb{R}^+\to\mathbb{R}^+$ is a continuous, nondecreasing function, and $c:\Omega\to\mathbb{R}^+$ with $c\in L^1(\Omega)$.
We consider the problem

$$\begin{cases} -\text{div}\, a(x,u,\nabla u)+g(x,u,\nabla u) = f & \text{in} \quad \Omega, \\ u=0 & \text{on} \quad \partial\Omega, \end{cases} \tag{3.7}$$

with $f\in W^{-1,p'(\cdot)}(\Omega)$ and later in $L^1(\Omega)$.

4 Some technical lemmas.

Lemma 4.1. *(see [5]) Let $g \in L^{r(\cdot)}(\Omega)$ and $g_n \in L^{r(\cdot)}(\Omega)$ with $\|g_n\|_{r(\cdot)} \le C$ for $1 < r(\cdot) < \infty$. If $g_n(x) \to g(x)$ a.e. on Ω, then $g_n \rightharpoonup g$ in $L^{r(\cdot)}(\Omega)$.*

Lemma 4.2. *Let $u \in W_0^{1,p(\cdot)}(\Omega)$ then $T_k(u) \in W_0^{1,p(\cdot)}(\Omega)$ for all $k > 0$. Moreover we have*

$$T_k(u) \longrightarrow u \quad \text{in } W_0^{1,p(\cdot)}(\Omega) \quad \text{as } k \to \infty.$$

Proof. We have $u \in W_0^{1,p(\cdot)}(\Omega)$, by the Proposition 2.7 it's clear that $T_k(u) \in W_0^{1,p(\cdot)}(\Omega)$, and

$$\begin{aligned}
&\int_\Omega |T_k(u) - u|^{p(x)} dx + \int_\Omega |\nabla T_k(u) - \nabla u|^{p(x)} dx \\
&\quad = \int_{\{|u| \le k\}} |T_k(u) - u|^{p(x)} dx + \int_{\{|u| > k\}} |T_k(u) - u|^{p(x)} dx \\
&\qquad + \int_{\{|u| \le k\}} |\nabla T_k(u) - \nabla u|^{p(x)} dx + \int_{\{|u| > k\}} |\nabla T_k(u) - \nabla u|^{p(x)} dx \\
&\quad = \int_{\{|u| > k\}} |T_k(u) - u|^{p(x)} dx + \int_{\{|u| > k\}} |\nabla u|^{p(x)} dx.
\end{aligned}$$

Since $T_k(u) \to u$ as $k \to \infty$ and by using the dominated convergence theorem, we obtain

$$\int_{\{|u| > k\}} |T_k(u) - u|^{p(x)} dx + \int_{\{|u| > k\}} |\nabla u|^{p(x)} dx \longrightarrow 0 \qquad \text{as} \quad k \to \infty.$$

Finally $\left\|T_k(u) - u\right\|_{1,p(\cdot)} \longrightarrow 0$ as $k \to \infty$.

Lemma 4.3. *Let $(u_n)_{n \in \mathbb{N}}$ be a bounded sequence in $W_0^{1,p(\cdot)}(\Omega)$.*

$$\text{If} \quad u_n \rightharpoonup u \quad \text{in } W_0^{1,p(\cdot)}(\Omega) \text{ (weak)} \qquad \text{then} \qquad T_k(u_n) \rightharpoonup T_k(u) \text{ in } W_0^{1,p(\cdot)}(\Omega) \text{(weak)}.$$

Proof. We have

$$\begin{aligned}
u_n \rightharpoonup u \text{ in } W_0^{1,p(\cdot)}(\Omega) \text{ (weak)} \quad &\Longrightarrow u_n \to u \text{ in } L^{q(x)}(\Omega) \text{ (strong)}, \qquad \forall 1 \le q(x) < p^*(x), \\
&\Longrightarrow u_n \to u \quad \text{a.e} \quad \text{in } \Omega, \\
&\Longrightarrow T_k(u_n) \to T_k(u) \quad \text{a.e} \quad \text{in } \Omega,
\end{aligned}$$

and since

$$T_k'(s) = \begin{cases} 1 & \text{if} \quad |s| \le k, \\ 0 & \text{if} \quad |s| > k, \end{cases}$$

then

$$\begin{aligned}
\int_\Omega \sum_{i=1}^N |\frac{\partial T_k(u_n)}{\partial x_i}|^{p(x)} dx &= \sum_{i=1}^N \int_{\{|u_n| \le k\}} |T_k'(u_n) \frac{\partial u_n}{\partial x_i}|^{p(x)} dx \\
&\le \sum_{i=1}^N \int_\Omega |\frac{\partial u_n}{\partial x_i}|^{p(x)} dx < \infty,
\end{aligned}$$

and we deduce that $(T_k(u_n))_n$ is bounded in $W_0^{1,p(\cdot)}(\Omega)$, then $T_k(u_n) \to v_k$ a.e in Ω, therefore $v_k = T_k(u)$ and we obtain

$$T_k(u_n) \rightharpoonup T_k(u) \quad \text{in} \quad W_0^{1,p(\cdot)}(\Omega) \text{ (weak)}.$$

Lemma 4.4. *(see [4]) Assuming that* $(3.2)-(3.4)$ *hold, and let* $(u_n)_{n\in\mathbb{N}}$ *be a sequence in* $W_0^{1,p(\cdot)}(\Omega)$ *such that* $u_n \rightharpoonup u$ *in* $W_0^{1,p(\cdot)}(\Omega)$ *and*

$$\int_\Omega (a(x,u_n,\nabla u_n)-a(x,u_n,\nabla u))\cdot(\nabla u_n-\nabla u)\,dx \longrightarrow 0, \tag{4.1}$$

then $u_n \to u$ *in* $W_0^{1,p(\cdot)}(\Omega)$ *for a subsequence.*

5 Strongly nonlinear problems.

5.1 The case of $f\in W^{-1,p'(\cdot)}(\Omega)$.

Definition 5.1. In the case of $f\in W^{-1,p'(\cdot)}(\Omega)$, A measurable function u is solution in the sense of distributions to the problem (3.7), if

$$\begin{cases} \displaystyle\int_\Omega a(x,u,\nabla u)\cdot\nabla v\,dx+\int_\Omega g(x,u,\nabla u)v\,dx=\int_\Omega fv\,dx \qquad \forall v\in W_0^{1,p(x)}(\Omega)\cap L^\infty(\Omega),\\ u\in W_0^{1,p(\cdot)}(\Omega),\quad g(x,u,\nabla u)\in L^1(\Omega),\quad g(x,u,\nabla u)u\in L^1(\Omega). \end{cases} \tag{5.1}$$

Theorem 5.2. *Assuming that* $(3.2)-(3.6)$ *holds and* $f\in W^{-1,p'(\cdot)}(\Omega)$. *Then the problem* (3.7) *has at least one solution in the sense of distributions.*

Proof of the Theorem 5.2.

Step 1 : Approximate problems.

We consider for all $n\geq 1$, the approximate problems

$$\begin{cases} Au_n+g_n(x,u_n,\nabla u_n)=f \quad \text{ in } \Omega.\\ u_n\in W_0^{1,p(\cdot)}(\Omega) \end{cases} \tag{5.2}$$

where $g_n(x,s,\xi)=\dfrac{g(x,s,\xi)}{1+\frac{1}{n}|g(x,s,\xi)|}$, note that

$$g_n(x,s,\xi)s\geq 0 \qquad , \qquad |g_n(x,s,\xi)|\leq |g(x,s,\xi)| \qquad \textit{and} \qquad |g_n(x,s,\xi)|\leq n.$$

We define the operator $G_n : W_0^{1,p(\cdot)}(\Omega)\longrightarrow W^{-1,,p'(\cdot)}(\Omega)$, by

$$\langle G_nu,v\rangle=\int_\Omega g_n(x,u,\nabla u)v\,dx \qquad \forall v\in W_0^{1,p(\cdot)}(\Omega),$$

by using the Hölder inequality, we have for all $u,\ v\in W_0^{1,p(\cdot)}(\Omega)$

$$\begin{aligned} \Big|\int_\Omega g_n(x,u,\nabla u)v\,dx\Big| &\leq (\frac{1}{p_-}+\frac{1}{p'_-})\,\big\|g_n(x,u,\nabla u)\big\|_{p'(\cdot)}\big\|v\big\|_{p(\cdot)}\\ &\leq (\frac{1}{p_-}+\frac{1}{p'_-})(\int_\Omega |g_n(x,u,\nabla u)|^{p'(x)}\,dx+1)^{\frac{1}{p'_-}}\|v\|_{1,p(\cdot)}\\ &\leq (\frac{1}{p_-}+\frac{1}{p'_-})(\int_\Omega n^{p'(x)}\,dx+1)^{\frac{1}{p'_-}}\|v\|_{1,p(\cdot)}\\ &\leq C_0\|v\|_{1,p(\cdot)}\,. \end{aligned} \tag{5.3}$$

Lemma 5.3. *The operator $B_n = A + G_n$ is pseudo-monotone from $W_0^{1,p(\cdot)}(\Omega)$ into $W^{-1,p'(\cdot)}(\Omega)$. Moreover, B_n is coercive in the following sense*

$$\frac{\langle B_n v, v\rangle}{||v||_{1,p(\cdot)}} \longrightarrow +\infty \qquad as \quad ||v||_{1,p(\cdot)} \longrightarrow +\infty \qquad for \quad v \in W_0^{1,p(\cdot)}(\Omega),$$

Proof of Lemma 5.3

Using the Hölder's inequality and the growth condition (3.2) we can show that the operator A is bounded, by using (5.3) we conclude that B_n is bounded. For the coercivity, in view of the Poincaré type inequality and (3.3)

$$\begin{aligned}\langle B_n u, u\rangle &= \int_\Omega a(x,u,\nabla u)\cdot\nabla u\,dx + \int_\Omega g_n(x,u,\nabla u)u\,dx \qquad \forall u \in W_0^{1,p(\cdot)}(\Omega)\\ &\geq \alpha \int_\Omega |\nabla u|^{p(x)}\,dx\\ &\geq \alpha ||\nabla u||^{\delta}_{p(\cdot)}\\ &\geq \alpha' ||u||^{\delta}_{1,p(\cdot)}\end{aligned}$$

with

$$\delta = \begin{cases} p_- & \text{if} \quad ||\nabla u||_{p(\cdot)} > 1,\\ p_+ & \text{if} \quad ||\nabla u||_{p(\cdot)} \leq 1,\end{cases}$$

then, we obtain

$$\frac{\langle B_n u, u\rangle}{||u||_{1,p(\cdot)}} \longrightarrow +\infty \qquad as \quad ||u||_{1,p(\cdot)} \longrightarrow +\infty.$$

It remain to show that B_n is pseudo-monotone. Let $(u_k)_{k\in\mathbb{N}}$ be a sequence in $W_0^{1,p(\cdot)}(\Omega)$ such that

$$\begin{cases} u_k \rightharpoonup u & \text{in } W_0^{1,p(\cdot)}(\Omega),\\ B_n u_k \rightharpoonup \chi & \text{in } W^{-1,p'(\cdot)}(\Omega),\\ \limsup\limits_{k\to\infty}\langle B_n u_k, u_k\rangle \leq \langle \chi, u\rangle. \end{cases} \tag{5.4}$$

We will prove that

$$\chi = B_n u \quad \text{ and } \quad \langle B_n u_k, u_k\rangle \longrightarrow \langle \chi, u\rangle \text{ as } k \to +\infty.$$

Firstly, since $W_0^{1,p(\cdot)}(\Omega) \hookrightarrow\hookrightarrow L^{p(\cdot)}(\Omega)$, then $u_k \to u$ in $L^{p(\cdot)}(\Omega)$ for a subsequence still denoted by u_k.

We have $(u_k)_{k\in\mathbb{N}}$ is a bounded sequence in $W_0^{1,p(\cdot)}(\Omega)$, then by the growth condition $a(x,u_k,\nabla u_k)$ is bounded in $(L^{p'(\cdot)}(\Omega))^N$, therefore there exists a function $\varphi \in (L^{p'(\cdot)}(\Omega))^N$ such that

$$a(x,u_k,\nabla u_k) \rightharpoonup \varphi \text{ in } \quad (L^{p'(\cdot)}(\Omega))^N \text{ as } k \to \infty. \tag{5.5}$$

Similarly, since $g_n(x,u_k,\nabla u_k)$ is bounded in $L^{p'(\cdot)}(\Omega)$, there exists a function $\psi_n \in L^{p'(\cdot)}(\Omega)$ such that

$$g_n(x,u_k,\nabla u_k) \rightharpoonup \psi_n \text{ in } \quad L^{p'(\cdot)}(\Omega) \text{ as } k \to \infty. \tag{5.6}$$

For all $v \in W_0^{1,p(\cdot)}(\Omega)$, we have

$$\begin{aligned}\langle \chi, v\rangle &= \lim_{k\to\infty}\langle B_n u_k, v\rangle \\ &= \lim_{k\to\infty}\int_\Omega a(x,u_k,\nabla u_k)\cdot\nabla v\,dx + \lim_{k\to\infty}\int_\Omega g_n(x,u_k,\nabla u_k)v\,dx \\ &= \int_\Omega \varphi\cdot\nabla v\,dx + \int_\Omega \psi_n v\,dx.\end{aligned} \tag{5.7}$$

Combining (5.4) and (5.7), we obtain

$$\begin{aligned}\limsup_{k\to\infty}\langle B_n u_k, u_k\rangle &= \limsup_{k\to\infty}\Big\{\int_\Omega a(x,u_k,\nabla u_k)\cdot\nabla u_k\,dx + \int_\Omega g_n(x,u_k,\nabla u_k)u_k\,dx\Big\} \\ &\le \int_\Omega \varphi\cdot\nabla u\,dx + \int_\Omega \psi_n u\,dx,\end{aligned} \tag{5.8}$$

using (5.6), we have

$$\int_\Omega g_n(x,u_k,\nabla u_k)u_k\,dx \longrightarrow \int_\Omega \psi_n u\,dx, \tag{5.9}$$

therefore

$$\limsup_{k\to\infty}\int_\Omega a(x,u_k,\nabla u_k)\cdot\nabla u_k\,dx \le \int_\Omega \varphi\cdot\nabla u\,dx. \tag{5.10}$$

On the other hand, thanks to (3.4) we have

$$\int_\Omega (a(x,u_k,\nabla u_k) - a(x,u_k,\nabla u))\cdot(\nabla u_k - \nabla u)\,dx \ge 0, \tag{5.11}$$

then

$$\int_\Omega a(x,u_k,\nabla u_k)\cdot\nabla u_k\,dx \ge \int_\Omega a(x,u_k,\nabla u_k)\cdot\nabla u\,dx + \int_\Omega a(x,u_k,\nabla u)\cdot(\nabla u_k - \nabla u)\,dx,$$

since $a(x,u_k,\nabla u) \to a(x,u,\nabla u)$ in $(L^{p'(\cdot)}(\Omega))^N$ and $\nabla u_k \rightharpoonup \nabla u$ in $(L^{p(\cdot)}(\Omega))^N$, it follows that

$$\int_\Omega a(x,u_k,\nabla u)\cdot(\nabla u_k - \nabla u)\,dx \longrightarrow 0 \qquad \text{as} \quad k\to\infty,$$

and by (5.5), we get

$$\liminf_{k\to\infty}\int_\Omega a(x,u_k,\nabla u_k)\cdot\nabla u_k\,dx \ge \int_\Omega \varphi\cdot\nabla u\,dx.$$

Using (5.10), we obtain

$$\lim_{k\to\infty}\int_\Omega a(x,u_k,\nabla u_k)\cdot\nabla u_k\,dx = \int_\Omega \varphi\cdot\nabla u\,dx. \tag{5.12}$$

By combining (5.7), (5.9) and (5.12), we deduce that

$$\langle B_n u_k, u_k\rangle \longrightarrow \langle \chi, u\rangle \ \text{ as } \ k\to+\infty.$$

Now, thanks to (5.12) we have

$$\lim_{k\to+\infty}\int_\Omega (a(x,u_k,\nabla u_k)-a(x,u_k,\nabla u))\cdot(\nabla u_k-\nabla u)\,dx=0,$$

in view of the Lemma 4.4,

$$u_k \longrightarrow u \quad \text{in} \quad W_0^{1,p(\cdot)}(\Omega) \qquad \text{and} \qquad \nabla u_k \to \nabla u \quad \text{a.e in } \Omega,$$

then

$$a(x,u_k,\nabla u_k) \rightharpoonup a(x,u,\nabla u) \quad \text{in} \quad (L^{p'(\cdot)}(\Omega))^N,$$

and

$$g_n(x,u_k,\nabla u_k) \rightharpoonup g_n(x,u,\nabla u) \quad \text{in} \quad L^{p'(\cdot)}(\Omega),$$

we deduce that $\chi = B_n u,$ which completes the proof the Lemma 5.3.

In view of Lemma 5.3, there exists at least one weak solution $u_n \in W_0^{1,p(\cdot)}(\Omega)$ of the problem (5.2), ($cf.$ [12]).

Step 2 : Weak convergence.

Taking u_n as a test function in (5.2), we obtain

$$\int_\Omega a(x,u_n,\nabla u_n)\cdot\nabla u_n\,dx+\int_\Omega g_n(x,u_n,\nabla u_n)u_n\,dx=\langle f,u_n\rangle_{W^{-1,p'(\cdot)}(\Omega),W_0^{1,p(\cdot)}(\Omega)}$$

since $g_n(x,u_n,\nabla u_n)u_n \geq 0,$ and by (3.3) we deduce that

$$\alpha\int_\Omega |\nabla u_n|^{p(x)}\,dx \leq \langle f,u_n\rangle_{W^{-1,p'(\cdot)}(\Omega),W_0^{1,p(\cdot)}(\Omega)}$$

thanks to the Hölder inequality, we get

$$\alpha\,\|\nabla u_n\|_{p(\cdot)}^{\gamma} \leq \Big(\frac{1}{p_-}+\frac{1}{p'_-}\Big)\|f\|_{-1,p'(\cdot)}\|u_n\|_{1,p(\cdot)} \quad \text{with} \quad \gamma=\begin{cases} p_- & \text{if} \quad \|\nabla u_n\|_{p(\cdot)}>1,\\ p_+ & \text{if} \quad \|\nabla u_n\|_{p(\cdot)}\leq 1,\end{cases}$$

therefore, by *the Poincaré type inequality*, we obtain

$$\|u_n\|_{1,p(\cdot)} \leq C_1. \tag{5.13}$$

with C_1 is a constant that does not depend on n. Then there exists a subsequence still denoted $(u_n)_{n\in\mathbb{N}}$ such that

$$\begin{cases} u_n \rightharpoonup u & \text{in} \quad W_0^{1,p(\cdot)}(\Omega)\\ u_n \to u & \text{in} \quad L^{p(\cdot)}(\Omega).\end{cases} \tag{5.14}$$

and in view of the Lemma 4.3, we get

$$\begin{cases} T_k(u_n) \rightharpoonup T_k(u) & \text{in} \quad W_0^{1,p(x)}(\Omega)\\ T_k(u_n) \to T_k(u) & \text{in} \quad L^{p(x)}(\Omega).\end{cases} \tag{5.15}$$

Step 3 : Strong convergence.

In the sequel, the functions of real numbers which converges to 0 as $n \to \infty$ will be denoted by $\varepsilon_i(n)$, $i = 1, 2, \dots$
Let $\varphi_k(s) = s.\exp(\gamma s^2)$ where $\gamma = (\frac{b(k)}{2\alpha})^2$, it is obvious that

$$\varphi_k'(s) - \frac{b(k)}{\alpha}|\varphi_k(s)| \geq \frac{1}{2} \qquad \forall s \in \mathbb{R},$$

we consider $h > k > 0$ and $M = 4k + h$, we set

$$\omega_n = T_{2k}(u_n - T_h(u_n) + T_k(u_n) - T_k(u)).$$

By taking $\varphi_k(\omega_n)$ as a test function in the approximate problem (5.2), we obtain

$$\int_\Omega a(x, u_n, \nabla u_n) \cdot \nabla \varphi_k(\omega_n)\, dx + \int_\Omega g_n(x, u_n, \nabla u_n)\varphi_k(\omega_n)\, dx = \int_\Omega f \varphi_k(\omega_n)\, dx.$$

it is clear that $\nabla \omega_n = 0$ on $\{|u_n| > M\}$, and since $g_n(x, u_n, \nabla u_n)\varphi_k(\omega_n) \geq 0$ on $\{|u_n| > k\}$, then

$$\int_\Omega a(x, T_M(u_n), \nabla T_M(u_n))\varphi_k'(\omega_n) \cdot \nabla \omega_n\, dx + \int_{\{|u_n| \leq k\}} g_n(x, u_n, \nabla u_n)\varphi_k(\omega_n)\, dx \leq \int_\Omega f \varphi_k(\omega_n)\, dx. \tag{5.16}$$

We have

$$\begin{aligned} &\int_\Omega a(x, T_M(u_n), \nabla T_M(u_n))\varphi_k'(\omega_n) \cdot \nabla \omega_n\, dx \\ &= \int_{\{|u_n| \leq k\}} a(x, T_k(u_n), \nabla T_k(u_n))\varphi_k'(\omega_n) \cdot \nabla T_{2k}(u_n - T_k(u))\, dx \\ &+ \int_{\{|u_n| > k\}} a(x, T_M(u_n), \nabla T_M(u_n))\varphi_k'(\omega_n) \cdot \nabla T_{2k}(u_n - T_h(u_n) + T_k(u_n) - T_k(u))\, dx. \end{aligned} \tag{5.17}$$

On the one hand, since $|u_n - T_k(u)| \leq 2k$ on $\{|u_n| \leq k\}$, then

$$\begin{aligned} &\int_{\{|u_n| \leq k\}} a(x, T_k(u_n), \nabla T_k(u_n))\varphi_k'(\omega_n) \cdot \nabla T_{2k}(u_n - T_k(u))\, dx \\ &= \int_\Omega a(x, T_k(u_n), \nabla T_k(u_n))\varphi_k'(\omega_n) \cdot (\nabla T_k(u_n) - \nabla T_k(u))\, dx \\ &+ \int_{\{|u_n| > k\}} a(x, T_k(u_n), \nabla T_k(u_n))\varphi_k'(\omega_n) \cdot \nabla T_k(u)\, dx. \end{aligned} \tag{5.18}$$

For the second term on the right hand side of (5.17), taking $z_n = u_n - T_h(u_n) + T_k(u_n) - T_k(u)$, then

$$\begin{aligned} &\int_{\{|u_n| > k\}} a(x, T_M(u_n), \nabla T_M(u_n))\varphi_k'(\omega_n) \cdot \nabla T_{2k}(u_n - T_h(u_n) + T_k(u_n) - T_k(u))\, dx \\ &= \int_{\{|u_n| > k\} \cap \{|z_n| \leq 2k\}} a(x, T_M(u_n), \nabla T_M(u_n))\varphi_k'(\omega_n) \cdot \nabla(u_n - T_k(u)) \cdot \chi_{\{|u_n| > h\}}\, dx \\ &- \int_{\{|u_n| > k\} \cap \{|z_n| \leq 2k\}} a(x, T_M(u_n), \nabla T_M(u_n))\varphi_k'(\omega_n) \cdot \nabla T_k(u) \cdot \chi_{\{|u_n| \leq h\}}\, dx \\ &\geq - \int_{\{|u_n| > k\}} |a(x, T_M(u_n), \nabla T_M(u_n))||\nabla T_k(u)|\varphi_k'(\omega_n)\, dx. \end{aligned} \tag{5.19}$$

By combining (5.17), (5.18) and (5.19), we get

$$\begin{aligned}
&\int_\Omega a(x,T_M(u_n),\nabla T_M(u_n))\varphi_k'(\omega_n)\cdot\nabla\omega_n\,dx\\
&\quad\geq\int_\Omega a(x,T_k(u_n),\nabla T_k(u_n))\varphi_k'(\omega_n)\cdot\nabla(T_k(u_n)-T_k(u))\,dx\\
&\quad+\int_{\{|u_n|>k\}} a(x,T_k(u_n),\nabla T_k(u_n))\varphi_k'(\omega_n)\cdot\nabla T_k(u)\,dx\\
&\quad-\int_{\{|u_n|>k\}} |a(x,T_M(u_n),\nabla T_M(u_n))|\,|\nabla T_k(u)|\varphi_k'(\omega_n)\,dx,
\end{aligned}\tag{5.20}$$

since $1\leq\varphi_k'(\omega_n)\leq\varphi_k'(2k)$, we get

$$\begin{aligned}
&\int_\Omega (a(x,T_k(u_n),\nabla T_k(u_n))-a(x,T_k(u_n),\nabla T_k(u)))\cdot(\nabla T_k(u_n)-\nabla T_k(u))\varphi_k'(\omega_n)\,dx\\
&\leq\int_{\{|u_n|>k\}} |a(x,T_M(u_n),\nabla T_M(u_n))|\,|\nabla T_k(u)|\varphi_k'(\omega_n)\,dx\\
&\quad-\int_{\{|u_n|>k\}} a(x,T_k(u_n),\nabla T_k(u_n))\varphi_k'(\omega_n)\cdot\nabla T_k(u)\,dx\\
&\quad+\int_\Omega a(x,T_M(u_n),\nabla T_M(u_n))\varphi_k'(\omega_n)\cdot\nabla\omega_n\,dx\\
&\quad-\int_\Omega a(x,T_k(u_n),\nabla T_k(u))\cdot(\nabla T_k(u_n)-\nabla T_k(u))\varphi_k'(\omega_n)\,dx\\
&\leq\varphi_k'(2k)\int_{\{|u_n|>k\}} |a(x,T_M(u_n),\nabla T_M(u_n))|\,|\nabla T_k(u)|\,dx\\
&\quad+\varphi_k'(2k)\int_{\{|u_n|>k\}} |a(x,T_k(u_n),\nabla T_k(u_n))|\,|\nabla T_k(u)|\,dx\\
&\quad+\int_\Omega a(x,T_M(u_n),\nabla T_M(u_n))\varphi_k'(\omega_n)\cdot\nabla\omega_n\,dx\\
&\quad+\varphi_k'(2k)\int_\Omega |a(x,T_k(u_n),\nabla T_k(u))|\,|\nabla T_k(u_n)-\nabla T_k(u)|\,dx.
\end{aligned}\tag{5.21}$$

We will study each term on the right hand side of the above inequality.
For the first and second terms on the right hand side of (5.21), we have $(|a(x,T_M(u_n),\nabla T_M(u_n))|)_n$ and $(|a(x,T_k(u_n),\nabla T_k(u_n))|)_n$ are bounded in $L^{p'(\cdot)}(\Omega)$, and since

$$|\nabla T_k(u)|^{p(x)}\chi_{\{|u_n|>k\}}\leq|\nabla T_k(u)|^{p(x)},$$

with

$$|\nabla T_k(u)|^{p(x)}\chi_{\{|u_n|>k\}}\longrightarrow 0,\quad \text{a.e. in}\quad\Omega\qquad\text{as}\quad n\to\infty,$$

in view of the Lebesgue dominated convergence theorem, we deduce that

$$|\nabla T_k(u)|\chi_{\{|u_n|>k\}}\longrightarrow 0\quad\text{in } L^{p(\cdot)}(\Omega)\qquad\text{as}\quad n\to\infty,$$

which implies that the first and second terms in the right hand side of (5.21) tends to 0 as n tends to ∞, we can write

$$\varepsilon_1(n)=\varphi_k'(2k)\int_{\{|u_n|>k\}} |a(x,T_M(u_n),\nabla T_M(u_n))||\nabla T_k(u)|\,dx\longrightarrow 0\qquad\text{as}\quad n\to\infty,\tag{5.22}$$

and

$$\varepsilon_2(n) = \varphi_k'(2k)\int_{\{|u_n|>k\}} |a(x,T_k(u_n),\nabla T_k(u_n))||\nabla T_k(u)|\,dx \longrightarrow 0 \qquad \text{as} \quad n\to\infty. \tag{5.23}$$

For the last term on the right hand side of (5.21), we have

$$|a(x,T_k(u_n),\nabla T_k(u))| \longrightarrow |a(x,T_k(u),\nabla T_k(u))| \quad \text{in} \qquad L^{p'(\cdot)}(\Omega),$$

and since $\nabla T_k(u_n)$ tends weakly to $\nabla T_k(u)$ in $(L^{p(\cdot)}(\Omega))^N$, we obtain

$$\varepsilon_3(n) = \varphi_k'(2k)\int_\Omega |a(x,T_k(u_n),\nabla T_k(u))|.|\nabla T_k(u_n)-\nabla T_k(u)|\,dx \longrightarrow 0 \qquad \text{as} \quad n\to\infty. \tag{5.24}$$

We conclude, by (5.21) that

$$\begin{aligned}&\int_\Omega (a(x,T_k(u_n),\nabla T_k(u_n))-a(x,T_k(u_n),\nabla T_k(u)))\cdot(\nabla T_k(u_n)-\nabla T_k(u))\varphi_k'(\omega_n)\,dx\\ &\quad\leq \int_\Omega a(x,T_M(u_n),\nabla T_M(u_n))\varphi_k'(\omega_n)\cdot\nabla\omega_n\,dx+\varepsilon_4(n).\end{aligned}\tag{5.25}$$

Now, we turn to the second term on the left hand side of (5.16), in view of the growth condition (3.6) we have

$$\begin{aligned}&\Big|\int_{\{|u_n|\leq k\}} g_n(x,u_n,\nabla u_n)\varphi_k(\omega_n)\,dx\Big| \leq \int_{\{|u_n|\leq k\}} b(|u_n|)(c(x)+|\nabla T_k(u_n)|^{p(x)})|\varphi_k(\omega_n)|\,dx\\ &\quad\leq b(k)\int_{\{|u_n|\leq k\}} c(x)|\varphi_k(\omega_n)|\,dx+\frac{b(k)}{\alpha}\int_\Omega a(x,T_k(u_n),\nabla T_k(u_n))\cdot\nabla T_k(u_n)|\varphi_k(\omega_n)|\,dx\\ &\quad\leq b(k)\int_{\{|u_n|\leq k\}} c(x)|\varphi_k(\omega_n)|\,dx\\ &\qquad+\frac{b(k)}{\alpha}\int_\Omega (a(x,T_k(u_n),\nabla T_k(u_n))-a(x,T_k(u_n),\nabla T_k(u)))\cdot(\nabla T_k(u_n)-\nabla T_k(u))|\varphi_k(\omega_n)|\,dx\\ &\qquad+\frac{b(k)}{\alpha}\int_\Omega a(x,T_k(u_n),\nabla T_k(u))\cdot(\nabla T_k(u_n)-\nabla T_k(u))|\varphi_k(\omega_n)|\,dx\\ &\qquad+\frac{b(k)}{\alpha}\int_\Omega a(x,T_k(u_n),\nabla T_k(u_n))\cdot\nabla T_k(u)|\varphi_k(\omega_n)|\,dx,\end{aligned}$$

then

$$\begin{aligned}&\frac{b(k)}{\alpha}\int_\Omega (a(x,T_k(u_n),\nabla T_k(u_n))-a(x,T_k(u_n),\nabla T_k(u)))\cdot(\nabla T_k(u_n)-\nabla T_k(u))|\varphi_k(\omega_n)|\,dx\\ &\quad\geq \Big|\int_{\{|u_n|\leq k\}} g_n(x,u_n,\nabla u_n)\varphi_k(\omega_n)\,dx\Big| - b(k)\int_{\{|u_n|\leq k\}} c(x)|\varphi_k(\omega_n)|\,dx\\ &\qquad-\frac{b(k)}{\alpha}\int_\Omega a(x,T_k(u_n),\nabla T_k(u))\cdot(\nabla T_k(u_n)-\nabla T_k(u))|\varphi_k(\omega_n)|\,dx\\ &\qquad-\frac{b(k)}{\alpha}\int_\Omega a(x,T_k(u_n),\nabla T_k(u_n))\cdot\nabla T_k(u)|\varphi_k(\omega_n)|\,dx.\end{aligned}\tag{5.26}$$

We have $|\varphi_k(\omega_n)|\cdot\chi_{\{|u_n|\leq k\}} \rightharpoonup |\varphi_k(T_{2k}(u-T_h(u)))|\cdot\chi_{\{|u|\leq k\}}$ weak-$*$ in $L^\infty(\Omega)$, then

$$\int_{\{|u_n|\leq k\}} c(x)|\varphi_k(\omega_n)|\,dx \longrightarrow \int_{\{|u|\leq k\}} c(x)|\varphi_k(T_{2k}(u-T_h(u)))|\,dx = 0 \qquad \text{as} \quad n\to\infty.$$

Concerning the third term on the right hand side of (5.26), we have

$$\int_\Omega a(x,T_k(u_n),\nabla T_k(u))\cdot(\nabla T_k(u_n)-\nabla T_k(u))|\varphi_k(\omega_n)|\,dx \\ \leq \varphi_k(2k)\int_\Omega |a(x,T_k(u_n),\nabla T_k(u))||\nabla T_k(u_n)-\nabla T_k(u)|\,dx,$$

and thanks to (5.24), we obtain

$$\int_\Omega a(x,T_k(u_n),\nabla T_k(u))\cdot(\nabla T_k(u_n)-\nabla T_k(u))|\varphi_k(\omega_n)|\,dx \longrightarrow 0 \quad \text{as} \quad n\to\infty. \tag{5.27}$$

For the last term of right hand side of (5.26), we have $(a(x,T_k(u_n),\nabla T_k(u_n)))_n$ is bounded in $(L^{p'(\cdot)}(\Omega))^N$ then there exists $\varphi\in(L^{p'(\cdot)}(\Omega))^N$ such that $a(x,T_k(u_n),\nabla T_k(u_n))\rightharpoonup\varphi$ in $(L^{p'(\cdot)}(\Omega))^N$, and since

$$\nabla T_k(u)|\varphi_k(\omega_n)| \longrightarrow \nabla T_k(u)|\varphi_k(T_{2k}(u-T_h(u)))| \quad \text{in} \quad (L^{p(\cdot)}(\Omega))^N,$$

it follows that

$$\int_\Omega a(x,T_k(u_n),\nabla T_k(u_n))\cdot\nabla T_k(u)|\varphi_k(\omega_n)|\,dx \longrightarrow \int_\Omega \varphi\nabla T_k(u)|\varphi_k(T_{2k}(u-T_h(u)))|\,dx=0. \tag{5.28}$$

Putting together (5.26) – (5.28), we get

$$\frac{b(k)}{\alpha}\int_\Omega (a(x,T_k(u_n),\nabla T_k(u_n))-a(x,T_k(u_n),\nabla T_k(u)))\cdot(\nabla T_k(u_n)-\nabla T_k(u))|\varphi_k(\omega_n)|\,dx \\ \geq \left|\int_{\{|u_n|\leq k\}} g_n(x,u_n,\nabla u_n)\varphi_k(\omega_n)\,dx\right| + \varepsilon_5(n), \tag{5.29}$$

By combining (5.25) and (5.29), we obtain

$$\begin{aligned}\int_\Omega &(a(x,T_k(u_n),\nabla T_k(u_n))-a(x,T_k(u_n),\nabla T_k(u)))\cdot(\nabla T_k(u_n)-\nabla T_k(u))\Big(\varphi_k'(\omega_n)-\frac{b(k)}{\alpha}|\varphi_k(\omega_n)|\Big)\,dx\\ &\leq \int_\Omega a(x,T_M(u_n),\nabla T_M(u_n))\varphi_k'(\omega_n)\cdot\nabla\omega_n\,dx - \left|\int_{\{|u_n|\leq k\}} g_n(x,u_n,\nabla u_n)\varphi_k(\omega_n)\,dx\right| + \varepsilon_6(n)\\ &\leq \int_\Omega f\varphi_k(\omega_n)\,dx+\varepsilon_6(n)\\ &\leq \int_\Omega f\varphi_k(T_{2k}(u-T_h(u)))\,dx+\varepsilon_7(n).\end{aligned} \tag{5.30}$$

since $\varphi_k(\omega_n)\rightharpoonup\varphi_k(T_{2k}(u-T_h(u)))$ in $W_0^{1,p(x)}(\Omega)$. Then by letting h tends to infinity in (5.30) , we conclude

$$\int_\Omega (a(x,T_k(u_n),\nabla T_k(u_n))-a(x,T_k(u_n),\nabla T_k(u)))\cdot(\nabla T_k(u_n)-\nabla T_k(u))\,dx\to 0 \quad \text{as} \quad n\to\infty \tag{5.31}$$

In view of the Lemma 4.4, we deduce that

$$\begin{cases} T_k(u_n)\longrightarrow T_k(u) & \text{in} \quad W_0^{1,p(\cdot)}(\Omega),\\ \nabla u_n\longrightarrow\nabla u \quad \text{a.e.} & \text{in} \quad \Omega.\end{cases} \tag{5.32}$$

Step 4 :Equi-integrability and passage to the limit.

Thanks to (5.32), we have

$$\begin{aligned} a(x,u_n,\nabla u_n) &\longrightarrow a(x,u,\nabla u) \quad a.e \ in \ \Omega, \\ g_n(x,u_n,\nabla u_n) &\longrightarrow g(x,u,\nabla u) \quad a.e \ in \ \Omega, \\ g_n(x,u_n,\nabla u_n)u_n &\longrightarrow g(x,u,\nabla u)u \quad a.e \ in \ \Omega, \end{aligned}$$

In the view of (5.13), we have $\|u_n\|_{1,p(\cdot)} \le C_1$, then $(a(x,u_n,\nabla u_n))_n$ is bounded in $(L^{p'(\cdot)}(\Omega))^N$, and by using the Lemma 4.1, we obtain

$$a(x,u_n,\nabla u_n) \rightharpoonup a(x,u,\nabla u) \qquad \text{in} \quad (L^{p'(\cdot)}(\Omega))^N \, (weak).$$

Now, we prove that

$$g_n(x,u_n,\nabla u_n) \longrightarrow g(x,u,\nabla u) \quad \text{strongly in} \quad L^1(\Omega),$$

using the Vitali convergence theorem, it is sufficient to show that $g_n(x,u_n,\nabla u_n)$ is uniformly equi-integrable. Indeed, taking $T_1(u_n - T_h(u_n))$ as a test function in (5.2), we obtain

$$\begin{aligned} \int_\Omega a(x,u_n,\nabla u_n)\cdot\nabla T_1(u_n - T_h(u_n))dx + \int_\Omega g_n(x,u_n,\nabla u_n)T_1(u_n - T_h(u_n))dx \\ = \int_\Omega f T_1(u_n - T_h(u_n))dx, \end{aligned}$$

it follows that

$$\int_{\{h\le|u_n|\}} g_n(x,u_n,\nabla u_n)T_1(u_n - T_h(u_n))\,dx \le \int_{\{h\le|u_n|\}} f T_1(u_n - T_h(u_n))\,dx, \tag{5.33}$$

then

$$\begin{aligned} \int_{\{h+1\le|u_n|\}} |g_n(x,u_n,\nabla u_n)|\,dx &\le \int_{\{h\le|u_n|\}} g_n(x,u_n,\nabla u_n)T_1(u_n - T_h(u_n))\,dx \\ &\le \int_{\{h\le|u_n|\}} f T_1(u_n - T_h(u_n))\,dx \\ &\le (\frac{1}{p_-} + \frac{1}{p'_-})\|f\|_{-1,p'(\cdot)}\|T_1(u_n - T_h(u_n))\|_{1,p(\cdot)} \longrightarrow 0 \quad \text{as} \quad h\to\infty, \end{aligned}$$

thus, for all $\eta > 0$, there exists $h(\eta) > 0$ such that

$$\int_{\{h(\eta)\le|u_n|\}} |g_n(x,u_n,\nabla u_n)|\,dx \le \frac{\eta}{2}. \tag{5.34}$$

On the other hand, for any measurable subset $E \subset \Omega$, we have

$$\int_E |g_n(x,u_n,\nabla u_n)|\,dx \le \int_E b(h)(c(x) + |\nabla T_h(u_n)|^{p(x)})\,dx + \int_{\{|u_n|\ge h(\eta)\}} |g_n(x,u_n,\nabla u_n)|\,dx, \tag{5.35}$$

there exists $\beta(\eta) > 0$ such that

$$b(h)\int_E (c(x) + |\nabla T_h(u_n)|^{p(x)})\,dx \le \frac{\eta}{2} \qquad \text{for} \quad meas(E) \le \beta(\eta). \tag{5.36}$$

By combining (5.34), (5.35) and (5.36), we obtain

$$\int_E |g_n(x,u_n,\nabla u_n)|\,dx \le \eta, \qquad \text{with} \quad meas(E) \le \beta(\eta), \tag{5.37}$$

in view of the Vitali convergence theorem we deduce that $g_n(x,u_n,\nabla u_n) \to g(x,u,\nabla u)$ in $L^1(\Omega)$. Now, taking $v \in W_0^{1,p(\cdot)}(\Omega)\cap L^\infty(\Omega)$ as a test function in (5.2), we have

$$\int_\Omega a(x,u_n,\nabla u_n)\,\nabla v\,dx + \int_\Omega g_n(x,u_n,\nabla u_n)\,v\,dx = \int_\Omega f v\,dx,$$

by letting n tends to ∞, we obtain

$$\int_\Omega a(x,u,\nabla u)\nabla v\,dx + \int_\Omega g(x,u,\nabla u)v\,dx = \int_\Omega f v\,dx \qquad \forall v \in W_0^{1,p(\cdot)}(\Omega)\cap L^\infty(\Omega). \tag{5.38}$$

Moreover, by using u_n as a test function in (5.2), we have

$$\int_\Omega a(x,u_n,\nabla u_n)\nabla u_n\,dx + \int_\Omega g_n(x,u_n,\nabla u_n)u_n\,dx = \int_\Omega f u_n\,dx$$

thanks to (5.13), we get

$$\int_\Omega g_n(x,u_n,\nabla u_n)u_n\,dx \le (\frac{1}{p_-}+\frac{1}{p'_-})\|f\|_{-1,p'(x)}\|u_n\|_{1,p(\cdot)} \le (\frac{1}{p_-}+\frac{1}{p'_-})\|f\|_{-1,p'(x)}C_1 = C_2,$$

since $g_n(x,u_n,\nabla u_n)u_n \ge 0$ and $g_n(x,u_n,\nabla u_n)u_n \to g(x,u,\nabla u)u$ a.e in Ω, in view of the *Fatou's lemma*, we deduce that

$$0 \le \int_\Omega g(x,u,\nabla u)u\,dx \le C_2 \qquad \text{then} \qquad g(x,u,\nabla u)u \in L^1(\Omega).$$

which ended the demonstration of the Theorem 5.2.

5.2 The case of $f \in L^1(\Omega)$.

Assuming that

$$\begin{cases} f \in L^1(\Omega), \\ \exists \rho_1,\rho_2 > 0, \text{ such that}: \quad \text{if} \quad |s| > \rho_1 \qquad \text{then} \qquad |g(x,s,\xi)| \ge \rho_2\,|\xi|^{p(x)}. \end{cases} \tag{5.39}$$

Definition 5.4. In the case of $f \in L^1(\Omega)$, A measurable function u is solution in the sense of distributions to the problem (3.7), if

$$\begin{cases} \displaystyle\int_\Omega a(x,u,\nabla u)\cdot\nabla v\,dx + \int_\Omega g(x,u,\nabla u)v\,dx = \int_\Omega f v\,dx \qquad \forall v \in W_0^{1,p(x)}(\Omega)\cap L^\infty(\Omega), \\ u \in W_0^{1,p(\cdot)}(\Omega), \quad g(x,u,\nabla u) \in L^1(\Omega). \end{cases} \tag{5.40}$$

Theorem 5.5. *Let* $f \in L^1(\Omega)$, *assuming that* (3.2)−(3.6) *and* (5.39) *holds, then the problem* (3.7) *has at least one solution in the sense of distributions.*

Proof of the Theorem 5.5.

Step 1 : Approximate problems.

Let $(f_n)_{n\in\mathbb{N}^*}$ a sequence in $W^{-1,p'(\cdot)}(\Omega)\cap L^1(\Omega)$ such that $f_n \longrightarrow f$ in $L^1(\Omega)$ with $|f_n|\leq|f|$ (for example $f_n = T_n(f)$). We consider the approximate problem

$$Au_n + g(x,u_n,\nabla u_n) = f_n \quad \text{in} \quad \Omega, \tag{5.41}$$

Thanks to the Theorem 5.2, there exists at least one solution in the sense of distributions $u_n \in W_0^{1,p(\cdot)}(\Omega)$ for the $p(x)$-elliptic problem (5.41).

Step 2 : Weak convergence.

Taking $T_k(u_n)$ as a test function in (5.41), we obtain

$$\int_\Omega a(x,u_n,\nabla u_n)\cdot\nabla T_k(u_n)\,dx + \int_\Omega g(x,u_n,\nabla u_n)T_k(u_n)\,dx = \int_\Omega f_n T_k(u_n)\,dx,$$

and since

$$\int_\Omega g(x,u_n,\nabla u_n)T_k(u_n)\,dx \geq 0,$$

then

$$\int_\Omega a(x,T_k(u_n),\nabla T_k(u_n))\cdot\nabla T_k(u_n)\,dx \leq k\int_\Omega |f_n|\,dx \leq k\|f\|_1,$$

and by (3.3), we obtain

$$\int_\Omega |\nabla T_k(u_n)|^{p(x)}\,dx \leq \frac{k}{\alpha}\|f\|_1, \tag{5.42}$$

also, we have

$$k\int_{\{|u_n|>k\}} |g(x,u_n,\nabla u_n)|\,dx \leq k\|f\|_1, \tag{5.43}$$

By combining (5.39), (5.42) and (5.43), for $k\geq\rho_1$ we deduce that

$$\begin{aligned}\int_\Omega |\nabla u_n|^{p(x)}\,dx &= \int_\Omega |\nabla T_k(u_n)|^{p(x)}\,dx + \int_{\{|u_n|>k\}} |\nabla u_n|^{p(x)}\,dx,\\ &\leq \frac{k}{\alpha}\|f\|_1 + \frac{1}{\rho_2}\int_{\{|u_n|>k\}} |g(x,u_n,\nabla u_n)|\,dx.\\ &\leq \frac{k}{\alpha}\|f\|_1 + \frac{\|f\|_1}{\rho_2} = C_3,\end{aligned}$$

using the Poincaré type inequality we obtain

$$\|u_n\|_{1,p(\cdot)}^{\gamma} \leq C_4 \qquad \text{with} \qquad \gamma = \begin{cases} p_- & \text{if} \quad \|\nabla u_n\|_{p(\cdot)} > 1,\\ p_+ & \text{if} \quad \|\nabla u_n\|_{p(\cdot)} \leq 1,\end{cases}$$

then

$$\|u_n\|_{1,p(\cdot)} \leq C_5,$$

and we conclude that

$$\begin{cases} T_k(u_n) \rightharpoonup T_k(u) & \text{in} \quad W_0^{1,p(\cdot)}(\Omega),\\ T_k(u_n) \longrightarrow T_k(u) & \text{in} \quad L^{p(\cdot)}(\Omega).\end{cases} \tag{5.44}$$

Step 3 : Strong convergence.

By taking $\varphi_k(\omega_n)$ as a test function in the approximate problem (5.41), we have

$$\int_\Omega a(x,u_n,\nabla u_n)\cdot\nabla\varphi_k(\omega_n)\,dx+\int_\Omega g(x,u_n,\nabla u_n)\varphi_k(\omega_n)\,dx=\int_\Omega f_n\varphi_k(\omega_n)\,dx,$$

it follows that

$$\int_\Omega a(x,T_M(u_n),\nabla T_M(u_n))\varphi_k'(\omega_n)\cdot\nabla\omega_n\,dx+\int_{\{|u_n|\le k\}} g(x,u_n,\nabla u_n)\varphi_k(\omega_n)\,dx\le\int_\Omega f_n\varphi_k(\omega_n)\,dx.$$

Using the same way as in (5.25) and (5.29), we can prove that

$$\begin{aligned}\int_\Omega (a(x,T_k(u_n),\nabla T_k(u_n))-a(x,T_k(u_n),\nabla T_k(u)))\cdot(\nabla T_k(u_n)-\nabla T_k(u))\varphi_k'(\omega_n)\,dx\\ \le\int_\Omega a(x,T_M(u_n),\nabla T_M(u_n))\varphi_k'(\omega_n)\cdot\nabla\omega_n\,dx+\varepsilon_8(n).\end{aligned}$$

and

$$\begin{aligned}\frac{b(k)}{\alpha}\int_\Omega (a(x,T_k(u_n),\nabla T_k(u_n))-a(x,T_k(u_n),\nabla T_k(u)))\cdot(\nabla T_k(u_n)-\nabla T_k(u))|\varphi_k(\omega_n)|\,dx\\ \ge\left|\int_{\{|u_n|\le k\}} g(x,u_n,\nabla u_n)\varphi_k(\omega_n)\,dx\right|+\varepsilon_9(n).\end{aligned}$$

We conclude that

$$\begin{aligned}&\int_\Omega (a(x,T_k(u_n),\nabla T_k(u_n))-a(x,T_k(u_n),\nabla T_k(u)))\cdot(\nabla T_k(u_n)-\nabla T_k(u))\Big(\varphi_k'(\omega_n)-\frac{b(k)}{\alpha}|\varphi_k(\omega_n)|\Big)\,dx\\ &\quad\le\int_\Omega a(x,T_M(u_n),\nabla T_M(u_n))\varphi_k'(\omega_n)\cdot\nabla\omega_n\,dx-\left|\int_{\{|u_n|\le k\}} g(x,u_n,\nabla u_n)\varphi_k(\omega_n)\,dx\right|+\varepsilon_{10}(n)\\ &\quad\le\int_\Omega f_n\varphi_k(\omega_n)\,dx+\varepsilon_{10}(n)\\ &\quad\le\int_\Omega f\varphi_k(T_{2k}(u-T_h(u)))\,dx+\varepsilon_{11}(n),\end{aligned}$$

since $f_n\to f$ in $L^1(\Omega)$ and $\varphi_k(\omega_n)\rightharpoonup\varphi_k(T_{2k}(u-T_h(u)))$ weak-$*$ in $L^\infty(\Omega)$. By letting h tends to infinity in the previews inequality, we get

$$\int_\Omega (a(x,T_k(u_n),\nabla T_k(u_n))-a(x,T_k(u_n),\nabla T_k(u)))\cdot(\nabla T_k(u_n)-\nabla T_k(u))\,dx\longrightarrow 0\quad\text{as}\quad n\to\infty,$$

using the Lemma 4.4, we deduce that

$$\begin{cases} T_k(u_n)\longrightarrow T_k(u) & \text{in}\quad W_0^{1,p(\cdot)}(\Omega),\\ \nabla u_n\longrightarrow\nabla u\quad\text{a.e.} & \text{in}\quad \Omega.\end{cases}\tag{5.45}$$

Step 4 : :Equi-integrability and passage to the limit.

Thanks to (5.45), we have

$$\begin{array}{lcl} a(x,u_n,\nabla u_n) & \longrightarrow & a(x,u,\nabla u) \quad a.e \ in\ \Omega, \\ g(x,u_n,\nabla u_n) & \longrightarrow & g(x,u,\nabla u) \quad a.e \ in\ \Omega, \end{array}$$

since $a(x,u_n,\nabla u_n)$ is bounded in $(L^{p'(\cdot)}(\Omega))^N$ and using the Lemma 4.1, we obtain

$$a(x,u_n,\nabla u_n) \rightharpoonup a(x,u,\nabla u) \qquad in \quad (L^{p'(\cdot)}(\Omega))^N\,(weak).$$

Now, let E be a measurable subset of Ω, for all $m>1$ we have

$$\begin{aligned} \int_E |g(x,u_n,\nabla u_n)|\,dx &= \int_{E\cap\{|u_n|\le m\}} |g(x,u_n,\nabla u_n)|\,dx + \int_{E\cap\{|u_n|>m\}} |g(x,u_n,\nabla u_n)|\,dx \\ &\le b(m)\int_E (c(x)+|\nabla T_m(u_n)|^{p(x)})\,dx + \int_{\{|u_n|>m\}} |g(x,u_n,\nabla u_n)|\,dx. \end{aligned}$$

For any $\varepsilon>0$, there exists $\beta(\varepsilon)>0$ such that

$$b(m)\int_E (c(x)+|\nabla T_m(u_n)|^{p(x)})\,dx \le \frac{\varepsilon}{2} \qquad \text{for} \quad meas(E)\le\beta(\varepsilon). \tag{5.46}$$

Taking $T_1(u_n - T_{m-1}(u_n))$ as a test function in (5.41), we obtain

$$\begin{aligned} \int_{\{|u_n|>m\}} |g(x,u_n,\nabla u_n)|\,dx &\le \int_{\{|u_n|>m-1\}} |f_n|\,dx \\ &\le \int_{\{|u_n|>m-1\}} |f|\,dx \longrightarrow 0 \qquad \text{as} \quad m\to\infty, \end{aligned}$$

then, there exists $m_0(\varepsilon)>0$ such that

$$\int_{\{|u_n|>m\}} |g(x,u_n,\nabla u_n)|\,dx < \frac{\varepsilon}{2} \qquad \forall m>m_0(\varepsilon). \tag{5.47}$$

Using (5.46) and (5.47), we deduce the equi-integrability of $g(x,u_n,\nabla u_n)$. In view of the Vitali convergence theorem, we obtain

$$g(x,u_n,\nabla u_n) \longrightarrow g(x,u,\nabla u) \qquad \text{strongly in} \quad L^1(\Omega).$$

Let $v\in W_0^{1,p(\cdot)}(\Omega)\cap L^\infty(\Omega)$, it is easy to pass to the limit in

$$\int_\Omega a(x,u_n,\nabla u_n)\,\nabla v\,dx + \int_\Omega g(x,u_n,\nabla u_n)\,v\,dx = \int_\Omega f_n v\,dx,$$

to obtain

$$\int_\Omega a(x,u,\nabla u)\nabla v\,dx + \int_\Omega g(x,u,\nabla u)v\,dx = \int_\Omega f v\,dx, \tag{5.48}$$

which completes our proof.

Example 5.6. We consider the following functions

$$a(x,u,\nabla u) = |\nabla u|^{p(x)-2}\nabla u \qquad \text{and} \qquad g(x,u,\nabla u) = (1+|\nabla u|^{p(x)})|u|^{p(x)-2}u\,,$$

it is clear that $a(x,u,\nabla u)$ and $g(x,u,\nabla u)$ verifies (3.2) – (3.4) and (3.5) – (3.6) respectively, then by the theorem 5.2, the problem

$$\begin{cases} -\mathrm{div}(|\nabla u|^{p(x)-2}\nabla u) + (1+|\nabla u|^{p(x)})|u|^{p(x)-2}u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{5.49}$$

has at least one solution in the sense of distributions for all $f \in W^{-1,p'(\cdot)}(\Omega)$.
Moreover, in the case of $f \in L^1(\Omega)$, since $g(x,u,\nabla u)$ verifies the condition (5.39) (taking $\rho_1 = \rho_2 = 1$), in view of the theorem 5.5, the problem (5.49) has at least one solution in the sense of distributions.

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